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On Wilson's Renormalization Group Structure

Giuseppe Iurato

Department of Physics, University of Palermo
giuseppe.iurato@unipa.it

Abstract

Very simple remarks on the elementary mathematical structure of pQFT Wilson's Renormalization Group are made in the light of the theory of Ehresmann's groupoids and their possible extensions.

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1 Some groupoid structures

In the papers [1,2], it has been introduced and used certain groupoid structures in some elementary quantum mechanics considerations, and talked about some further possible applications of these algebraic structures in Physics, one of which will be briefly exposed in the present paper.
Precisely, in this work we want to compare these algebraic structures with the Kenneth G. Wilson and J. Kogut Renormalization Group structure of perturbative Quantum Field Theory (pQFT, for short) in such a way to obtain more general structures on it, with possible, further physical interpretations.

Following [1, Section 1], an *Ehresmann groupoid* (or an *E-groupoid*) is an algebraic system of the type $(G, G^{(0)}, r, s, \star)$ where $G, G^{(0)}$ are non-empty sets such that $G^{(0)} \subseteq G$, $r, s : G \to G^{(0)}$ and $\star : G^{(2)} \to G$ with $G^{(2)} = \{(g_1, g_2); g_1, g_2 \in G, s(g_1) = r(g_2)\}$, satisfying the following conditions:

1. $s(g_1 \star g_2) = s(g_2), r(g_1 \star g_2) = r(g_1), \quad \forall (g_1, g_2) \in G^{(2)}$;
2. $s(g) = r(g) = g, \quad \forall g \in G^{(0)}$;
3. $g \star \alpha(s(g)) = \alpha(r(g)) \star g = g, \quad \forall g \in G$;
4. $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3), \quad \forall g_1, g_2, g_3 \in G$;
5. $\forall g \in G, \exists g^{-1} \in G : g \star g^{-1} = \alpha(r(g)), g^{-1} \star g = \alpha(s(g))$,

where $\alpha : G^{(0)} \hookrightarrow G$ is the immersion of $G^{(0)}$ into G. The maps r, s are called, respectively, *range* (or *target*) and *source,* G is the *support* and $G^{(0)}$ is the *set of units* of the groupoid. g^{-1} is said to be the (bilateral) *inverse* of g, so that we have an inversion map of the type $i_G : g \to g^{-1}$, defined on the whole of G; instead, the map $\star$ is a partial map defined on $G^{(2)} \subseteq G \times G$, and not on the whole of $G \times G$. However, from now on, for simplicity we'll suppress the symbol α in 3. and 5., writing only $r(g), s(g)$ instead of $\alpha(r(g)), \alpha(s(g))$.

An E-groupoid is a more general structure than the first notion of groupoid historically introduced by W. Brandt and A. Baer, that we call *Baer-Brandt groupoid* (or *BB-groupoid*), since it may be defined as an E-groupoid satisfying the further condition

6. $\forall g, g'' \in G, \exists g' \in G$ such that $(g, g') \in G^{(2)}, (g', g'') \in G^{(2)}$,

so that, in conclusion, we'll call *Ehresmann-Bauer-Brandt groupoid* (or *EBB-groupoid*) an algebraic system of the type $(G, G^{(0)}, r, s, \star)$ satisfying the above conditions 1., 2., 3., 4., 5. and 6.

If X is an abstract (non-void) set, let $G = X^2$, $G^{(0)} = \Delta(X^2) = \{(x, x); x \in X\}$, $r = pr_1 : (x, y) \to x$, $s = pr_2 : (x, y) \to y$, $\forall x, y \in X$ and $(x, y) \star (y, z) = (x, z)$. Hence, it is easy to verify that $(G, G^{(0)}, r, s, \star)$ is an EEB-groupoid, with $(x, y)_d^{-1} = (x, y)_s^{-1} = (y, x)$, called the *natural* (or *paid,* or *coarse*) *EBB-groupoid* on X, and denoted by $\mathcal{G}_{EBB}(X)$.

Following [2, Section 2], an *Ehresmann semigroupoid* (or an *E-semigroupoid*) is an algebraic system of the type $(G, G^{(0)}, G^{(1)}, r, s, \star)$ where $G, G^{(0)}, G^{(1)}$ are non-empty sets such that $G^{(0)}, G^{(1)} \subseteq G$, $r, s : G \to G^{(0)}$, $i : G^{(1)} \to G^{(1)}$ and $\star : G^{(2)} \to G$ with $G^{(2)} = \{(g_1, g_2); g_1, g_2 \in G, s(g_1) = r(g_2)\}$, satisfying the following conditions:

1'. $s(g_1 \star g_2) = s(g_2), r(g_1 \star g_2) = r(g_1), \quad \forall (g_1, g_2) \in G^{(2)}$;

2'. $s(g) = r(g) = g, \quad \forall g \in G^{(0)}$;

3'. $g \star \alpha(s(g)) = \alpha(r(g)) \star g = g, \quad \forall g \in G$;

4'. $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3), \quad \forall g_1, g_2, g_3 \in G$;

5'. $\forall g \in G^{(1)}, \exists g^{-1} \in G^{(1)} : g \star g^{-1} = \alpha(r(g)), g^{-1} \star g = \alpha(s(g))$,

where $\alpha : G^{(0)} \hookrightarrow G$ is the immersion of $G^{(0)}$ into G and $i : g \to g^{-1}$. The maps r, s are called, respectively, *range* (or *target*) and *source,* G is the *support,* $G^{(0)}$ is the *set of units* and $G^{(1)}$ is the *set of inverses,* of the given E-semigroupoid; if, moreover, it is also verified the above condition 6., then we speak of an *EEB-semigroupoid.* We obtain an E-groupoid when $G^{(1)} = G$, whereas we obtain a monoid when $G^{(0)} = \{e\}$.

