Principally $\delta_M$–Lifting Modules

Ali Omer Alattass

Department of Mathematics, Faculty of Science
Hadramout University of Science and Technology
P.O. Box 50663, Mukalla, Yemen
alattassali@yahoo.com

Abstract

Let $M$ be a right $R$–module. In this paper principally $\delta_M$–Lifting modules are defined and several properties of these modules are investigated.

Mathematics Subject Classification: 16D80, 16D90, 16D99

Keywords: $\delta_M$– small submodules, $\delta_M$–lifting modules, principally $\delta_M$–lifting modules

1 Introduction

Throughout this paper, $R$ denotes an associative ring with unity, $Mod–R$ is the category of all unitary right $R$–modules and $M$ is a unitary right $R$–module. The notations $K \subseteq M$, $K \subseteq^@ M$ and $K \subseteq_e M$ are reserved for a submodule $K$, direct summand $K$ and essential submodule $K$ of $M$ respectively. An $R$–module $N$ is called $M$–generated ( or generated by $M$ ) if there exists an epimorphism $f : M^{(\Lambda)} \to N$, for some indexed set $\Lambda$. Any right $R$–module $N$ is said to be subgenerated by $M$ if $N$ is isomorphic to a submodule of an $M$–generated module. We denote by $\sigma[M]$ the full subcategory of $Mod–R$ whose objects are all $R$–modules subgenerated by $M$. A module $N \in \sigma[M]$ is said to be $M$–singular if $N \cong L/K$, for some $L \in \sigma[M]$ and $K \leq_e L$. The class of all $M$–singular modules is closed under submodules, homomorphic images, and direct sums (see [4]). A submodule $K$ of a module $M$ is called small in $M$ ( notation $K \ll M$ ) if, whenever $K + X = M$, we have $X = M$. Now we consider a generalization of the notation ”small”. Zhou [12] generalized the notation of a small submodule to a $\delta$-small submodule. Zhou called a submodule $K$ of a module $M$ is $\delta$-small in $M$ ( notation $K \ll_{\delta} M$ ) if, whenever $K + X = M$ with $M/X$ singular, we have $X = M$. An epimorphism $f : P \to M$ is called a $\delta$–cover of $M$ if $Ker(f) \ll_{\delta} P$ and if moreover $P$ is
projective, then it is called a projective $\delta$--cover. Following [3] (resp. [6]) a module $M$ is called lifting (resp. $\delta$--lifting) module if, for all $K \leq M$ there exists a decomposition $M = A \oplus B$ such that $A \leq K$ and $K \cap B \ll M$ (resp. $K \cap B \ll_{\delta} M$). Let $L$ and $K$ be two submodules of $M$. $L$ is called a supplement (resp. $\delta$--supplement) of $K$ in $M$ if $M = L + K$ and $L \cap K \ll L$ (resp. $L \cap K \ll_{\delta} L$).

Now we consider the notation $\delta$--small in $\sigma[M]$. For a module $N$ in $\sigma[M]$, Özcan and Alkan [8] call a submodule $L$ of $N$ is $\delta - M$ (or $\delta_M$)--small submodule, written $L \ll_{\delta_M} N$, in $N$ if $L + K \neq N$, for any proper submodule $K$ of $N$ with $N/K$ $M$--singular. Clearly if $L$ is $\delta$--small, then $L$ is a $\delta_M$--small submodule. Hence $\delta_M$--small submodules are the generalization of $\delta$--small submodules in the category $Mod - R$. An epimorphism $f : P \rightarrow N$ is called a $\delta_M$--cover of $N$ if $Ker(f) \ll_{\delta_M} P$ and if moreover $P$ is projective in $\sigma[M]$, then it is called a projective $\delta_M$--cover. Also Özcan and Alkan [8] consider the following submodule of a module $N$ in $\sigma[M]$, $\delta_M(N) = \cap \{K \leq N : N/K$ is $M$--singular simple $\}$.

We collect in the following Lemma some known results (see 2.3, 2.4 in [8] and 2.1 in [10]).

**Lemma 1.1** Let $N \in \sigma[M]$.

(1) For modules $K$ and $L$ with, $K \leq L \leq N$, we have $L \ll_{\delta_M} N$ if and only if $K \ll_{\delta_M} N$ and $L/K \ll_{\delta_M} N/K$.

(2) For submodules $K$ and $L$ of $N$, $K + L \ll_{\delta_M} N$ if and only if $K \ll_{\delta_M} N$ and $L \ll_{\delta_M} N$.

