A Note on Semigroups of Linear Transformations with Invariant Subspaces

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Abstract

Let $V$ be a vector space over a field and $T(V)$ the semigroup of all linear transformations from $V$ into $V$. For a fixed subspace $W$ of $V$, denote

$$S(V,W) = \{ \alpha \in T(V) : W\alpha \subseteq W \}.$$ 

Then $S(V,W)$ is a subsemigroup of $T(V)$. In this note, we show that the semigroup $S(V,W)$ is always abundant.

Keywords: semigroups; Green’s-starred relations; abundant semigroups; vector spaces; linear transformations

1 Introduction

Let $V$ be a vector space over a field and $T(V)$ the semigroup (under composition) consisting of all linear transformations from $V$ into $V$. Let $W$ be a fixed subspace of $V$. In [4], R. P. Sullivan studied a kind of linear transformation semigroups involved with the subspace $W$,

$$T(V,W) = \{ \alpha \in T(V) : V\alpha \subseteq W \}.$$ 

She described Green’s relations and ideals for the semigroup $T(V,W)$. In [6] and [7], the authors consider another kind of subsemigroups of $T(V)$ involved with a fixed subspace $W$, that is,

$$S(V,W) = \{ \alpha \in T(V) : W\alpha \subseteq W \}.$$ 

In [6], Green’s relations and ideals on $S(V,W)$ are characterized. In [7], the regular elements in $S(V,W)$ are described and it is proved that $S(V,W)$ is regular if and only if $W = V$ or $W = \{0\}$. 
Let $S$ be a semigroup. We say that $a, b \in S$ are $\mathcal{L}^*$-related ($\mathcal{R}^*$-related) if they are $\mathcal{L}$-related ($\mathcal{R}$-related) in a semigroup $T$ such that $S$ is a subsemigroup of $T$. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations on $S$. A semigroup $S$ is called abundant if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of $S$ contains an idempotent. It is well-known that regular semigroups are abundant. However, the converse is not true. For example, Umar showed in [5] that the semigroups $S^-(X)$ of order-decreasing transformations on a totally ordered finite set $X$ is abundant but not regular. Also, letting $T(X)$ be the full transformation semigroup on a set $X$, in [3], the authors find out some equivalences $E$ on $X$ for which the semigroup

$$T_E(X) = \{\alpha \in T(X) : (x\alpha, y\alpha) \in E \text{ for all } (x, y) \in E\}$$

is abundant but not regular.

In this note, we first describe the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on $S(V, W)$. Then we show that the semigroups $S(V, W)$ is always abundant.

For convenience, we adopt the convention used in [4]. We write a subset $\{e_i : i \in I\}$ of $V$ as $\{e_i\}$, letting the subscript denote an (unspecified) index set $I$. The subspace $U$ of $V$ generated by a linearly independent subset $\{e_i\}$ is denoted by $\langle e_i \rangle$ and we write $\dim U = |I|$. Often it is necessary to construct some $\alpha \in T(V)$ by first choosing a basis $\{e_i\}$ of $V$ and some $\{u_i\} \subseteq V$, and then letting $e_i\alpha = u_i$ for each $i \in I$ and extending this action by linearity to the whole of $V$. To abbreviate matters, we simply say that given $\{e_i\}$ and $\{u_i\}$ within context, then $\alpha \in T(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}.$$

For undefined notation and concepts one can consult [1].

## 2 Main results

The following Lemma 1 and Lemma 2 come from [2], and Lemma 3 comes from [1, p.63, Exercise 19].

**Lemma 1** Let $S$ be a semigroup and $\alpha, \beta \in S$. Then the following statements are equivalent:

1. $(\alpha, \beta) \in \mathcal{R}^*$.
2. For all $\sigma, \gamma \in S^1$, $\sigma \alpha = \gamma \alpha$ if and only if $\sigma \beta = \gamma \beta$.

