On Weakly Commutative SCI-Rings and Generalized Commutative SCS-Rings

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Abstract

Let $R$ be a ring. A left $R$-module $M$ is said to be weakly Co-Hopfian (resp. generalized Hopfian) if every injective (resp. surjective) endomorphism of $M$ is essential (resp. superfluous). The ring $R$ is called weakly left SCI-ring (resp. generalized left SCS-ring) if every weakly left Co-Hopfian (resp. generalized left Hopfian) module is finitely cogenerated. In this note we will prove that the following conditions are equivalent: (i) $R$ is an artinian commutative principal ideal ring; (ii) $R$ is a weakly commutative SCI-ring; (iii) $R$ is a generalized commutative SCS-ring.

Keywords: Co-Hopfian module, Hopfian module, weakly Co-Hopfian module, generalized Hopfian module, module finitely cogenerated, weakly SCI-ring, generalized SCS-ring

1 Introduction

Let $R$ be a commutative ring with $1 \neq 0$. An $R$-module $M$ is said to be Co-Hopfian (resp. Hopfian) if every injective (resp. surjective) endomorphism of $M$ is an automorphism. $M$ is said to be finitely cogenerated if its socle is
essential in \( M \) and finitely generated. The ring \( R \) is called commutative SCI-ring (resp. SCS-ring) if every Co-Hopfian (resp. Hopfian) module is finitely cogenerated. It have been proved in [5] that commutative artinian principal ideal rings characterize commutative SCI-rings and commutative SCS-rings. An \( R \)-module is said to be weakly Co-Hopfian (resp. generalized Hopfian) if every injective (resp. surjective) endomorphism of \( M \) is essential (resp. superfluous). The ring \( R \) is called a weakly commutative SCI-ring (resp. a generalized commutative SCS-ring) if every weakly Co-Hopfian (resp. generalized Hopfian) module is finitely cogenerated. The purpose of this note is to prove that commutative artinian principal ideal rings characterize weakly commutative SCI-rings and generalized commutative SCS-rings. In this note all rings are associative with \( 1 \neq 0 \) and all modules are unitary. The reader may refer to [1] for all notion or notation not defined in this paper.

2 Characterization of weakly commutative SCI-Rings and generalized commutative SCS-rings

**Proposition 2.1 (5). theorem 2.5**

Let \( R \) be a commutative ring. Then the following conditions are equivalent

(i) \( R \) is an artinian principal ideal ring

(ii) \( R \) is a SCI-ring

(iii) \( R \) is a SCS-ring

**Proposition 2.2 (1). proposition 10 - 18**

For each ring the following statements are equivalent

(i) \( R \) is left artinian

(ii) Every finitely generated left \( R \)-module is finitely cogenerated.

**Proposition 2.3.** If \( M \) is a direct sum of an infinite countable family \((M_n)_{n \in \mathbb{N}}\) of submodules of \( M \) such that any two of them are isomorphic, then \( M \) is neither weakly co-Hopfian nor generalized hopfian.

**Proof.** For every integer (resp. nonzero integer) \( n \) let \( \varphi_n \) (resp. \( \xi_n \)) be an isomorphism of \( M_n \) onto \( M_{n+1} \) (resp. \( M_{n-1} \)), \( \xi_0 \) the zero endomorphism of \( M_0 \) and \( \varphi \) (resp. \( \xi \)) the endomorphism of \( M \) such that \( \varphi/M_n = \)
\varphi_n (\text{resp. } \xi/M_n = \xi_n). \text{ Then } \varphi \ (\text{resp. } \xi) \text{ is a monomorphism (resp. an epimorphism) of } M \text{ such that } \text{Im } f = \bigoplus_{n \geq 1} M_n \ (\text{resp. } \text{Ker } \xi = M_n) \text{ which is not essential (resp. superfluous) in } M. \tag*{\square}

**Proposition 2.4.** A direct summand of a generalized Hopfian (resp. weakly Co-Hopfian) module is a genralized Hopfian (resp. weakly Hopfian) module.