(3) If $K \ll_{\delta_M} N$ and $f : N \rightarrow L$ is a homomorphism in $\sigma[M]$, then $f(K) \ll_{\delta_M} L$. In particular, if $K \ll_{\delta_M} L$ and $L \leq N$, then $K \ll_{\delta_M} N$.

(4) If $K \leq L \leq N$ and $K \ll_{\delta_M} N$, then $K \ll_{\delta_M} L$.

(5) $\delta_M(N) = \sum \{L \leq N : L \ll_{\delta_M} N\}$.

(6) Let $K$ be a submodule of $N$. Then $K \ll_{\delta_M} N$ if and only if $N = X \oplus Y$ for a projective semisimple submodule $Y$ in $\sigma[M]$ with $Y \leq K$ whenever $X + K = N$.

A module $N$ in $\sigma[M]$ is called $\delta_M$--lifting if, for all $K \leq N$, there exists a decomposition $N = A \oplus B$ such that $A \leq K$ and $K \cap B \ll_{\delta_M} N$(see [7]). Now, let $N \in \sigma[M]$ and $L, K \leq N$. $L$ is called a $\delta_M$--supplement of $K$ in $N$ if $N = K + L$ and $K \cap L \ll_{\delta_M} L$.

The following Proposition has been proved in ([1],Proposition 2.4).

**Proposition 1.2** Let $N$ be in $\sigma[M]$ and $K, L, X \leq N$ such that $X \ll_{\delta_M} N$.

(1) If $K$ is a $\delta_M$-supplement of $L$ in $N$, then $K$ is a $\delta_M$-supplement of $L + X$ in $N$.

(2) If $K$ is a $\delta_M$-supplement of $L + X$ in $N$, then $N$ has a direct summand
\textit{Y which is semisimple, projective in }\sigma[M]\textit{ and } K + Y \textit{ is a } \delta_M \textit{-supplement of } L \textit{ in } N. \\

In this work principally } \delta_M \textit{-lifting modules are defined as analog of principally } \delta \textit{-lifting modules ( see [5] ). A module } N \textit{ in } \sigma[M] \textit{ is called a principally } \delta_M \textit{-lifting module if for any cyclic submodule } K \textit{ of } N, \textit{ there exists a decomposition } N = A \oplus B \textit{ such that } A \leq K \textit{ and } K \cap B \ll \delta_M N. \textit{ We start by giving some examples of these modules, where some of them are not lifting. Then we characterize these modules. Also we observe, by giving an example, that the class of principally } \delta_M \textit{-lifting modules need not be closed under direct sums besides we consider when direct sum of two principally } \delta_M \textit{-lifting modules is principally } \delta_M \textit{-lifting module. Furthermore we defined finitely } \delta_M \textit{-lifting modules, } \delta_M \textit{-hollow modules, finitely } \delta_M \textit{-hollow modules as well as principally } \delta_M \textit{-hollow modules. After that we investigate the interconnections between these modules and principally } \delta_M \textit{-lifting modules. General background material can be found in [2] and [11].}

\section{Principally } \delta_M \textit{-Lifting Modules.}

In this section we define and study principally } \delta_M \textit{-lifting modules.

\textbf{Definition 2.1 .} A module } N \textit{ in } \sigma[M] \textit{ is called a principally } \delta_M \textit{-lifting if for any cyclic submodule } K \textit{ of } N, \textit{ there exists a decomposition } N = A \oplus B \textit{ such that } A \leq K \textit{ and } K \cap B \ll \delta_M N, \textit{ i.e for each } n \textit{ in } N, \textit{ there exists a decomposition } N = A \oplus B \textit{ such that } A \leq nR \textit{ and } nR \cap B \ll \delta_M N.

\textbf{Example 2.2 (1)} Every semisimple module in } \sigma[M] \textit{ is principally } \delta_M \textit{-lifting.

(2) For every prime number } p \textit{ and every positive integer } n \textit{ the } \mathbb{Z} \textit{-module } N = \mathbb{Z}/p^n\mathbb{Z} \textit{ is principally } \delta_\mathbb{Z} \textit{-lifting.}

Clearly, every lifting module in } \sigma[M] \textit{ is principally } \delta_M \textit{-lifting. But the converse is not true. We give below an example of principally } \delta_M \textit{-lifting which is not lifting.

\textbf{Example 2.3 (1)} The } \mathbb{Z} \textit{-module } \mathbb{Q} \textit{ is principally } \delta_\mathbb{Z} \textit{-lifting but not lifting } \mathbb{Z} \textit{-module.

