Changing and Unchanging Domination Number of a Commutative Ring

J. Ravi Sankar

Department of Mathematics, Saradha Gangadharan College
Puducherry - 605 004, India
ravisankar.maths@gmail.com

S. Meena

Department of Mathematics, Government Arts College
Chidambaram - 608 104, India
meenasaravanan14@gmail.com

Abstract

Let \( R \) be a commutative ring and let \( \Gamma(Z_n) \) be the zero divisor graph of a commutative ring \( R \), whose vertices are non-zero zero divisors of \( Z_n \), and such that the two vertices \( u, v \) are adjacent if \( n \) divides \( uv \). In this paper, we evaluate the changing and unchanging domination number of \( \Gamma(Z_n) \) and we evaluate the same result in \( \Gamma(Z_{2p}), \Gamma(Z_{3p}), \Gamma(Z_{p^2}), \Gamma(Z_{2^n}) \) and \( \Gamma(Z_{3^n}) \).

Our main result is, Brigham et al. [4] studied \( \gamma \)-critical graphs and posed the following questions; (1) "If \( G \) is a \( \gamma \)-critical graph, \( G \) has \( (\Delta(G) + 1)(\gamma(G) - 1) + 1 \) vertices, is \( G \) regular?". (2) "If \( G \) is a \( \gamma \)-critical graph, is \( |V| \geq (\delta(G) + 1)(\gamma(G) - 1) + 1 \) vertices?". J. Fulman et al. [7] show that the first question has a positive answer and the second question has a negative answer. But we show that the first question has a negative answer and the second question has a positive answer.

Mathematics Subject Classification: 05C25

Keywords: Commutative ring, Zero divisor graph, Changing and unchanging domination number

1 Introduction

Let \( R \) be a commutative ring and let \( Z(R) \) be its set of zero-divisors. We associate a graph \( \Gamma(R) \) to \( R \) with vertices \( Z(R)^* = Z(R) - \{0\} \), the set of non-zero zero divisors of \( R \) and for distinct \( x, y \in Z(R)^* \), the vertices \( x \) and \( y \)
are adjacent if and only if \( xy = 0 \). Thus \( \Gamma(R) \) is the empty graph if and only if \( R \) is an integral Domain. Throughout this paper, we consider the commutative ring \( R \) by \( \mathbb{Z}_n \) and zero divisor graph \( \Gamma(R) \) by \( \Gamma(\mathbb{Z}_n) \). Let graph \( G=(V, E) \) be a graph of order \( n \). A set \( D \subseteq V \) is a dominating set if every vertex in \( V-D \) is adjacent to at least one vertex in \( D \). The domination number \( \gamma(\Gamma(\mathbb{Z}_n)) \) is the minimum cardinality of a dominating set of \( \Gamma(\mathbb{Z}_n) \). The private neighbor set of a vertex \( v \) with respect to a set \( D \), denoted by \( \text{pn}[v, D] \), is \( N[v] - N[D - \{v\}] \) and each \( u \in \text{pn}[v, D] \) is called a private neighbor of \( v \) with respect to \( D \).

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [3], where he was mainly interested in colorings. The zero divisor graph is very useful to find the algebraic structures and properties of rings. For notation and graph theory terminology, we in general follow [1, 8, 9, 11, 12] and the basic for commutative ring theory is [10]. We examine references the effects on the domination number, when the graph is modified by deleting a vertex or deleting or adding an edge. The semi-expository paper by Carrington, Harary, and Haynes [5] surveyed the problems of characterizing the graphs \( G \) in the following six classes. Let \( G - v \) (respectively, \( G - e \)) denote the graph formed by removing vertex \( v \) (respectively edge \( e \)) from \( G \). We use acronyms to denote the following classes of graphs (C represents changing; U-unchanging; V-vertex; E-edge; R-removal; A-addition).

\[
(CVR) \implies \gamma(G - v) \neq \gamma(G), \forall v \in V. \tag{1}
\]
\[
(CER) \implies \gamma(G - e) \neq \gamma(G), \forall e \in E. \tag{2}
\]
\[
(CVA) \implies \gamma(G + e) \neq \gamma(G), \forall e \in E(G). \tag{3}
\]
\[
(UVR) \implies \gamma(G - v) = \gamma(G), \forall v \in V. \tag{4}
\]
\[
(UER) \implies \gamma(G - e) = \gamma(G), \forall e \in E. \tag{5}
\]
\[
(UVA) \implies \gamma(G + e) = \gamma(G), \forall e \in E(G). \tag{6}
\]

These six problems have been approached individually in the literature with other terminology. It is useful to partition the vertices of \( G \) into three sets according to how their removal affects \( \gamma(G) \).

