On the Structure of Some Groups

Containing \( L_2(9) \text{wr} M_{11} \)

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Abstract

In this paper, we will show the structure of some groups containing the wreath product \( L_2(9) \text{wr} M_{11} \). Some symmetric and alternating groups are constructed in view of such wreath product. Some other related cases are also included.

Mathematics Subject Classification: 20B99

Keywords: Mathieu group; wreath product of groups; Linear group

1 Introduction

The Linear groups \( L_2(9) \) and the Mathieu group \( M_{11} \) are two of the well known simple groups. In [9], they are fully described. In a matter of fact, they can be faintly presented in different ways. One of the well known presentations of them as follows:

\[
L_2(9) = < X_1, X_2, X_3, X_4 | X_i^3 = (X_iX_j)^2 = 1, i \neq j >. \tag{1}
\]

\[
M_{11} = < X, Y, Z | X^{11} = Y^4 = (XZ)^3 = 1, X^Y = X^4, Y^Z = Y^2 >. \tag{2}
\]
In[2], Al-Muhaimeed showed that $L_2(9)$ can be presented as follows:

\[ L_2(9) = < X, Y \mid X^5 = Y^4 = [X, Y]^2 = (XY)^3 = 1 >. \] \hspace{1cm} (3)

She also showed that $L_2(9)$ can be generated using two permutations of orders 5 and 4 as follows:

\[ L_2(9) = < (1, 2, 3, 4, 5)(6, 7, 8, 9, 10), (3, 5, 7, 6)(4, 8, 9, 10) >. \] \hspace{1cm} (4)

where $M_{11}$ can be generated using two permutations of order 11 and 4 as follows:

\[ M_{11} = < (1, 2, 3, \ldots, 11)(1, 2, 3, 7, 6)(4, 8, 5, 9, 10) >. \] \hspace{1cm} (5)

In this paper, we will generate some forms of wreath product containing $L_2(9)$ and $M_{11}$. Some symmetric and alternating groups are going to be generated using some of these wreath products.

2 PRELIMINARY RESULTS

DEFINITION 2.1. Let $A$ and $B$ be groups of permutations on non empty sets $\Omega_1$ and $\Omega_2$, respectively, where $\Omega_1 \cap \Omega_2 = \phi$. The wreath product of $A$ and $B$ is denote by $A \wr B$ and defined as $A \wr B = A^{\Omega_2} \times_{\theta} B$, i.e., the direct product of $|\Omega_2|$ copies of $A$ and a mapping $\theta$, where $\theta : B \rightarrow \text{Aut}(A^{\Omega_2})$ is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that

\[ |A \wr B| = (|A|)^{|\Omega_2|}|B|. \] \hspace{1cm} (6)

THEOREM 2.1 [6] Let $G$ be the group generated by the n-cycle $(1, 2, \cdots, n)$ and the 2-cycle $(n, a)$. If $1 < a < n$, is an integer with $n = am$, then

\[ G \cong S_m \wr C_a. \] \hspace{1cm} (7)

THEOREM 2.2 [6] Let $1 \leq a \neq b < n$ be any integers. Let $n$ be an odd integer and let $G$ the group generated by the n-cycle $(1, 2, \cdots, n)$ and the 3-cycle $(n, a, b)$. If $\text{hcf}(n, a, b)$, then $G \cong A_n$. While if $n$ can be even then

\[ G \cong S_n. \] \hspace{1cm} (8)
THEOREM 2.3 [6] Let \( 1 \leq a \neq b < n \) be any integers. Let \( n \) be an even integer and let \( G \) the group generated by the \( n \)-cycle \((1, 2, \ldots, n)\) and the 3-cycle \((n, a, b)\). Then

\[
G \cong A_n. \tag{9}
\]

3 THE RESULTS

THEOREM 3.1 The wreath product \( L_2(9) \wr M_{11} \) can be generated using three elements of orders 55, 4 and 4.

Proof: Let \( X = (1, \ldots, 55)(56, \ldots, 110) \), \( Y = (33, 55, 77, 66)(44, 88, 99, 110) \) and

\[
\begin{align*}
Z &= (3, 7, 11, 8)(4, 10, 5, 6)(14, 18, 22, 19)(15, 21, 16, 17) \\
&\quad (102, 106, 110, 107)(103, 109, 104, 105),
\end{align*}
\]

be three permutation of orders 55, 4 and 4.