(2) Let } Q = \prod_{i=1}^{\infty} F_i \textit{ where } F_i = \mathbb{Z}_2. \textit{ Let } R \textit{ be the subring of } Q \textit{ generated by } \bigoplus_{i=1}^{\infty} F_i \textit{ and } 1_Q \textit{ and } S = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a \in R, b \in Soc(R) \right\}. \textit{ Note that } S \textit{ is a ring under the matrix addition and multiplication. By example 3.8 in [5], } S_\mathbb{A} \textit{ is principally } \delta \textit{-lifting but not } \delta \textit{-lifting. Hence if } M = S_\mathbb{A} \textit{ then } M \textit{ is principally } \delta_M \textit{-lifting but not } \delta_M \textit{-lifting.}
Theorem 2.4 The following are equivalent for any module $N$ in $\sigma[M]$.

(a) $N$ is principally $\delta_M-$lifting.
(b) For any cyclic submodule $K$ of $N$ there exists a decomposition $N = A \oplus B$ such that $A \leq K$ and $K \cap B \ll_{\delta_M} B$.
(c) Every cyclic submodule $K$ of $N$ has a $\delta_M-$supplement $L$ in $N$ such that $K \cap L$ is a direct summand of $K$.
(d) Every cyclic submodule $K$ of $N$ can be written as $K = A \oplus B$ such that $A \leq N$ and $B \ll_{\delta_M} N$.
(e) For any cyclic submodule $K$ of $N$, there exists a direct summand $A$ of $N$ such that $A \leq K$ and $K/A \ll_{\delta_M} N/A$.
(f) For every cyclic submodule $K$ of $N$, there is a direct summand $A$ of $N$ and a submodule $X$ of $N$ with $A \leq K$, $K = A + X$ and $X \ll_{\delta_M} N$.
(g) For every cyclic submodule $K$ of $N$, there is an idempotent $e \in \text{End}(N)$ with $e(N) \leq K$ and $(1 - e)(K) \ll_{\delta_M} (1 - e)(N)$.
(h) For each $n \in N$, there exist ideals $I$ and $J$ of $R$ such that $nR = nI \oplus nJ$ where $nI$ is a direct summand of $N$ and $nJ \ll_{\delta_M} N$.

Proof. $(a) \Leftrightarrow (b)$ is by Lemma 1.1 (3), (4).

$(b) \Rightarrow (c)$: Let $K$ be a cyclic submodule of $N$. Then there exists a decomposition $N = A \oplus B$ such that $A \leq K$ and $K \cap B \ll_{\delta_M} B$. This implies $B$ is a $\delta_M-$supplement of $K$ in $N$ as $N = K + B$. Moreover $K = K \cap N = K \cap (A \oplus B) = A \oplus (K \cap B)$. Hence (c) follows.

$(c) \Rightarrow (d)$: Let $K$ be any cyclic submodule of $N$. By (c), there is a submodule $L$ of $N$ such that $N = L + K$, $L \cap K \ll_{\delta_M} L$ and $K = (K \cap L) \oplus V$ for some submodule $V$ of $K$. Hence $N = L + (L \cap K) + V = L + V$. But $L \cap V = L \cap (K \cap V) = (L \cap K) \cap V = 0$. So $N = L \oplus V$.

$(d) \Rightarrow (e)$: Let $K$ be any cyclic submodule of $N$. By the hypothesis, there exists a decomposition $K = A \oplus B$ such that $A \leq N$ and $B \ll_{\delta_M} N$. If $\eta : N \rightarrow N/A$ is the natural homomorphism, then by Lemma 1.1(3), $\eta(K) \ll_{\delta_M} N/A$ and so $K/A \ll_{\delta_M} N/A$.

$(e) \Rightarrow (f)$: Let $K$ be any cyclic submodule of $N$. Then there exists a direct summand $A$ of $N$ such that $A \leq K$ and $K/A \ll_{\delta_M} N/A$. If $N = A \oplus B$, then $K = A + X$, where $X = K \cap B$. Since $N/A \cong B$, let $\phi : N/A \rightarrow B$ be the obvious isomorphism. Then, since $K/A \ll_{\delta_M} N/A$ and $\phi(k/a) = k \cap B = X$, $X \ll_{\delta_M} B$ by Lemma 1.1(3). Hence $X \ll_{\delta_M} N$.

$(f) \Rightarrow (a)$: Let $K$ be any cyclic submodule of $N$. Then there is a direct summand $A$ of $N$ and a submodule $X$ of $N$ with $A \leq K$, $K = A + X$ and $X \ll_{\delta_M} N$. Let $N = A \oplus B$. Then $B$ is a $\delta_M-$supplement of $A$ in $N$ and hence a $\delta_M-$supplement of $K = A + X$ by Lemma 1.2 (1). So $B \cap K = B \cap (A + X) \ll_{\delta_M} B$.