Let \( V = V^o \cup V^+ \cup V^- \) for,

\[
V^o = \{ v \in V : \gamma(G - v) = \gamma(G) \}. \tag{7}
\]
\[
V^+ = \{ v \in V : \gamma(G - v) > \gamma(G) \}. \tag{8}
\]
\[
V^- = \{ v \in V : \gamma(G - v) < \gamma(G) \}. \tag{9}
\]

Similarly, the edge set can be partitioned into,

\[
E^o = \{ uv \in E : \gamma(G - uv) = \gamma(G) \}. \tag{10}
\]
\[
E^+ = \{ uv \in E : \gamma(G - uv) > \gamma(G) \}. \tag{11}
\]
If $\gamma(G-v) \neq \gamma(G)$ for all $v \in V$, then $\gamma(G-v) = \gamma(G) - 1$ for all $v \in V$, and so the graphs in CVR are precisely the vertex critical graphs introduced by Brigham et al. [4] and studied in e.g. [7]. The graphs in CEA are precisely the edge critical graphs introduced by Sumner and Blitch [14]. The three classes of graphs with unchanging domination number were investigated in [5] among others, but no constructive characterizations of the classes UVR and UEA have been given. On the other hand, the graphs in UER were independently characterized by Baner et al. [2] and Wallkar and Acharya [15].

2 Changing and unchanging vertex deletion

Theorem 2.1 For any graph $\Gamma(Z_{2p})$, where $p$ is any odd prime number, then there exists a non-pendent vertex $v \in V$ such that $\gamma(\Gamma(Z_{2p}) - v) > \gamma(\Gamma(Z_{2p}))$.

**Proof:** Since, $\Gamma(Z_{2p})$ has at least one vertex $v$ with $\deg(v) \geq 2$ that is adjacent to at least two end vertex. If $v$ is adjacent to two or more end vertices, let $x$ and $y$ are any two end vertices of $v$. Clearly, $v$ is every $\gamma$-set of $\Gamma(Z_{2p})$ and $\gamma(\Gamma(Z_{2p}) - x) = \gamma(\Gamma(Z_{2p}))$. Let $\Gamma'(Z_{2p}) = \Gamma(Z_{2p}) - x$. For any graph $\Gamma(Z_{2p})$, if $\deg(x) = 1$ then $\gamma(\Gamma(Z_{2p}) - x) = \gamma(\Gamma(Z_{2p}))$, and let $\Gamma'(Z_{2p}) = \Gamma(Z_{2p}) - v$ then $\gamma(\Gamma(Z_{2p}) - v) > \gamma(\Gamma(Z_{2p}))$, where $\deg(v) \geq 2$.

Theorem 2.2 If $p$ is any odd prime number, then there exists a pendent vertex $u \in V$ such that $\gamma(\Gamma(Z_{2p}) - u) = \gamma(\Gamma(Z_{2p}))$.

Theorem 2.3 For any graph $\Gamma(Z_{2p})$, where $p$ is any odd prime then

(a) If $v \in V^+$, then $v$ is not an isolate and is in every $\gamma$-set of $\Gamma(Z_{2p})$.

(b) If $v \in V^+$, then no subset $S \subseteq V - N[v]$ with cardinality $\gamma(\Gamma(Z_{2p}))$ dominates $\Gamma(Z_{2p}) - v$.

(c) If $v \in V^+$, then for every $\gamma$-set of $\Gamma(Z_{2p})$, then $v \in S$ and $pn[v, S]$ contains at least two non-adjacent vertices.

(d) If $v \in V^+$ and $u \in V^o$, then $v$ and $u$ are adjacent.

(e) $|V^o| > 2|V^+|$.

(f) $\gamma(\Gamma(Z_{2p})) \neq \gamma(\Gamma(Z_{2p}) - v)$ for all $v \in V$ iff $V = V^+$ and $\gamma(\Gamma(Z_{2p})) = \gamma(\Gamma(Z_{2p}) - v)$ for all $v \in V$ iff $V = V^o$.

(g) If $v \in V^o$ and $v$ is not an isolate in $\Gamma(Z_{2p})$, then there exists a $\gamma$-set $S$ of $\Gamma(Z_{2p})$ such that $v \notin S$.

**Proof:** (a) Using theorem (2.1), we know that $v$ is every $\gamma$-set of $\Gamma(Z_{2p})$ and $\gamma(\Gamma(Z_{2p}) - v) > \gamma(Z_{2p})$.

(b) Clearly, if $v \in V^+$, then no subset $S \subseteq V - N[v]$ with $|\gamma(\Gamma(Z_{2p}))|$ dominates $\Gamma(Z_{2p}) - v$.