2 On Renormalization Group

Now, we briefly recall the main elementary formal aspects of the (Wilson) Renormalization Group structure of perturbative Quantum Field Theory.

Following [3], the formal structure of the Renormalization Group can be described, in the Wilson's framework, as follows. Let S be the general space of Theories (for instance, it is the space of all the Lagrangians of certain types of Fields) and M the set of scales of momenta (the energy, for example), isomorphic (but not canonically) to $\mathbb{R}_+$; let μ, Λ be scales of momenta with $\mu < \Lambda$. For each theory $\mathcal{L} \in S$, one finds another (renormalized) theory $R_{\Lambda\mu}\mathcal{L}$ which is the effective theory, at the (final) scale μ, for the original theory $\mathcal{L}$ at the (initial) scale Λ. Thus, we have a map (*renormalization scheme*) $R_{\Lambda\mu} : S \to S$ such that $R_{\Lambda_1\Lambda_2}R_{\Lambda_2\Lambda_3} = R_{\Lambda_1\Lambda_3}$, hence an action of the semigroup $\mathbb{R}_+^{\leq 1}$ on the space $S \times M$, given by $\lambda \circ (\mathcal{L}, \Lambda) = (R_{\Lambda,\lambda\Lambda}\mathcal{L}, \lambda\Lambda)$, that, abusing mathematical terminology, physicists call this semigroup the *Renormalization Group* (*RG*). In this paper we are mainly interested in the formal structure underlying to the set of functions R.

In [4, Section 6.1] (see also [5, Section 14.5]), as a brief explanation of the origin of the renormalization group equations, it is considered an unspecified renormalization scheme R related to the R-normalized one-particle irreducible Green's function Γ_R (renormalized quantity by R) whose relation to the bare Green's function Γ_B (unrenormalized quantity) is of the type $\Gamma_R = Z(R)\Gamma_B$, where $Z(R)$ denotes the appropriate product of the renormalization constants defined within the R scheme; if we choose another renormalization scheme R', since the unrenormalized quantity Γ_B was independent by the renormalization scheme, then it is $\Gamma_{R'} = Z(R')\Gamma_B$. There exists a relation between both renormalized Green's functions, namely $\Gamma_{R'} = Z(R', R)\Gamma_R$, where $Z(R', R) = Z(R')/Z(R)$. Considering the set of all possible $Z(R', R)$ for arbitrary R and R', among its elements, the following composition law subsists

$$Z(R'', R) = Z(R'', R')Z(R', R) \tag{1}$$

and, in addition, for each element $Z(R', R)$, there exists an (unique) inverse $Z^{-1}(R', R) = Z(R, R')$; at last, we can also define the unit element as $Z(R, R) = 1$. The composition law (1) is not always defined, however, for arbitrary pairs of functions Z, that is to say, the product

$$Z(R_i, R_j)Z(R_k, R_l) \tag{2}$$

is not in general an element of the set $Z(R', R)$ unless $R_j = R_k$, whence it follows that this set is just equipped with a structure of natural EBB-groupoid.

A simple physical example confirming the above considerations is provided by the photon wave function renormalization constant Z_3 given in [6].

However, the composition law (1) plays certain key roles from the physical viewpoint since, for example (see [4, Section 6.1]), supposing we are working in a μ-subtraction scheme and we want to relate two renormalizations performed at subtraction points μ_1 and μ_2, then the relevant quantity will be $Z(\mu_1, \mu_2)$ which, in the one-electron loop approximation, reads

$$Z(\mu_1, \mu_2) = 1 + \frac{\alpha}{2\pi} \int_0^1 2x(1-x) \ln \frac{\mu_2^2 x(1-x) + m^2}{\mu_1^2 x(1-x) + m^2} dx \tag{3}$$

and this representation of Z obeys the composition law (1) but the composition law (2) is not obeyed unless $m = 0$, so that, from this, it emerges the importance of considering the way with which the functions Z may combine among them.

On the other hand, in many points[1] of the basic paper [7], it is recalled the main elementary formal aspects of the Renormalization Group structure, in particular pointing out on the possible existence of no invertible elements in this structure.

Moreover, for further claims (besides similar to those here treated) on the importance of the convolution law (1) in pQFT renormalization, see also [8] and references therein.

Thus, it would be of a certain interest, also from the physical viewpoint, to analyze what role a more general E-groupoid or E-semigroupoid[2] structure could have from the formal viewpoint in the Renormalization Group Theory.

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[1] See, above all, its Footnotes [3,78,112].

[2] The latter being more suitable than the former due to the major degree of noninvertibility (above all, of the Rs and not of the Zs) which plays a fundamental role in Renormalization (to this regards, see Footnote [3] of [7]).

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