**Lemma 2** Let $S$ be a semigroup and $\alpha, \beta \in S$. Then the following statements are equivalent:

1. $(\alpha, \beta) \in \mathcal{L}^*$.
2. For all $\sigma, \gamma \in S^1$, $\alpha \sigma = \alpha \gamma$ if and only if $\beta \sigma = \beta \gamma$. 

Let $V$ be a vector space and $\alpha \in T(V)$. Denote $\ker\alpha = \{v \in V : v\alpha = 0\}$.

**Lemma 3** Let $\alpha, \beta \in T(V)$. Then 
(1) $(\alpha, \beta) \in \mathcal{L}$ if and only if $V\alpha = V\beta$; 
(2) $(\alpha, \beta) \in \mathcal{R}$ if and only if $\ker\alpha = \ker\beta$.

First we need the next observation. Let $\alpha \in S(V,W)$ and \{zi\} be a basis for $V\alpha \cap W$. Extend this, respectively, to a basis \{zi, zj\} for $W$, to a basis \{zi, zi\} for $V\alpha$. Then we have

**Lemma 4** \{zi, zj, zi\} is linearly independent.

**Proof.** Let $z_i = z'_i \alpha, \ z_l = z'_l \alpha$ for some $z'_i, z'_l \in V$. Assume

$$\sum k_i z_i + \sum k_j z_j + \sum k_l z_l = 0$$

for some scalars $k_i, k_j, k_l$. Then

$$\sum k_j z_j = -\sum k_i z_i - \sum k_l z_l = -\sum k_i z'_i \alpha - \sum k_l z'_l \alpha = (-\sum k_i z'_i - \sum k_l z'_l)\alpha$$

which implies that $\sum k_j z_j \in V\alpha \cap W$. Therefore, $\sum k_j z_j$ can be expressed by \{zi\}. Suppose $\sum k_j z_j = \sum h_i z_i$ for some scalars $h_i$. Then, since \{zi, zj\} is linearly independent, we have $k_j = 0$ for each $j$. Furthermore, $\sum k_i z_i + \sum k_l z_l = 0$. Notice that \{zi, zi\} is linearly independent, it must be the case that $k_i = 0$ for each $i$ and $k_l = 0$ for each $l$. Consequently, \{zi, zj, zi\} is linearly independent. \hfill $\square$

**Theorem 1** Let $\alpha, \beta \in S(V,W)$. Then $(\alpha, \beta) \in \mathcal{L}^*$ if and only if $V\alpha = V\beta$.

**Proof.** If $V\alpha = V\beta$, then $(\alpha, \beta) \in \mathcal{L}$ in the semigroup $T(V)$ by Lemma 3. Hence $(\alpha, \beta) \in \mathcal{L}^*$ in $S(V,W)$, and the sufficiency follows. Now suppose $(\alpha, \beta) \in \mathcal{L}^*$. If $V\alpha \neq V\beta$, without loss of generality, we may assume $V\alpha \backslash V\beta \neq \emptyset$. Take $z \in V\alpha \backslash V\beta$. There are two cases to consider: $z \in W$ or $z \notin W$.

Case 1. $z \in W$. Suppose \{zi\} is a basis for $V\beta \cap W$. Extend this to a basis \{zi, zj\} for $W$, to a basis \{zi, zl\} for $V\beta$, respectively. Then \{zi, zi, zi\} is linearly independent by Lemma 4. Extend it to a basis \{zi, zj, zl, zm\} for $V$. Define $\sigma, \gamma \in T(V)$ by letting

$$\sigma = \begin{pmatrix} z_i & z_j & z_l & zm & z \\ zi & z_j & z_l & zm & 0 \end{pmatrix}, \ \ \ \ \gamma = \begin{pmatrix} z_i & z_j & z_l & zm & z \\ zi & z_j & z_l & zm & z \end{pmatrix}.$$ 

Then $\sigma, \gamma \in S(V,W)$ and $\beta \sigma = \beta \gamma$ holds. However, for each $x \in z\alpha^{-1}$, $x\sigma = z\sigma = 0$ and $x\alpha \gamma = z\gamma = z$. So $\sigma \neq \alpha \gamma$ contradicting Lemma 1.