(c) From (a) and (b), we know that each $v \in V^+$ is not isolated vertex and is in every $\gamma$-set $S$ of $\Gamma(Z_{2p})$. If $pn[v, S] = \{v\}$, then for any $u \in N(v)$, $S - \{v\} \cup \{u\}$ is a $\gamma(\Gamma(Z_{2p}))$ set, contradicting the fact that $v$ in every $\gamma(\Gamma(Z_{2p}))$
set. Hence, \( v \) has a private neighbor in \( V-S \). But, In \((\Gamma(Z_{2p}))\), \( \gamma(\Gamma(Z_{2p}) - v) > \gamma(\Gamma(Z_{2p})) \) and \( v \in V^+ \). Clearly, \( \gamma(\Gamma(Z_{2p}) - v) \) is totally disconnected graph. Therefore, \( v \in S \) and \( pn[v, S] \) contains atleast two non adjacent vertices.

(d) Suppose that \( v \in V^+ \) and \( u \in V^o \) and \( uv \in E(\Gamma(Z_{2p})) \). Let \( S_u \) be a dominating set of \( \Gamma(Z_{2p}) - u \) with cardinality \( \gamma(\Gamma(Z_{2p})) \). Clearly, \( \gamma(\Gamma(Z_{2p})) = \gamma(\Gamma(Z_{2p}) - u) \) for all \( u \in V^o \) and \( \gamma(\Gamma(Z_{2p})) < \gamma(\Gamma(Z_{2p}) - v) \) for all \( v \in V^+ \). But, in \( \Gamma(Z_{2p}) \), only one vertex has \( \text{deg} \geq 2 \). That is, \( \text{deg}(v) \geq 2 \). Clearly, \( v \) is adjacent to remaining vertices in \( \Gamma(Z_{2p}) \). That is, \( \text{diam}(u, v) = 1 \), for all \( u \in V(\Gamma(Z_{2p})) \). Therefore, any vertex \( u \neq v \) in \( \Gamma(Z_{2p}) \), then \( u \) is an adjacent to \( v \).

(e) Using (a) and (b), \( |V^o| \geq 2|V^+| \).

(f) Obviously, we know that, the vertices of \( \Gamma(Z_{2p}) \) can be partition into three sets according to how their removal affects \( \gamma(\Gamma(Z_{2p})) \).

Let \( V = V^o \cup V^+ \cup V^- \) for,

\[
V^o = \{v \in V : \gamma(\Gamma(Z_{2p}) - v) = \gamma(\Gamma(Z_{2p}))\}. \tag{12}
\]

\[
V^+ = \{v \in V : \gamma(\Gamma(Z_{2p}) - v) > \gamma(\Gamma(Z_{2p}))\}. \tag{13}
\]

\[
V^- = \{v \in V : \gamma(\Gamma(Z_{2p}) - v) < \gamma(\Gamma(Z_{2p}))\}. \tag{14}
\]

But, In \( \Gamma(Z_{2p}) \) has only two types of partition. That is, there exists a vertex \( v \in V(\Gamma(Z_{2p})) \) either \( v \) is a center vertex or end vertex. If \( v \) is a center vertex in \( \Gamma(Z_{2p}) \), then \( \gamma(\Gamma(Z_{2p}) - v) > 2 \gamma(\Gamma(Z_{2p})) \) and \( v \in V^+ \). Similarly \( v \) is an end vertex in \( \Gamma(Z_{2p}) \), then \( \gamma(\Gamma(Z_{2p}) - v) = \gamma(\Gamma(Z_{2p})) \) and \( v \in V^o \). Clearly the vertex set \( V \) can be partition into \( V^+ \) and \( V^o \). That is \( V = V^+ \cup V^o \). If \( v \in V \), then \( V \in V^+ \cup V^o \). So, \( v \) belongs to either \( V^+ \) or \( V^o \).

(g) This follows directly from (a) and [13].

Note that the removing a vertex can increase the domination number by more than one, but can decrease it by at most one. For example removing the center vertex of a \( \Gamma(Z_{2p}) \) with \( p \) is any odd prime, increases the domination number by more than one. The path \( P_{3k+1} \), for \( k \geq 1 \), is another example of a graph for which the removal of an end vertex decreases the domination number by one. Furthermore, if \( S \) is a \( \gamma \)-set, then removing any vertex in \( V-S \) cannot increase the domination number, So \( |V^+| \leq \gamma(G) \).

**Corollary 2.4** A graph \( \Gamma(Z_n) \in CVR \) iff for each vertex \( v \in V \), \( pn[v, S] - \{v\} \) for same \( \gamma \)-set \( S \) containing \( v \).

**Theorem 2.5** If a graph \( \Gamma(Z_{2p}) \in CVR \), where \( p \) is any odd prime, and \( \gamma(\Gamma(Z_{2p}))=2 \), then \( \text{diam}(\Gamma(Z_{2p})) \leq 2(\gamma(\Gamma(Z_{2p})) - 1) \).