Let \( \alpha = Y X^{-22}(Y^{-1}X^{11})^2 \). It is not difficult to show that \( H_1 = \langle \alpha, Y \rangle \cong L_2(9) \). Hence, by conjugation by powers of \( X \) we get the direct product \( H_1 \times H_2 \times \cdots \times H_{11} \subseteq G = \langle X, Y, Z \rangle \), where each \( H_i \) is a copy of \( L_2(9) \).

Since the element \( t = (11, 22, 33, 44, 55)(66, 77, 88, 99, 110) \) \( \in H_1 \), then \( \langle t^{-1}X, Z \rangle \cong M_{11} \), where it conjugates each \( H_i \) into \( H_{i+1} \). Hence we get the wreath product \( L_2(9) \wr M_{11} \subseteq G \). The other direction is clear.

COROLLARY 3.1 The wreath product \((L_2(9) \wr M_{11}) \wr C_K\) can be generated using four elements of order 110, 4, 4 and \( K \), for all integers \( K \geq 1 \).

Proof:

Let \( X = (1, \ldots, 55k)(55k+1, \ldots, 110k) \), \( Y = (33k, 55k, 77k, 66k)(44k, 88k, 99k, 110k) \),

\[
\begin{align*}
Z &= (3k, 7k, 11k, 8k)(4k, 10k, 5k, 6k)(14k, 18k, 22k, 19k)(15k, 21k, 16k, 17k) \\
&\quad (102k, 106k, 110k, 107k)(103k, 109k, 104k, 105k),
\end{align*}
\]

and \( T = (1, 2, \ldots, k)(k+1, \ldots, 2k)(109k+1, \ldots, 110k) \).

Also, let \( \beta = X^k \) and \( \alpha = Y \beta^{-22}(Y^{-1}\beta^{11})^2 \).

It is not difficult to show that \( H_1 = \langle \alpha, Y \rangle \cong L_2(9) \).

Hence, by conjugation by powers of \( X^k \) we get the direct product \( H_1 \times H_2 \times \cdots \times H_k \subseteq G = \langle X, Y, Z, T \rangle \), where each \( H_i \) is a copy of \( L_2(9) \).
Since \( u = (11k, 22k, 33k, 44k, 55k)(66k, 77k, 88k, 99k, 110k) \in H_1 \), then \( u^{-1}X^k, Z \cong M_{11} \), where it conjugates each \( H_i \) into \( H_{i+1} \). Hence we get the wreath product \( L_2(9) \wr M_{11} \subseteq G \).

Hence, \( <L_2(9)\wr M_{11}, T> \cong (L_2(9)\wr M_{11}) \wr C_K \subseteq G \). The other direction is clear. ♦

**COROLLARY 3.2** The wreath product \((L_2(9)\wr M_{11}) \wr S_K\) can be generated using \((L_2(9)\wr M_{11}) \wr C_K\) and an element of order 2, for all \( K \geq 2 \).

**Proof:** Let \( H = (L_2(9)\wr M_{11}) \wr C_K \) and \( u = (1, 2)(k + 1, k + 2)...(109k + 1, 109k + 2) \).

Since \( T = (1, 2, \ldots, k)(k+1, \ldots, 2k)\ldots(109k+1, \ldots, 110k) \in (L_2(9)\wr M_{11}) \wr C_K \), then \( \langle T, u \rangle \cong S_K \) and so the result is true. ♦

**COROLLARY 3.3** The wreath product \((L_2(9)\wr M_{11}) \wr A_k\) can be generated using \((L_2(9)\wr M_{11}) \wr C_K\) and an element of order 3, for all odd integers \( k \geq 3 \).

**Proof:** The proof is similar to the previous one. ♦

**COROLLARY 3.4** The wreath product \((L_2(9)\wr M_{11}) \wr (S_M \wr C_a)\) can be generated using \((L_2(9)\wr M_{11}) \wr C_K\) and an element of order 2, for all \( k = am \) with \( 1 < a < k \).