$(a) \Rightarrow (g)$: If $K$ is a cyclic submodule of $N$, then there exists a decomposition $N = A \oplus B$ such that $A \leq K$ and $K \cap B \ll_{\delta_M} N$. Let
$e : N \rightarrow A$ be the projection map along $B$. As $A \leq K$, $e(N) \leq K$ and $(1 - e)(K) = K \cap (1 - e)(N) = K \cap B \leq_{\delta_M} B = (1 - e)(N)$ as required.

$(g) \Rightarrow (h)$. Let $n \in N$. By $(g)$, there exists an idempotent $e \in \text{End}(N)$ such that $e(N) \leq nR$, $N = e(N) \oplus (1 - e)(N)$ and $(1 - e)(nR) \leq_{\delta_M} (1 - e)(N)$. Let $r \in R$ such that $nr = (1 - e)m$, for some $m \in N$. Then $m = nr + em \in nR$ because $e(N) \leq nR$. This implies $nR \cap (1 - e)(N) \leq nR \cap (1 - e)(nR)$. Thus $nR \cap (1 - e)(N) = (1 - e)(nR)$. So $nR = e(N) \oplus (1 - e)(nR)$. Let $I = \{r \in R : nr \in e(N)\}$ and $J = \{s \in R : ns \in (1 - e)(nR)\}$. Then $nR = nI \oplus nJ$.

$(h) \Rightarrow (a)$. Let $n \in N$. Then there exist ideals $I$ and $J$ of $R$ such that $nR = nI \oplus nJ$, where $nI$ is a direct summand of $N$ and $nJ \leq_{\delta_M} N$. Suppose that $N = nI \oplus K$, where $K \leq N$. Hence $nR = nI \oplus (nR \cap K)$. Thus $N$ is principally $\delta_M$-lifting as $nR \cap K \cong nJ$ and $nJ \leq_{\delta_M} N$. 

\textbf{Proposition 2.5}. Every direct summand of a principally $\delta_M$-lifting module $N$ in $\sigma[M]$ is also principally $\delta_M$-lifting.

\textbf{Proof.} Let $A$ be a direct summand of $N$. Then there exists a submodule $B$ of $N$ such that $N = A \oplus B$. To show that $A$ is principally $\delta_M$-lifting, let $K$ be a cyclic submodule of $A$. Hence $K$ is a cyclic submodule of $N$. Science $N$ is principally $\delta_M$-lifting, there exists a decomposition $N = C \oplus D$ with $C \leq K$ and $K \cap D \leq_{\delta_M} N$. Consequently $A = C \oplus (A \cap D)$ and $K \cap (A \cap D) = K \cap D \leq_{\delta_M} N$. So $K \cap (A \cap D) \leq_{\delta_M} A$ by Lemma 1.1(4). Therefore $A$ is a principally $\delta_M$-lifting module.

The following example shows that the class of principally $\delta_M$-lifting modules need not be closed under direct sums.

\textbf{Example 2.6} Consider the $\mathbb{Z}$-modules $N_1 = \mathbb{Z}/2\mathbb{Z}$ and $N_2 = \mathbb{Z}/8\mathbb{Z}$. Note that $N_1$ and $N_2$ are in $\sigma[\mathbb{Z}]$. It is clear that $N_1$ and $N_2$ are principally $\delta_Z$-lifting. If $N = N_1 \oplus N_2$, then $N = A + B$, where $A = (\bar{1}, 2)\mathbb{Z}$, $B = (\bar{1}, 1)\mathbb{Z}$. Hence $A$ is not a direct summand of $N$ and does not contain a nonzero direct summand of $N$. It follows that $N$ is not principally $\delta_Z$-lifting since nonzero factor modules of $N$ are $\mathbb{Z}$-singular.

Next we give sufficient condition for a direct sum of two principally $\delta_M$-lifting modules to be principally $\delta_M$-lifting module.

Recall that a submodule $X$ of a module $M$ is called fully invariant if for every $f \in \text{End}(M)$, $f(X) \leq X$. A module $M$ is called a duo module if every submodule of $M$ is fully invariant.
Lemma 2.7 (Lemma 2.1 in [9]) Let $M = \oplus_{i \in I} M_i$ be a direct sum of submodules $M_i$ ($i \in I$) and let $X$ be a fully invariant submodule of $M$. Then $X = \oplus_{i \in I} (X \cap M_i)$.