**Proof:** Let, \( x, y \in \Gamma(Z_{2p}) \) and assume that \( x \) and \( y \) are distinct and non-zero. If \( xy \neq 0 \), then there exists \( z \in \Gamma(Z_{2p}) \) such that \( xz = zy = 0 \). Since, \( \text{diam}(\Gamma(Z_{2p}))=2 \). Suppose, \( xy = 0 \) and there exists \( z \in \Gamma(Z_{2p}) \) such that
Changing and unchanging domination number

\[(xy)z = 0,\] which implies that, \(x(yz) = 0\) or \((xz)y = 0\). That is either \(yz = 0\) or \(xz = 0\).

Since \(\Gamma(Z_{2p}) \in CVR\), then there exists a non pendent vertex \(v \in V(\Gamma(Z_{2p}))\) such that \(\gamma(\Gamma(Z_{2p})-v) > \gamma(\Gamma(Z_{2p}))\). That is, there exists a vertex \(v \in V^+\) with \(\deg(v) \geq 2\) such that \(xv, vy\) is a maximal path length 2. Clearly \(v\) is adjacent to all the vertices in \(\Gamma(Z_{2p})\), by Theorem (1), and hence, \(\gamma(\Gamma(Z_{2p})) = 2\).

Baner et al.[2] explored a related problem of determining the minimum number of vertices whose removal changes \(\gamma(G)\). Let \(\mu^+(G)\) denote the minimum number of vertices whose removal increases \(\gamma\) and \(\mu^-(G)\) the corresponding number of vertices whose removal decreases the domination number. Note that \(\mu^+(G)\) may not exist, for example, \(G = K_n\).

**Theorem 2.6** If \(p\) is any prime then \(\Gamma(Z_{p^2}) \in UVR\).

**Proof:** If \(n = p^2\), then the vertex set of \(\Gamma(Z_{p^2})\) is \(\{p, 2p, 3p, 4p, ....(p-1)p\}\). Clearly \(p\) is adjacent to all the vertices in \(V(\Gamma(Z_{p^2}))\). Also note that, any two vertices in \(\Gamma(Z_{p^2})\) is adjacent and hence \(\Gamma(Z_{p^2})\) is a complete graph.

Let \(\gamma(\Gamma(Z_{p^2})) = 1\) and let \(v \in V(\Gamma(Z_{p^2}))\) then, \(\gamma(\Gamma(Z_{p^2}) - v) = 1 = \gamma(\Gamma(Z_{p^2}))\). Clearly, every vertex \(v\) is in \(V^o\) which implies that \(\Gamma(Z_{p^2}) \in UVR\).

Since, a graph \(G \in UVR\) iff \(G\) has no isolated vertices and for each vertex \(v\), either (a) there is an \(\gamma\)-set \(S'\) such that \(v \notin S'\) and for each \(\gamma\)-set \(S\) such that \(v \in S'\) and \(pn[v,S]\) contains atleast one vertex from \(V-S\) or (b) \(v\) is an every \(\gamma\)-set and there is a subset of \(\gamma(G)\) vertices in \(G - N[v]\) that dominates \(G - v\).

**Theorem 2.7** If \(p > 3\) is any prime, then there exists a non pendent vertex \(v\) with maximum degree such that \(\gamma(\Gamma(Z_{3p}) - v) < \gamma(\Gamma(Z_{3p}))\).

**Proof:** Let, a vertex \(v \in \Gamma(Z_{3p})\) with \(\deg(v) = \Delta\). Clearly the number of vertices in \(\Gamma(Z_{3p})\) is \(p+1\). Suppose \(u\) be another vertex with \(\deg(u) = \Delta\) in \(\Gamma(Z_{3p})\). Clearly, either \(u = p\) and \(v = 2p\) or, \(u = 2p\) and \(v = p\).

Then, \(uv = 2p\). \(p = 2p^2\) which does not divide by \(3p\). Therefore \(u\) and \(v\) are non adjacent vertices in \(\Gamma(Z_{3p})\). But any other vertex \(w\) in \(\Gamma(Z_{3p})\), \(uw = wv = 0\). That is, remaining vertices in \(\Gamma(Z_{3p})\) are adjacent to both \(u\) and \(v\). Clearly, \(\gamma(\Gamma(Z_{3p})) = 2\) and \(\gamma(\Gamma(Z_{3p}) - v) = 1 = \gamma(\Gamma(Z_{3p})) - 1\). Hence \(\gamma(\Gamma(Z_{3p}) - v) < \gamma(\Gamma(Z_{3p}))\).