Case 2. $z \notin W$. Also, suppose \{zi\} is a basis for $V\beta \cap W$. Extend this to a basis \{zi, zj\} for $W$, to a basis \{zi, zi\} for $V\beta$, respectively. Then \{zi, zj, zi\} is
linearly independent by lemma 4. There are also two subcases: \( z \in \langle z_i, z_j, z_l \rangle \) or \( z \not\in \langle z_i, z_j, z_l \rangle \).

If the former is the case, extend \( \{z_i, z_j, z_l\} \) to a basis \( \{z_i, z_j, z_l, z_m\} \) for \( V \) and define \( \sigma, \gamma \in T(V) \) by letting

\[
\sigma = \begin{pmatrix} z_i & z_j & z_l & z_m \\ z_i & 0 & z_l & z_m \end{pmatrix}, \quad \gamma = \begin{pmatrix} z_i & z_j & z_l & z_m \\ z_i & z_j & z_l & z_m \end{pmatrix}.
\]

It is routine to verify that \( \sigma, \gamma \in S(V,W) \) and \( \beta \sigma = \beta \gamma \) holds. By hypothesis, \( z \in \langle z_i, z_j, z_l \rangle \), we can assume \( z = \sum k_i z_i + \sum k_j z_j + \sum k_l z_l \) for some scalars \( k_i, k_j, k_l \) with \( k_{j_0} \neq 0 \) for some \( j_0 \) (If \( k_j = 0 \) for all \( j \) then \( z = \sum k_i z_i + \sum k_l z_l \in V \beta \), contradicting the hypothesis). For each \( x \in z \alpha^{-1} \),

\[
x \alpha \sigma = z \sigma = \sum k_i z_i + \sum k_l z_l \neq z = z \gamma = x \alpha \gamma.
\]

Thus, \( \alpha \sigma \neq \alpha \gamma \), a contradiction.

If the latter is the case, that is, \( z \not\in \langle z_i, z_j, z_l \rangle \). Extend \( \{z_i, z_j, z_l\} \) to a basis \( \{z_i, z_j, z_l, z_m, z\} \) for \( V \) and define \( \sigma, \gamma \in S(V,W) \) as the same as in Case 1. Then we also have \( \beta \sigma = \beta \gamma \) and \( \alpha \sigma \neq \alpha \gamma \), a contradiction. Consequently, it follows that \( V \alpha = V \beta \) and the necessity holds. \( \square \)

**Theorem 2** Let \( \alpha, \beta \in S(V,W) \). Then \((\alpha, \beta) \in R^* \) if and only if \( \ker \alpha = \ker \beta \).

**Proof.** The sufficiency follows immediately from Lemma 3, since \( \ker \alpha = \ker \beta \) implies \((\alpha, \beta) \in R \) in \( T(V) \). Now suppose \((\alpha, \beta) \in R^* \) in \( S(V,W) \). If \( \ker \alpha \neq \ker \beta \) then, without loss of generality, we may assume there exists some \( z \in \ker \alpha \setminus \ker \beta \). Suppose \( \{u_l\} \) is a basis for \( W \alpha \). Extend this to a basis \( \{u_l, u_m\} \) for \( V \alpha \cap W \), and further extend to a basis \( \{u_l, u_m, u_n\} \) for \( V \alpha \). Choose \( z_l \in u_l \alpha^{-1} \cap W \) for each \( l \), \( z_m \in u_m \alpha^{-1} \) for each \( m \) and \( z_n \in u_n \alpha^{-1} \) for each \( n \).

There are two possibilities: \( z \in W \) or \( z \not\in W \).