Proposition 2.8 Let $N = N_1 \oplus N_2 \in \sigma[M]$ be a duo module. Then $N$ is principally $\delta_M$-lifting if and only if $N_1$ and $N_2$ are principally $\delta_M$-lifting modules.

Proof. The necessity follows by Proposition 2.5.
Conversely, assume that $N_1$ and $N_2$ are principally $\delta_M$-lifting modules and let $n \in N$. By Lemma 2.7, $nR = (nR \cap N_1) \oplus (nR \cap N_2)$. Since $nR \cap N_1$ and $nR \cap N_2$ are cyclic submodules of $N_1$ and $N_2$ respectively, there exist decompositions $N_1 = A_1 \oplus B_1$ and $N_2 = A_2 \oplus B_2$ such that $A_1 \leq nR \cap N_1$, $B_1 \cap (nR \cap N_1) = B_1 \cap nR \ll \delta_M B_1$ and $A_2 \leq nR \cap N_2$ such that $B_2 \cap (nR \cap N_2) = B_2 \cap nR \ll \delta_M B_2$. Hence $N = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$ and $A_1 \oplus A_2 \leq nR$ and $nR \cap (B_1 \oplus B_2) = (nR \cap B_1) \oplus (nR \cap B_2) \ll \delta_M N_1 \oplus N_2$. Thus $N = N_1 \oplus N_2$ is principally $\delta_M$-lifting.

Lemma 2.9 (Lemma 2.6 in [6]) Let $N_1, N_2$ be modules and $N = N_1 \oplus N_2$. Then the following are equivalent:
(1) $N_1$ is $N_2$-projective.
(2) For every submodule $K$ of $N$ with $N = K + N_2$, there exists a submodule $K'$ of $K$ such that $N = K' \oplus N_2$.

Proposition 2.10 Let $N_1, N_2 \in \sigma[M]$ be such that $N_1$ is semisimple and $N_2$ is principally $\delta_M$-lifting. If $N_1$ is $N_2$-projective, then $N = N_1 \oplus N_2$ is principally $\delta_M$-lifting.

Proof. Let $n$ be a nonzero element in $N$ and let $L = N_1 \cap (nR + N_2)$.
Case (i) If $L \neq 0$, then $N_1 = L \oplus L_1$ for some submodule $L_1$ of $N_1$ and so $N = L \oplus L_1 \oplus N_2$. Hence $L$ is $N_2 \oplus L_1$-projective. By Lemma 2.9, there exists a submodule $K_1$ of $nR$ such that $N = K_1 \oplus (N_2 \oplus L_1)$ since $N = nR + (N_2 \oplus L_1)$.
We may assume $nR \cap (N_2 \oplus L_1) \neq 0$. Clearly $nR \cap (X + L_1) = X \cap (nR + L_1)$, for any submodule $X$ of $N$. In particular $nR \cap (N_2 + L_1) = N_2 \cap (nR + L_1)$. Since $N = K_1 \oplus (nR \cap (N_2 \oplus L_1))$, there exist $n_1 \in K_1$ and $n_2 \in nR \cap (N_2 \oplus L_1)$ such that $n = n_1 + n_2$. So $K_1 = n_1 R$ and $nR \cap (N_2 \oplus L_1) = n_2 R$. Since $N_2$ is principally $\delta_M$-lifting, there exists a submodule $A$ of $nR \cap (N_2 + L_1) = N_2 \cap (nR + L_1)$ such that $N_2 = A \oplus B$ and $B \cap (nR + L_1) \ll \delta_M N_2$.
Hence $N = (K_1 \oplus A) \oplus (B \oplus L_1)$. Moreover Lemma 1.1 (4), (3) implies that $nR \cap (B \oplus L_1) = B \cap (nR + L_1) \ll \delta_M B \oplus L_1$.

Case (ii) Let $L = 0$. Then $nR \leq N_2$. Since $N_2$ is principally $\delta_M$-lifting, by
Theorem 2.4, there exists a decomposition \( N_2 = A \oplus B \) such that \( A \leq nR \) and \( nR \cap B \ll_{\delta_M} B \). Hence, \( N = A \oplus (N_1 \oplus B) \) and \( nR \cap (N_1 \oplus B) = nR \cap B \ll_{\delta_M} B \). By Lemma 1.1 (3), \( nR \cap (N_1 \oplus B) \) is \( \delta_M \)-small in \( N_1 \oplus B \). Thus \( N \) is principally \( \delta_M \)-lifting by Theorem 2.4. \( \square \)

Recall that a module \( N \) is called principally semisimple if every cyclic submodule of \( N \) is a direct summand of \( N \). Hence principally semisimple modules in \( \sigma[M] \) are principally \( \delta_M \)-lifting modules.