**Proof.** Let \( M \) be a module and \( N \) a direct summand of \( M \). We can write \( M = N \oplus K \) where \( K \) is a submodule of \( M \).

If \( M \) is a generalized Hopfian module and \( f \) a surjective endomorphism of \( N \), then

\[
\varphi : M = N \oplus K \rightarrow M = N \oplus K \quad n + k \rightarrow f(n) + k
\]

is a surjective endomorphism of \( M \). Therefore, \( \text{Ker } \varphi = \text{Ker } f \) is surperfluous in \( M \). If \( L \) is a submodule of \( N \) such that \( \text{Ker } f + L = N \), then \( M = \text{Ker } f + L + K \) and consequently \( M = L + K \). Thus, \( N = N \cap M = N \cap (L + K) = L + N \cap K = L + 0 = L \).

If \( M \) is a weakly co-hopfian module and \( g \) an injective endomorphism of \( N \), then

\[
\xi : M = N \oplus K \rightarrow M = N \oplus K \quad n + k \rightarrow g(n) + k
\]

is an injective endomorphism of \( M \). Then \( \text{Im } \xi \subseteq N \) i.e. \( \text{Im } g \oplus K \subseteq N \oplus K \) which implies that \( \text{Im } g \subseteq N \). (We can also see \cite{6} corollary 1.3 and \cite{7} corollary 1.3). \tag*{\square}

**Proposition 2.5.** If \( R \) is a weakly commutative SCI-ring (resp. generalized commutative SCS-ring) then \( R \) is a commutative SCI-ring (resp. commutative SCS-ring).

**Proof.** Let \( R \) be a weakly commutative SCS-ring (resp. generalized commutative SCS-ring) and \( M \) a Co-Hopfian module (resp. Hopfian module). Then \( M \) is weakly Co-Hopfian (resp. generalized Hopfian) and consequently \( M \) is finitely cogenerated. \tag*{\square}

**Theorem 2.6.** Let \( R \) be a commutative ring. Then the following conditions are equivalent
(i) $R$ is an artinian principal ideal ring
(ii) $R$ is a weakly SCI-ring
(iii) $R$ is a generalized SCS-ring.

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$

Let $R$ be a commutative artinian principal ideal ring. Following [2] every $R$-module is a direct sum of cyclic submodules. Let now $M$ be a weakly Co-Hopfian (resp. generalized Hopfian) module which is not finitely cogenerated. Then by proposition 2.2 $M$ is not finitely generated. We can write $M = \bigoplus_{i \in I} M_i$ where the $M_i$ are cyclic submodule of $M$. Since there is only a finite number of non isomorphic cyclic $R$-modules, then there is an infinite countable family $(M_n)$ of the family $(M_i)_{i \in I}$ such that any two of them are isomorphic. Therefore, we can write

$$M = K \oplus L \text{ where } L = \bigoplus_{n \in \mathbb{N}} M_n$$

Following proposition 2.4 $L$ is weakly Co-Hopfian (resp. generalized Hopfian) and following proposition 2.3 $L$ is not weakly Co-Hopfian (resp. generalized Hopfian). This is a contradiction.

$(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$

If $R$ is a weakly commutative SCI-ring (resp. generalized commutative SCS-ring) then by proposition 2.5 $R$ is a commutative SCI-ring (resp. commutative SCS-ring) and consequently, by proposition 2.1 $R$ is an artinian commutative principal ideal ring.

Corollary 2.7. Let $R$ a commutative ring. Then the following conditions are equivalent:

(1) $R$ is an artinian principal ideal ring
(2) $R$ is a weakly commutative SCS-ring
(3) $R$ is a generalized commutative SCS-ring
(4) $R$ is a commutative SCI-ring
(5) $R$ is a commutative SCS-ring

Proof. Proposition 2.1 and theorem 2.6 prove the corollary.
References


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