**Proof:** Let \( H = (L_2(9)\wr M_{11}) \wr C_K \) and \( u = (a, k)(2k, k+a)\ldots(119k, 109k + a) \).

Since \( T = (1, 2, \ldots, k)(k+1, \ldots, 2k)\ldots(109k+1, \ldots, 110k) \in (L_2(9)\wr M_{11}) \wr C_K \), then by theorem 2.2 \( \langle T, u \rangle \cong S_m \wr C_a \) and so the result is true.

**THEOREM 3.2** \( S_{110k+1} \) and \( A_{110k+1} \) can be generated using the wreath product \((L_2(9)\wr M_{11}) \wr C_K\) and a transposition in \( S_{110k+1} \) for all integers \( k \geq 1 \) and an element of order 11 in \( A_{110k+1} \) for all odd integers \( k > 1 \).

**Proof:** Let \( \sigma = (1, 2, \ldots, 110k) \),

\[ \tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k) \\
(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k, 39k) \\
(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k) \\
(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k) \],

\( \mu = (110k + 1, 1) \) and \( \mu^\perp = (1, k, 110k + 1) \) be four Permutations, of order 77k,2,2 and 3 respectively.

Let \( H = <\sigma, \tau> \). By theorem 3.2 \( H \cong (L_2(9)\wr M_{11}) \wr C_K \).

**Case 1:** Let \( G = <\sigma, \tau, \mu^\perp> \). Let \( \sigma = \sigma \mu \), then \( \alpha = (1, 2, \ldots, 110k, 110k + 1) \) which is a cycle of order 110k+1. By theorem 2.4 \( G = <\sigma, \tau, \mu^\perp> = <\alpha> \).
\(\alpha, \mu >\approx S_{110K+1}\).

**Case 2:** Let \(G =<\sigma, \tau, \mu>\). By theorem 2.5 \(<\sigma, \mu>\approx A_{110K+1}\). Since \(\tau\) is an even permutation, then \(G \approx A_{110K+1}\).

**THEOREM 3.3** \(S_{110K+1}\) and \(A_{110K+1}\) can be generated using the wreath product \(L_2(9)\wr M_{11}\) and an element of order \(k+1\) in \(S_{110K+1}\) and \(A_{110K+1}\) for all integers \(k \geq 1\).

**Proof:** Let \(G =<\sigma, \tau, \mu>\), where

\[
\begin{align*}
\sigma & = (1, 2, 3, ..., 110)(110(k - (k - 1)) + 1, ..., 110(k - (k - 1)) + 110) \\
& \quad \ldots (110(k - 1) + 1, ..., 110(k - 1) + 110), \\
\tau & = (1, 9)(2, 6)(4, 57, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28) \\
& \quad (26, 27)(29, 30)(34, 42, 56, 64) (35, 39)(37, 38)(40, 41)(45, 53)(46, 50) \\
& \quad (48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) \\
& \quad (110(k - 1) + 1, 110(k - 1) + 9) \ldots (110(k - 1) + 73, 110(k - 1) + 74),
\end{align*}
\]

\[\mu = (110, 220, ..., 110k, 110k + 1)\], where \(k - i > 0\), be three permutations of order 110, 4 and \(k + 1\) respectively.

Let \(H =<\sigma, \tau>\). Define the mapping \(\theta\) as follows

\[\theta(11(k - i) + j) = j \forall 1 \leq j \leq 11\]

\[H =<\sigma, \tau>\approx L_2(9)\wr M_{11}\]. Let \(\alpha = \mu\sigma\) it is easy to show that \(\alpha = (1, 2, ..., 110k, 110k + 1)\), which is acyclic of order \(110k+1\).

Let \(\mu\nu = \mu^\sigma = (1, 111, ..., 110(k - 1) + 1, 110k + 1)\) and \(\beta = [\mu, \mu^\sigma] = (1, 110, 110k + 1)\). Since \(h.c.f(1, 110, 110k + 1) = 1\), then by theorem 2.3 \(G =<\sigma, \tau, \mu>\approx S_{110K+1}\) or \(A_{110K+1}\) depending on whether \(k\) is an odd or an even integer respectively.

**ACKNOWLEDGEMENTS.** The authors wish to thank the Editor and referees for valuable comments.

**References**


Received: March, 2012