**Proposition 2.11** Let \( N \in \sigma[M] \) be a principally \( \delta_M \)-lifting module. Then

1. \( N/\delta_M(N) \) is principally semisimple.
2. Every cyclic submodule of \( N \) has a \( \delta_M \)-supplement which is a direct summand of \( N \).
3. If \( N = N_1 + N_2 \) such that \( N_1 \cap N_2 \) is cyclic, then \( N_2 \) contains a \( \delta_M \)-supplement of \( N_1 \) in \( N \).

**Proof.** (1) Let \( n \in N \). By the hypothesis, there exists a decomposition \( N = A \oplus B \) such that \( A \leq nR \) and \( nR \cap B \ll_{\delta_M} N \). Hence \( N/\delta_M(N) = (A + \delta_M(N))/\delta_M(N) + (B + \delta_M(N))/\delta_M(N) = (nR + \delta_M(N))/\delta_M(N) + (B + \delta_M(N))/\delta_M(N) \cap (B + \delta_M(N)) = \delta_M(N) \). Thus \( N/\delta_M(N) = (nR + \delta_M(N))/\delta_M(N) \oplus (B + \delta_M(N))/\delta_M(N) \) and so \( N/\delta_M(N) \) is principally semisimple.

(2) Let \( K \) be a cyclic submodule of \( N \). By Theorem 2.4 (b), there is a decomposition \( N = A \oplus B \) with \( A \leq K \) and \( K \cap B \ll_{\delta_M} B \). Therefore \( B \) is a \( \delta_M \)-supplement of \( K \) in \( N \) as \( N = K + B \).

(3) Assume that \( N = N_1 + N_2 \) and \( N_1 \cap N_2 \) is cyclic. By Theorem 2.4, \( N_1 \cap N_2 = A \oplus B \), where \( A \) is a direct summand of \( N \) and \( B \ll_{\delta_M} N \). Let \( N = A \oplus C \) and \( N_2 = A \oplus (N_2 \cap C) \). It follows that \( N_1 \cap N_2 = A \oplus B = A \oplus (N_1 \cap N_2 \cap C) \). Let \( \pi : N_2 \to C \) be the natural projection map. Since \( B \ll_{\delta_M} N, \pi(B) = N_1 \cap N_2 \cap C \ll_{\delta_M} C \) by Lemma 1.1 (3). Then we have \( N = N_1 + (N_2 \cap C) \), \( N_2 \cap C \leq N_2 \) and \( N_1 \cap N_2 \cap C \ll_{\delta_M} N_2 \cap C \leq N_2 \). This means \( N_1 \cap N_2 \) contains a \( \delta_M \)-supplement \( N_2 \cap C \) of \( N_1 \) in \( N \). \( \square \)

A module \( N \) in \( \sigma[M] \) is said to be finitely \( \delta_M \)-lifting if for any finitely generated submodule \( K \) of \( N \), there exists a decomposition \( N = A \oplus B \) such that \( A \leq K \) and \( K \cap B \ll_{\delta_M} N \). We call a nonzero module \( N \) in \( \sigma[M] \) is \( \delta_M \)-hollow if every proper submodule is \( \delta_M \)-small, and \( N \) is called principally (finitely) \( \delta_M \)-hollow if every proper cyclic (finitely generated) submodule of \( N \) is \( \delta_M \)-small.

**Theorem 2.12** Let \( N \) be in \( \sigma[M] \). The following are equivalent:

(a) \( N \) is finitely \( \delta_M \)-lifting.
(b) \( N \) is principally \( \delta_M \)-lifting.

(c) For every finitely generated submodule \( K \) of \( N \), there is an idempotent 
\( e \in \text{End}(N) \) with \( e(N) \leq K \) and \( (1-e)(K) \ll_{\delta_M} (1-e)(N) \).

If \( N \) is indecomposable, then (a), (b) and (c), are also equivalent to:

(d) \( N \) is principally \( \delta_M \)-hollow.

(e) \( N \) is finitely \( \delta_M \)-hollow.

Proof. (a) \( \Rightarrow \) (b) is obvious.