**Theorem 2.8** In \(\Gamma(Z_{3p})\) where \(p\) is any prime with \(> 3\), then \(V = V^o \cup V^-\).

**Proof:** Using Theorem (2.7), there exists \(u, v \in V(\Gamma(Z_{3p}))\), then \(\deg(u) = \deg(v) = \Delta = p - 1\). Clearly, \(\gamma(\Gamma(Z_{3p}) - u) < \gamma(\Gamma(Z_{3p}))\) and \(\gamma(\Gamma(Z_{3p}) - v) < \gamma(\Gamma(Z_{3p}))\). Let \(w\) be any other vertex which is not equal to \(u\) and \(v\) in \(\Gamma(Z_{3p})\), then \(w\) is adjacent to only \(u\) and \(v\). Therefore, \(\gamma(\Gamma(Z_{3p}) - w) = \gamma(\Gamma(Z_{3p}))\), which implies, \(w \in V^o\) and any vertex in \(V(\Gamma(Z_{3p}))\) except \(u\) and \(v\) belong to \(V^o\) and \(u, v\) belong to \(V^-\). Hence, \(V = V^o \cup V^-\).
Theorem 2.9 Any odd prime \( p \) then, \( \mu^+(\Gamma(Z_{2p})) = 1 \).

Proof: Let \( v \) be a non pendent vertex in \( \Gamma(Z_{2p}) \). Let \( u \neq v \in V(\Gamma(Z_{2p})) \), then \( uv = 0 \) and \( w \) be any other vertex \( w \neq u \) and \( w \neq v \in V(\Gamma(Z_{2p})) \) then \( uw \neq 0 \). But same time \( uv = 0 \). Clearly, \( v \) is adjacent to all the vertices in \( \Gamma(Z_{2p}) \) and \( \gamma(\Gamma(Z_{2p})) = 1 \). Furthermore, the removal of \( v \) from \( \Gamma(Z_{2p}) \) increase the domination number of \( \Gamma(Z_{2p}) \) which implies that \( \gamma(\Gamma(Z_{2p}) - v) > \gamma(\Gamma(Z_{2p})) \), where \( v \in V^+ \). That is \( \gamma(\Gamma(Z_{2p}) - v) = p - \gamma(\Gamma(Z_{2p})) \) and hence \( \gamma(\Gamma(Z_{2p}) - v) \) is totally disconnected and \( \mu^+(\Gamma(Z_{2p})) = 1 \).

Corollary 2.10 In \( \Gamma(Z_8) \), \( \mu^+(\Gamma(Z_8)) = 1 \).

Proof: Using above theorem, \( \mu^+(\Gamma(Z_8)) = 1 \).

3 Changing and unchanging edge addition and deletion

Theorem 3.1 Any graph \( \Gamma(Z_{2p}) \) where \( p \neq 2 \) any prime, then \( \Gamma(Z_{2p}) \in \text{UEA} \) iff \( V^- \) is empty.

Proof: Let \( V^- \) is empty and suppose \( \Gamma(Z_{2p}) \) has a vertex \( v \in V^- \). Thus \( \gamma(G - v) < \gamma(G) \). Using theorem (2.1), (2.2) and (2.3), we get \( \gamma(\Gamma(Z_{2p}) - v) \geq \gamma(\Gamma(Z_{2p})) \) where either \( v \in V^o \) or \( v \in V^+ \), which is contradiction to our assumption \( v \in V^- \).

To prove the converse, let \( \Gamma(Z_{2p}) \in \text{UEA} \) and suppose \( \Gamma(Z_{2p}) \) has a vertex \( v \) with \( \deg(v) \geq 2 \). Let \( D \) be a \( \gamma^- \) set of \( \Gamma(Z_{2p}) - v \). Then adding edge \( uv \) for any \( u \in D \) gives \( \gamma(\Gamma(Z_{2p}) + uv) < \gamma(\Gamma(Z_{2p})) \) contrary to the fact that \( \Gamma(Z_{2p}) \in \text{UEA} \).

Theorem 3.2 If \( p \) is any prime then \( \Gamma(Z_{p^2}) \in \text{UEA} \).

Proof: Using theorem (2.7), \( \deg(u) = \deg(v) = \Delta \), \( u \) and \( v \) are not relatively prime numbers. That is \( \gcd(u, v) = p \). Since, \( u \) and \( v \) are non-adjacent vertices in \( \Gamma(Z_{p^2}) \), then there exist a edge \( uv \in E(\Gamma(Z_{p^2})) \) such that \( \gamma(\Gamma(Z_{p^2}) + uv) = 1 \). That is, \( \gamma(\Gamma(Z_{p^2}) + uv) < \gamma(\Gamma(Z_{p^2})) = 2 \) and hence \( \Gamma(Z_{p^2}) \in \text{UEA} \).

Theorem 3.3 If \( p \) is any odd prime with \( > 3 \) then, \( \Gamma(Z_{3p}) \in \text{CEA} \).

