On Subgroups Lattice of Quasidihedral Group

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Abstract
In this paper, we have determined the number of subgroups of finite non-abelian group defined by a presentation $G = \langle x, y \mid x^2 = y^{16} = 1, yx = xy^7 \rangle$. We identify the form and the order of elements of the group $G$. There are 26 non-trivial subgroups of $G$ and the subgroups lattice diagram is then presented.

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1 Introduction

One of the most important problems of fuzzy theory is to classify the fuzzy subgroups of a finite group[4]. Several problems have treated the particular case finite abelian group. Laszlo studied the construction of fuzzy subgroups of groups of the orders one to six. Zhang and Zhou[15] determined the number of fuzzy subgroups of cyclic groups of the order $p^n$ where $p$ is a prime number. Murali and Makamba [8], considering a similar problem, found the number of fuzzy subgroups of groups of the order $p^nq^m$ where $p$ and $q$ are distinct primes under different equivalence relation. In [14], Tarnauceanu and Bentea established the recurrence relation verified by the number of fuzzy subgroups of finite cyclic groups. Their result improves Murali’s work in [10, 9]. In [12] the authors calculated the number of fuzzy subgroups of finite abelian groups.

Since all of them have worked on finite abelian groups. It is interesting to investigate the number of fuzzy subgroups of nonabelian groups. Raden Sulaiman and Abd Ghafur Ahmed [4], worked on nonabelian groups $S_2, S_3$ and $A_4$ and constructed their fuzzy subgroups. The number of fuzzy subgroups of $S_4$ is also calculated [13, 7]. We have determined the subgroup lattice diagram of group $G$ defined by the presentation $G = \langle x, y \mid x^2 = y^{16} = 1, yx = xy^7 \rangle$ which will help in determining number of fuzzy subgroups of $G$. 
2 Preliminaries

In this section, we review the definitions and preliminary results that are required later in this paper. Throughout the section, a group \( G \) is assumed to be a finite non-abelian group defined by presentation \( G = \langle x, y \mid x^2 = y^{16} = 1, yx = xy^7 \rangle \) and it is also called Quasidihedral group.

Definition 2.1 [11] A partial ordered on a non-empty set \( P \) is a binary relation \( \leq \) on \( P \) that is reflexive, antisymmetric and transitive. The pair \( \langle P, \leq \rangle \) is called a partially ordered set or poset. Poset \( \langle P, \leq \rangle \) is totally ordered if every \( x, y \in P \) are comparable, that is \( x \leq y \) or \( y \leq x \). A non-empty subset \( S \) of \( P \) is a chain in \( P \) if \( S \) is totally ordered by \( \leq \).

Definition 2.2 [11] Let \( \langle P, \leq \rangle \) be a poset and let \( S \subseteq P \). An upper bound for \( S \) is an element \( x \in P \) for which \( s \leq x, \forall s \in S \). The least upper bound of \( S \) is called the supremum or join of \( S \). A lower bound of \( S \) is called the supremum or join of \( S \). A non-empty subset \( S \) of \( P \) is a chain in \( P \) if \( S \) is totally ordered by \( \leq \).

Definition 2.3 [5] (Lagrange Theorem) If \( G \) is a finite group and \( H \) is a subgroup of \( G \), then order of \( H \) is a divisor of order of \( G \).

Definition 2.4 [5] If \( G \) is a finite group and \( a \in G \), then order of \( a \) is a divisor of order of \( G \).

Definition 2.5 [6] (The First Sylow Theorem) Let \( G \) be a finite group and let \( |G| = p^n m \) where \( n \leq 1 \), \( p \) is a prime number and \( (p, m) = 1 \). Then \( G \) contains a subgroup of order \( p \) for each \( i \) where \( 1 \leq i \leq n \).

3 Main Results

Consider \( G = \langle x, y \mid x^2 = y^{16} = 1, yx = xy^7 \rangle \) be a non-abelian group, then the elements of \( G \) can be written as \( x, y, y^2, y^3, \ldots, y^{15}, xy, xy^2, xy^3, \ldots, xy^{15} \). The order of elements of \( G \) are given below:

<table>
<thead>
<tr>
<th>order</th>
<th>Elements</th>
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<tbody>
<tr>
<td>1</td>
<td>{e}</td>
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<tr>
<td>2</td>
<td>( x, y^8, xy^2, xy^4, xy^6, xy^8, xy^{10}, xy^{12}, xy^{14} )</td>
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<tr>
<td>4</td>
<td>( y^4, y^{12}, xy, xy^3, xy^5, xy^7, xy^9, xy^{11}, xy^{13}, xy^{15} )</td>
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<td>8</td>
<td>( y^2, y^6, y^8, y^{14} )</td>
</tr>
<tr>
<td>16</td>
<td>( y, y^5, y^9, y^{11}, y^{13}, y^{15} )</td>
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</tbody>
</table>
According to the Lagrange theorem, non-trivial subgroups of $G$ must have order 2, 4, 8, 16. We will determine all subgroups of $G$. Clearly, the subgroup of order 1 is the trivial subgroup $H_1 = e$. The order of non-trivial subgroups of $G$ are given by:

**Subgroups of order 2:**

Let $H$ be a subgroup of order 2. Since 2 is a prime number, then $H$ is cyclic. Therefore, $H$ is generated by an element of $G$ of order 2. Thus, all subgroups of order 2 are generated by elements of order 2 i.e.,

$$\langle x \rangle, \langle y^8 \rangle, \langle xy^2 \rangle, \langle xy^4 \rangle, \langle xy^6 \rangle, \langle xy^{10} \rangle, \langle xy^{12} \rangle, \langle xy^{14} \rangle.$$  

**Subgroups of order 4:**

Let $M$ be any arbitrary subgroup of $G$ of order 4. According to the theorem 2.4, the elements of $M$ must have order 1, 2 or 4. If $M$ consists an element of order 4, then $M$ is a cyclic group. The generators of these subgroups are $M_1 = \langle y^4 \rangle = \langle y^{12} \rangle = \{e, y^4, y^8, y^{12}\}, M_2 = \langle xy^3 \rangle = \langle xy^{11} \rangle = \{e, xy^3, y^8, xy^{11}\}, M_3 = \langle xy^7 \rangle = \langle xy^{15} \rangle = \{e, xy^7, y^8, xy^{15}\}, M_4 = \langle xy \rangle = \langle xy^9 \rangle = \{e, xy, y^8, xy^9\}, M_5 = \langle xy^5 \rangle = \langle xy^{13} \rangle = \{e, xy^5, y^8, xy^{13}\}.$

If $M$ is generated by an elements of order 2, then $M$ does not have elements of order 4. Therefore, the order of elements of $M$ is 2, except identity element. We will try to multiply all of the combinations of elements of order 2. The following is the multiplication table of elements of order 2:

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<th></th>
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<th>$xy^2$</th>
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We observe that the subgroups of order 4 generated by elements of order 2 are $M_6 = \langle x, y^8 \rangle = \langle x, xy^2 \rangle = \{e, x, y^8, xy^2\}, M_7 = \langle xy^2, y^8 \rangle = \langle xy^{10}, xy^{14} \rangle = \{e, xy^2, y^8, xy^{10}\}, M_8 = \langle xy^4, y^8 \rangle = \langle xy^{12}, y^8 \rangle = \{e, xy^4, y^8, xy^{12}\}, M_9 = \langle xy^6, y^8 \rangle = \langle xy^{14}, y^8 \rangle = \{e, xy^6, y^8, xy^{14}\}.$ Thus we have 9 subgroups of order 4.

**Subgroups of order 8:**

Let $N$ be any arbitrary subgroup of $G$ of order 8. If $N$ contains an element of order 8, then $N$ is cyclic. The elements of order 8 i.e., $\langle y^2 \rangle = \langle y^6 \rangle = \langle y^{10} \rangle = \langle y^{14} \rangle$ generates the same subgroup $N_1 = \{e, y^2, y^4, y^6, y^8, y^{10}, y^{12}, y^{14}\}.$ Since only the
elements of order 2 and 4 together will form the subgroup of order 8. Therefore, checking all possible combinations of them, we have $N_2 = \langle x, y^4 \rangle = \langle x, y^{12} \rangle = \{e, x, y^4, xy^4, y^8, xy^8, xy^{12}\}$, $N_3 = \langle xy^2, y^4 \rangle = \langle xy^6, y^4 \rangle = \langle xy^{10}, y^4 \rangle = \langle xy^2, y^{12} \rangle$, $N_4 = \langle xy^4, y^4 \rangle = \langle xy^8, y^4 \rangle = \langle xy^{12}, y^4 \rangle$, $N_5 = \langle xy^6, y^4 \rangle$.

**Subgroups of order 16:**

Let $K$ be any arbitrary subgroup of $G$ of order 16 then the elements of $K$ must have order 1, 2, 4, 8 or 16. If $K$ contains an element of order 16, then $K$ is generated by an element of order 16. The elements of order 16 i.e., $\{y^3 = y^5 = y^7 = y^9 = y^{11} = y^{13} = y^{15}\}$ generates the same subgroup $K_1 = \{e, x, y^2, y^4, y^6, y^8, y^{10}, y^{12}, y^{14}, xy^2, xy^4, xy^6, xy^8, xy^{10}, xy^{12}, xy^{14}\}$. Since only the elements of order 2 and 8 together will form the subgroup of order 16, therefore, checking all possible combinations of them, we have $K_2 = \langle x, y^2 \rangle$ and $K_3 = \langle xy, y^2 \rangle$ the subgroups of order 16.

**References**


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