(b) \( \Rightarrow \) (c). The proof is obtained by induction on the number of generating elements of the submodules of \( N \). By (b) and Theorem 2.4, the assertion is true for every submodule with one generator. Suppose that \( n \) is a positive integer such that the assertion is true for every submodule of \( N \) with \( n \) generators and let \( K = k_1R + \ldots + k_{n+1}R \) be a finitely generated submodule of \( N \). By theorem 2.4, there is an idempotent \( f \in \text{End}(N) \) with \( f(N) \leq k_{n+1}R \) and \( k_{n+1}R \cap (1-f)(N) = (1-f)(k_{n+1}R) \ll_{\delta_M} (1-f)(N) \). If \( L = \sum_{i=1}^{n}(1-f)(k_iR) \), then by induction hypothesis, there is an idempotent \( g \in \text{End}(N) \) with \( g(N) \leq L \) and \( L \cap (1-g)(N) = (1-g)(L) \ll_{\delta_M} (1-g)(N) \). Since \( g(N) \leq L \leq (1-f)(N) \),
\[ (1-f)g = g \text{ and hence } fg = 0. \]
Now let \( e = f + g - gf \). Then \( e \) is an idempotent, \( e(N) \leq f(N) + (1-e)(K) = (1-e)(K) = (1-g)(1-f)(K) \ll_{\delta_M} (1-e)(N) \). Since \( (1-e)(N) \ll_{\delta_M} (1-e)(N) \), this establishes (c).

(c) \( \Rightarrow \) (a). Let \( K \) be a finitely generated submodule of \( N \). Then there is an idempotent \( e \in \text{End}(N) \) with \( e(N) \leq K \) and \( (1-e)(K) \ll_{\delta_M} (1-e)(N) \). Letting \( A = e(N) \) and \( B = (1-e)(N) \). Hence we have \( N = A \oplus B \), \( K \leq A \)
and \( K \cap B = (1-e)(K) \ll_{\delta_M} (1-e)(N) \leq N \). Therefore \( N \) is finitely \( \delta_M \)-lifting.

Let \( N \) be an indecomposable module.

(b) \( \Rightarrow \) (d). Let \( K \) be a proper cyclic submodule of \( N \). Then there is a decomposition \( N = A \oplus B \) such \( A \leq K \) and \( K \cap B \ll_{\delta_M} N \). Since \( N \) is indecomposable, \( A = 0 \) and \( B = N \). So \( K \cap B = K \ll_{\delta_M} B = N \).

(d) \( \Rightarrow \) (b). Let \( K \) be a cyclic submodule of \( N \). If \( K = N \), then \( N = K \oplus (0) \) and \( K \cap (0) = (0) \ll_{\delta_M} N \). Suppose that \( K \neq N \). Then, by (d), \( K \ll_{\delta_M} N \). Since \( K = K \oplus (0) \) and \( (0) \) is a direct summand of \( N \), \( N \) is principally \( \delta_M \)-lifting by Theorem 2.4.

(d) \( \Rightarrow \) (e) follows since finite sum of \( \delta_M \)-small submodules is \( \delta_M \)-small.

(e) \( \Rightarrow \) (d) is obvious.

\[ \square \]

**Theorem 2.13** Let \( N \) be projective in \( \sigma[M] \) with \( \delta_M(N) \ll_{\delta_M} N \). Then the following are equivalent:

(a) \( N \) is principally \( \delta_M \)-lifting.

(b) \( N/\delta_M(N) \) is principally semisimple and each cyclic submodule of \( N/\delta_M(N) \)
can be lefts to a cyclic direct summand of \( N \).
Proof. (a) \( \Rightarrow \) (b). \( N/\delta_M(N) \) is principally semisimple by Proposition 2.11 (1). Note that for any submodule \( A \) of \( N \) we write \( \bar{A} \) for \( (A+\delta_M(N))/\delta_M(N) \). Hence any cyclic submodule of \( N/\delta_M(N) \) is of the form \( nR \), where \( n \in N \). Let \( n \in N \). By Theorem 2.4 there exists a decomposition \( nR = A \oplus B \) such that \( A \) is a direct summand of \( N \) and \( B \leq \delta_M(N) \). Hence \( B \leq \delta_M(N) \). So \( \bar{nR} = (nR + \delta_M(N))/\delta_M(N) = (A + \delta_M(N))/\delta_M(N) = \bar{A} \).