Proof: Using theorem (2.6), \( \deg(u) = \deg(v) = \Delta \), \( u \) and \( v \) are not relatively prime numbers. That is \( \gcd(u, v) = p \). Since, \( u \) and \( v \) are non-adjacent vertices in \( \Gamma(Z_{3p}) \), then there exist a edge \( uv \in E(\Gamma(Z_{3p})) \) such that \( \gamma(\Gamma(Z_{3p}) + uv) = 1 \). That is, \( \gamma(\Gamma(Z_{3p}) + uv) < \gamma(\Gamma(Z_{3p})) = 2 \) and hence \( \Gamma(Z_{3p}) \in \text{CEA} \).

Theorem 3.4 If \( p \) is any odd prime with \( > 3 \) and \( \Gamma(Z_{3p}) \in \text{CEA} \) then, \( |V^-| = \gamma(\Gamma(Z_{3p})) \).

Proof: Assume \( \Gamma(Z_{3p}) \in \text{CEA} \) is connected graph. Using theorem (2.8), \( V = V^o \cup V^- \). Obviously, if \( V^o = \phi \), the theorem holds. Suppose \( |V^o| \geq 1 \) and \( |V^-| = k < \gamma(\Gamma(Z_{3p})) \). Since, \( \Gamma(Z_{3p}) \) is connected and not completed, using[6] implies that \( k \geq 1 \) and at least one vertex \( v \in V^- \) has a neighbor \( w \) in \( V^o \). It also follows from [6], that \( w \) dominates every vertex in \( V^o \). Hence,
\{w\} \cup (V^- - \{v\}) is a dominating set of \(\Gamma(Z_{3p})\) with cardinality \(k < \gamma(\Gamma(Z_{3p}))\) is a contradiction. Therefore, \(|V^-| = \gamma(\Gamma(Z_{3p}))\).

For example in \(\Gamma(Z_{15})\), \(\gamma(\Gamma(Z_{15})) = 2\) and \(V^- = \{5, 10\}\). That is \(|V^-| = 2\) and hence \(|V^-| = \gamma(\Gamma(Z_{15})) = 2\).

4 Relationships among the classes

**Theorem 4.1** For any graph \(\Gamma(Z_{p^2}) \in UVR\), then \(\Gamma(Z_{p^2}) \in UEA\).

**Proof:** We know that a graph \(\Gamma(Z_{p^2}) \in UEA\) iff \(V^-\) is empty. Using the theorem (2.6) and (3.2) and hence \(\Gamma(Z_{p^2}) \in UEA\).

**Theorem 4.2** For any graph \(\Gamma(Z_{3p}) \in CVR\) then \(\Gamma(Z_{3p}) \in CER\).

**Proof:** Using theorem (2.8), \(\Gamma(Z_{3p}) \in CVR\), then for each \(v \in V\) there exists an \(\gamma\)- set \(D\) such that \(v\) is in \(D\) only to dominate itself. That is \(\gamma(\Gamma(Z_{3p})) - 1\) vertices dominate \(\Gamma(Z_{3p}), \forall v \in V\). Using theorem (3.3), removing an edge incident to any vertex cannot increase the domination number and hence the theorem.

**Theorem 4.3** For any graph \(\Gamma(Z_{2n})\), where \(n > 3\), then

(a) There exists a non pendant vertex \(v \in V\) with \(deg(v) = \Delta\) such that \(\gamma(\Gamma(Z_{2n}) - v) > \gamma(\Gamma(Z_{2n}))\).

(b) There exists a non pendant vertex \(u \in V\) with \(\delta < \deg(u) < \Delta\) such that \(\gamma(\Gamma(Z_{2n}) - u) = \gamma(\Gamma(Z_{2n}))\).

(c) There exists a vertex \(u \in V\) with \(\deg(u) = 1\) such that \(\gamma(\Gamma(Z_{2n}) - u) = \gamma(Z_{2n})\).

(d) There exists a vertex \(v \in V\) such that either \(v \in V^o\) or \(v \in V^+\) then \(V = V^o \cup V^+\).

(e) There exists a vertex \(v \in V(\Gamma(Z_{2n}))\), such that \(\gamma(\Gamma(Z_{2n}) - v) > \gamma(\Gamma(Z_{2n}))\) which implies \(v \in V^+\).

(f) There exists an edge \(uv\) in \(\Gamma(Z_{2n})\) such that \(\gamma(\Gamma(Z_{2n})) = \gamma(\Gamma(Z_{2n}) + uv)\) which implies \(\Gamma(Z_{2n}) \in UEA\).

(g) There exists an edge \(uv\) in \(\Gamma(Z_{2n})\) such that \(\gamma(\Gamma(Z_{2n}) - uv) > \gamma(\Gamma(Z_{2n}))\) which implies \(\Gamma(Z_{2n}) \in CER\).