Case 1. \( z \in W \). Take a basis \( \{z, z_i\} \) for \( \ker \alpha \cap W \) and extend this to a basis \( \{z, z_i, z_j\} \) for \( \ker \alpha \). Then \( \{z, z_i, z_l\} \) is a basis for \( W \) while \( \{z, z_i, z_j, z_l, z_m, z_n\} \) is a basis for \( V \). Define \( \sigma, \gamma \in T(V) \) by letting

\[
\sigma = \begin{pmatrix} z & z_i & z_j & z_l & z_m & z_n \\ 0 & z_i & z_j & z_l & z_m & z_n \end{pmatrix}, \quad \gamma = \begin{pmatrix} z & z_i & z_j & z_l & z_m & z_n \\ z & z_i & z_j & z_l & z_m & z_n \end{pmatrix}.
\]

One routinely verifies that \( \sigma, \gamma \in S(V,W) \), \( \sigma \alpha = \gamma \alpha \) and \( z \sigma \beta = 0 \neq z \beta = z \gamma \). Thus \( \sigma \beta = \gamma \beta \), a contradiction.

Case 2. \( z \not\in W \). Take a basis \( \{z_i\} \) for \( \ker \alpha \cap W \) and extend this to a basis \( \{z_i, z_j\} \) for \( \ker \alpha \). Then \( \{z_i, z_l\} \) is a basis for \( W \) while \( \{z, z_i, z_j, z_l, z_m, z_n\} \) is a basis for \( V \). Define \( \sigma, \gamma \in S(V,W) \) as the same as in Case 1. Then we also have \( \sigma \alpha = \gamma \alpha \) and \( \sigma \beta = \gamma \beta \), a contradiction. Consequently, we have \( \ker \alpha = \ker \beta \) and the proof is complete. \( \square \)
Theorem 3  For each $\alpha \in S(V,W)$, there exists an idempotent $\beta \in S(V,W)$ such that $V\alpha = V\beta$. Consequently, each $L^*$ class contains an idempotent.

**Proof.** Suppose $\{z_i\}$ is a basis for $V\alpha \cap W$, $\{z_i, z_j\}$ is a basis for $W$ and $\{z_i, z_l\}$ is a basis for $V\alpha$. Then $\{z_i, z_j, z_l\}$ is linearly independent by Lemma 4. Extend this to a basis $\{z_i, z_j, z_l, z_m\}$ for $V$, and define $\beta \in T(V)$ by letting

$$\beta = \begin{pmatrix} z_i & z_j & z_l & z_m \\ z_i & 0 & z_l & 0 \end{pmatrix}.$$ 

Obviously, $\beta$ is an idempotent in $S(V,W)$ and $V\beta = \langle z_i, z_l \rangle = V\alpha$. The remaining conclusion follows from Theorem 1. $\Box$

Theorem 4  For each $\alpha \in S(V,W)$, there exists an idempotent $\beta \in S(V,W)$ such that $\ker \alpha = \ker \beta$. Consequently, each $R^*$ class contains an idempotent.

**Proof.** Suppose $\{z_i\}$ is a basis for $\ker \alpha \cap W$, $\{z_i, z_j\}$ is a basis for $\ker \alpha$. Let $\{u_l\}$ be a basis for $W\alpha$ and $\{u_l, u_m\}$ a basis for $V\alpha$. Take $z_l \in u_l \alpha^{-1} \cap W$ for each $l$, $z_m \in u_m \alpha^{-1}$ for each $m$. Then $\{z_i, z_l\}$ is a basis for $W$ and $\{z_l, z_j, z_l, z_m\}$ is a basis for $V$. Define $\beta \in T(V)$ by letting

$$\beta = \begin{pmatrix} z_i & z_j & z_l & z_m \\ 0 & 0 & z_l & z_m \end{pmatrix}.$$ 

Then $\beta$ is an idempotent in $S(V,W)$. Moreover, $\ker \beta = \langle z_i, z_j \rangle = \ker \alpha$. The remaining conclusion follows from Theorem 2. $\Box$

The following conclusion readily follows from Theorems 3 and 4.

**Corollary**  The semigroup $S(V,W)$ is always abundant.

Consequently, if $W$ is a non-trivial subspace of $V$ (namely, $W$ is neither $\{0\}$ nor $V$), then $S(V,W)$ is abundant but not regular.

**References**


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