(b) \( \Rightarrow \) (a). Let \( n \in N \). Then \( \bar{nR} \) is a cyclic submodule of \( N/\delta_M(N) \). So \( N/\delta_M(N) = \bar{nR} \oplus K/\delta_M(N) \), for some submodule \( K \) of \( N \) containing \( \delta_M(N) \). By the hypothesis there exists a decomposition \( N = A \oplus B \) such \( \bar{nR} = \bar{A} \). Hence \( N = A + B + \delta_M(N) = nR + B + \delta_M(N) \). Since \( \delta_M(N) \leq \delta_M(N) \), \( N = (nR + B) \oplus \bar{Y} \) for a projective semisimple submodule \( \bar{Y} \) in \( \sigma[M] \) with \( Y \leq \delta_M(N) \) by Lemma 1.1(6). Since \( N \) is projective in \( \sigma[M] \), \( (nR + B) \) is also projective in \( \sigma[M] \). By Lemma 2.9, \( nR + B = X \oplus B \) with \( X \leq nR \). So \( N = (X + B) \oplus \bar{Y} \). Moreover, \( nR \cap (B \oplus \bar{Y}) = nR \cap B \leq \delta_M(N) \leq \delta_M(N) \). Hence \( nR \cap (B \oplus \bar{Y}) \leq \delta_M(N) \). Thus \( N \) is principally \( \delta_M \)-lifting.

We end this paper by defined principally \( \delta_M \)-semiperfect modules and we show that for projective modules in \( \sigma[M] \) the concepts of principally \( \delta_M \)-semiperfect and principally \( \delta_M \)-lifting modules are equivalent. Let \( N, P \in \sigma[M] \). An epimorphism \( f : P \to N \) is called a projective \( \delta_M \)-cover of \( N \) if \( P \) is projective in \( \sigma[M] \) and \( \text{Ker}(f) \leq \delta_M \) \( P \).

**Definition 2.14** A module \( N \) in \( \sigma[M] \) is called a principally \( \delta_M \)-semiperfect module if every factor module of \( N \) by a cyclic submodule has a projective \( \delta_M \)-cover.

**Theorem 2.15** Let \( N \) be projective in \( \sigma[M] \). Then the following are equivalent:
(a) \( N \) is principally \( \delta_M \)-semiperfect.
(b) \( N \) is principally \( \delta_M \)-lifting.

Proof. (a) \( \Rightarrow \) (b). Let \( n \in N \) and \( f : P \to N/nR \) be a projective \( \delta_M \)-cover. If \( \eta : N \to N/nR \) is the natural epimorphism, then there exists a homomorphism \( g : N \to P \) such that \( fg = \eta \). Hence \( P = g(N) + \text{Ker}(f) \). Since \( \text{Ker}(f) \leq \delta_M \) \( P \), by Lemma 1.1 (6), there exists a semisimple submodule \( \bar{Y} \) of \( \text{Ker}(f) \) such that \( P = g(N) \oplus \bar{Y} \). So \( g(N) \) is projective in \( \sigma[M] \). Hence \( N = \text{Ker}(g) \oplus K \) for some submodule \( K \) of \( N \). Since \( \text{Ker}(g) \leq nR \), \( N = nR + K \). Next we prove \( K \cap nR \leq \delta_M \) \( K \). Since \( \text{Ker}(f) \leq \delta_M \) \( P \), \( \text{Ker}(f) \cap g(K) \leq \delta_M \) \( P \) by Lemma 1.1
Therefore $g(K \cap nR) \ll_{\delta_M} P$. This implies $K \cap nR \ll_{\delta_M} g^{-1}(P) = K$ as $g^{-1}$ is isomorphism from $g(N)$ onto $K$.

(b) ⇒ (a). Assume that $N$ is principally $\delta_M$-lifting module. Let $n \in N$. Then there is a decomposition $N = A \oplus B$ such $A \leq nR$ and $nR \cap B \ll_{\delta_M} N$. If $\eta : B \to N/nR$ is the natural epimorphism defined by $\eta(b) = b + nR$, for all $b \in B$. Since $B$ is projective in $\sigma[M]$ and $\ker(\eta) = B \cap nR \ll_{\delta_M} B$, $\eta : B \to N/nR$ is a projective $\delta_M$-cover of $N/nR$. Therefore, $N$ is principally $\delta_M$-semiperfect module.

In [5] a ring $R$ is called a principally $\delta$–semiperfect ring if every factor module of the $R$-module $R$ by a cycle submodule has a projective $\delta$-cover. Then, since $R$ is a projective module as an $R$-module, by Theorems 2.15 and 2.13, we get the following corollary:

**Corollary 2.16** The following are equivalent for every ring $R$.

(a) $R$ is principally $\delta$–semiperfect.

(b) $R$ is principally $\delta$–lifting.

If $\delta(R) \ll_{\delta} R$, then (a) and (b) are also equivalent to:

(c) $R/\delta(R)$ is principally semisimple and each cyclic submodule of $R/\delta(R)$ can be lefts to a cyclic direct summand of $R$.

Note: The equivalence of (a), (b) in the above Corollary has been observed by I. Hatice. H. Sait and A. Harmanci in [5].

**References**


Received: May, 2012