**Proof:** (a) Let \(v \in \Gamma(Z_{2n})\) has maximum degree \(\Delta\) which implies \(\deg(v) = 2^{n-1} - 2\). Since, \(\gamma(\Gamma(Z_{2n})) = 1\). Clearly, the vertex \(v\) is adjacent to all vertices in \(\Gamma(Z_{2n})\) and \(\gamma(\Gamma(Z_{2n}) - v) = 2^{n-2} + 1\). Hence, \(\gamma(\Gamma(Z_{2n})) < \gamma(\Gamma(Z_{2n}) - v)\).

(b) Let \(u \in V(\Gamma(Z_{2n}))\) has degree lies between \(\delta\) and \(\Delta\). But using (a), there exists a vertex \(v\) is adjacent to all vertices in \(\Gamma(Z_{2n})\) including \(u\). Therefore \(\gamma(\Gamma(Z_{2n}) - u) = \gamma(\Gamma(Z_{2n}))\).

(c) Using (b) and theorem (2.2) the theorem is true.

(d) Using (a), \(\gamma(\Gamma(Z_{2n}) - v) > \gamma(\Gamma(Z_{2n}))\). Clearly, \(v \in V^+\). Using (b) and using (c), \(\gamma(\Gamma(Z_{2n}) - u) = \gamma(\Gamma(Z_{2n}))\) which implies \(u \in V^o\). Hence, \(V = V^o \cup V^+\).
(e) Using (a), there exists a vertex \( v \in V(\Gamma(Z_{2^n})) \) such that \( \gamma(\Gamma(Z_{2^n})) < \gamma(\Gamma(Z_{2^n}) - v) \) and hence \( \Gamma(Z_{2^n}) \in CVR. \)

(f) Using (a), we know that \( v \) is adjacent to all vertices. Let \( u \) and \( w \) are two non-adjacent vertices in \( \Gamma(Z_{2^n}) \). Then, add an edge \( e = uw \) in \( \Gamma(Z_{2^n}) \).

Clearly, \( \gamma(\Gamma(Z_{2^n}) + uw) = \gamma(\Gamma(Z_{2^n})) \) and hence \( \Gamma(Z_{2^n}) \in UEA. \)

(g) Let \( u, v \) be any two adjacent vertices in \( \Gamma(Z_{2^n}) \). Using (a), \( \gamma(\Gamma(Z_{2^n}) - uv) = 2. \) That is \( \gamma(\Gamma(Z_{2^n}) - uv) > \gamma(\Gamma(Z_{2^n})) \) and hence \( \Gamma(Z_{2^n}) \in CER. \)

**Theorem 4.4** For any graph \( \Gamma(Z_{3^n}) \), where \( n \geq 3 \), then

(a) There exist a non-pendent vertex \( v \in V \) then, \( \gamma(\Gamma(Z_{3^n}) - v) = \gamma(\Gamma(Z_{3^n})) \).

(b) There exist a vertex \( v \in V \) such that \( v \in V^o \) and \( \Gamma(Z_{3^n}) \in UVR. \)

(c) There exist an edge \( e = uw \) where \( u \) and \( w \) are non-adjacent vertices, then \( \gamma(\Gamma(Z_{3^n}) + uw) = \gamma(\Gamma(Z_{3^n})) \) and \( \Gamma(Z_{3^n}) \in UEA. \)

(d) There exist an edge \( e = uw \) where \( u \) and \( w \) are adjacent in \( \Gamma(Z_{3^n}) \), then \( \gamma(\Gamma(Z_{3^n}) - uv) = \gamma(\Gamma(Z_{3^n})) \) and \( \Gamma(Z_{3^n}) \in UER. \)

**Proof:** (a) Since, \( \Gamma(Z_{3^n}) \) has no pendent vertex and there exists two vertices \( v \) and \( w \) are adjacent to all the vertices in \( \Gamma(Z_{3^n}) \). That is, there exists any vertex \( u \in V \) such that \( u \) is adjacent to both \( v \) and \( w \) and hence, \( \gamma(\Gamma(Z_{3^n})) = 1 = \gamma(\Gamma(Z_{3^n}) - v) = \gamma(\Gamma(Z_{3^n}) - w). \)

(b) Using (a), any vertex \( v \in \Gamma(Z_{3^n}) \), after removing \( v \) from \( \Gamma(Z_{3^n}) \), then the domination number never changed. So that is \( \gamma(\Gamma(Z_{3^n}) - v) = 1 \) which implies that \( v \in V^o \) and hence \( \Gamma(Z_{3^n}) \in UVR. \)

(c) Using (a) and (b), we get \( \Gamma(Z_{3^n}) \in UEA. \)

(d) Similarly using (a), (b) and (c), we get \( \Gamma(Z_{3^n}) \in UER. \)

### 5 Main Results

A graph \( G \) is vertex domination-critical or \( \gamma \)-critical or simply critical, if any vertex \( v \) of \( G, \gamma(G - v) < \gamma(G). \) In [4], Brigham et al., studied \( \gamma \)-critical graphs and posed the questions, ‘If a \( \gamma \)-critical graph \( G \) has \( (\Delta + 1)(\gamma - 1) + 1 \) vertices, is \( G \) regular?’ and ‘If \( G \) is a \( \gamma \)-critical graph is \( |V| \geq (\delta + 1)(\gamma - 1) + 1?’. In [7], J.Fulman, D.Hasan and G.Macgillivray show that, if \( G \) is \( \gamma \)-critical with \( |V| = (\Delta + 1)(\gamma - 1) + 1 \), then \( G \) is regular and second question has negative answer.

But we show that the first question has a negative answer and the second question has a positive answer. That is, \( G \) is \( \gamma \)-critical with \( |V| = (\Delta + 1)(\gamma - 1) + 1 \), then \( G \) is non-regular and \( G \) is a \( \gamma \)-critical is \( |V| \geq (\delta + 1)(\gamma - 1) + 1. \)

**Theorem 5.1** For any graph \( \Gamma(Z_{3^n}) \), where \( p \) is prime \( > 3 \), and \( \Gamma(Z_{3^n}) \in CVR \) has order \( |V| = (\Delta + 1)(\gamma - 1) + 1 \), then \( \Gamma(Z_{3^n}) \) is non-regular.

**Proof:** Since, we know that there exist almost two vertices \( u \) and \( v \) in \( \Gamma(Z_{3^n}) \) have maximum degree. That is, \( \text{deg}(v) = \text{deg}(u) = \Delta. \) The number of vertices in \( \Gamma(Z_{3^n}) \) is \( p+1. \) Using Theorem (2.7) and (2.8), there exists any
other vertex \(w\) which is not equal to \(u\) and \(v\) in \(\Gamma(Z_p)\) such that \(w\) is adjacent to both \(u\) and \(v\).

Let \(D\) be the domination set of \(\Gamma(Z_p)\). Clearly, \(D = \{u, v\} = \{p, 2p\}\), and \(uv = p \times 2p = 2p^2\) implies that \(3p\) does not divide \(2p^2\). Clearly, we know that \(u\) and \(v\) are non adjacent vertices in \(\Gamma(Z_p)\). Since, there exist any vertex \(w\) in \(\Gamma(Z_p)\) such that \(uw = vw = 0\) implies that \(3p/uv\) and \(3p/vw\). Therefore \((p + 1) = p - 1\) vertices are adjacent to \(u\) and \(v\).

Clearly \(\deg(u) = \deg(v) = p - 1\) and remaining vertices in \(\Gamma(Z_p)\) has degree 2. Since, \(\gamma(\Gamma(Z_p)) = |D| = 2\), then

\[
(\Delta + 1)(\gamma - 1) + 1 = ((p - 1) + 1)(2 - 1) + 1 = p + 1 = |V(\Gamma(Z_p))| \quad (15)
\]

Since, \(\deg(u) = \deg(v) \neq \deg(w)\), where \(w\) be any other vertex in \(\Gamma(Z_p)\) and hence \(\Gamma(Z_p)\) is non-regular.

**Theorem 5.2** For any graph \(\Gamma(Z_p)\) where \(p\) is any prime \(> 3\) is a \(\gamma\)-critical, then \(|V| \geq (\delta + 1)(\gamma - 1) + 1\).

**Proof:** Using Theorem (5.1), we know that the minimum degree of \(\Gamma(Z_p)\) is 2 and the domination number of \(\gamma(\Gamma(Z_p))\) is 2. Then,

\[
|V| \geq (2 + 1)(2 - 1) + 1 = (3 + 1) = 4 \quad (16)
\]

Since, \(|V| = p + 1\) and any prime, \(p > 3\). Hence, \(|V| > 4\).

For example, \(\Gamma(Z_{13})\) where \(p = 13\), is a \(K_{2,12}\) graph. In \(\Gamma(Z_{39})\), only two vertices have degree 12 and remaining vertices have degree 2. Clearly, \(\Gamma(Z_{39})\) is non-regular and \(\delta = 2, \gamma = 2\).

\[
|V(\Gamma(Z_{39}))| = 13 \geq (\delta + 1)(\gamma - 1) + 1 = (2 + 1)(2 - 1) + 1 = 4 \quad (17)
\]

Finally, it may be concluded that, Brigham et al [4] two questions given need not be sufficient.

**References**


Received: September, 2012