A Symmetric MacWilliams Identity

of the Linear Code over $F_2 + uF_2 + vF_2 + uvF_2$

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Abstract. The symmetric weight enumerator and the Hamming weight enumerator over $F_2 + uF_2 + vF_2 + uvF_2$ were defined. By means of Lee Weight enumerator over $F_2 + uF_2 + vF_2 + uvF_2$ and the Gray map from $F_2 + uF_2 + vF_2 + uvF_2$ to $F_2$, a symmetric MacWilliams identity for Hamming weight enumerators and the symmetric weight enumerators between the linear code and its dual code over $F_2 + uF_2 + vF_2 + uvF_2$ was given.

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In 1994, Hammons et al mainly relate Lee weight of codeword over $Z_4$ to their Gray image, proving that non-linear binary code satisfied MacWilliams identity [1]. So the MacWilliams which describes the relation between different type of weight distribution for linear codes has drawn increasing research interest [2-4]. In this paper, the symmetric weight enumerator over $F_2 + uF_2 + vF_2 + uvF_2$
was defined. Using the relation between the symmetric weight enumerator and Lee weight enumerator, we obtained the MacWilliams identity for the symmetric enumerator. Finally, an example which illustrates the correctness and function of the identity were given.

1. Preliminaries

\[ R = F_2 + uF_2 + vF_2 + uvF_2 \]
\[ = \left\{ a + ub + vc + uvd \mid a, b, c \in F_2, u^2 = v^2 = 0, uv = vu \right\} \]

is defined as a commutative ring with unit element.

Let \( R^n = \left\{(x_1, \cdots, x_n) \mid x_i \in R, i = 1, 2, \cdots, n\right\} \). If \( C \) is an additive submodule of the \( R \)-module \( R^n \), then \( C \) is called as a linear code of length \( n \) over \( R \). Let \( x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \) be any two elements over \( R^n \). We may define an inner product over \( R \) by \( x \cdot y = x_1y_1 + \cdots + x_ny_n \). Then, if \( x \cdot y = 0 \), we can say \( x \) and \( y \) are orthogonal.

If \( C \) is a linear code of length \( n \) over \( R \), suppose
\[ C^\perp = \left\{ x \in R^n \mid x \cdot y = 0, \forall y \in C \right\} \]

It is easy to prove that \( C^\perp \) is also a linear code of length \( n \) over \( R \), defined as the dual code of \( C \). Furthermore, if \( C = C^\perp \), then \( C \) is called as self-dual.

The following concepts and results can be found in [5].

Map \( \Phi : R^n \to F_2^{4n} \)

\[ a + ub + vc + uvd \to (a + b + c + d, c + d, b + d, d) \]

can be defined as the Gray map from \( R^n \) to \( F_2^{4n} \).

For \( a + ub + vc + uvd \in R \), we define
\[ W'_f(a + ub + vc + uvd) = W_n(a + b + c + d, c + d, b + d, d) \]
where \( W_h \) denotes the Hamming weight for binary codes. So

\[
W_h(a + b + c + d, c + d, b + d, d) = \begin{cases} 
0 & a = b = c = d \\
1 & \text{otherwise}
\end{cases}
\]

Obviously

\[
W_L(0) = 0, \quad W_L(1) = W_L(1+u) = W_L(1+v) = W_L(1+u+v+uv) = 1.
\]

\[
\]

\[
W_L(1 + uv) = W_L(1 + u + uv) = W_L(1 + v + uv) = W_L(uv) = 4.
\]

Let \( C \) be a linear code of length \( n \) over \( R \). \( \forall x = (x_1, \cdots, x_n) \in C \). Then

\[
\sum_{i=1}^{n} W_h(x_i) \text{ is the Hamming weight for the code } x, \text{ denoting as } W_h(x). \text{ While}
\]

\[
\sum_{i=1}^{n} W_L(x_i) \text{ is Lee weight for the code } x, \text{ denoting as } W_L(x).
\]

Obviously, if \( c \in C \), then \( W_L(c) = W_h(\Phi(c)) \).

Let \( n_i(x) \) denote the number of Lee weight for coordinates of \( x \) which are to \( i (i = 0, 1, 2, 3, 4) \). Then

\[
W_L(x) = n_0(x) + 2n_1(x) + 3n_2(x) + 4n_3(x).
\]

**Lemma 1.1** Let \( C \) be a linear code over \( R \) and \( C^\perp \) be the dual code of \( C \), then \( \Phi(C^\perp) = \Phi(C)^\perp \).

2. Results

**Definition 1** Let \( C \) be a linear code of length \( n \) over \( R \), then

\[
Lee_c(s, t) = \sum_{x \in C} s^{4n-W_L(x)} W_L(x).
\]
can be called as the Lee weight enumerator of $C$. While
\[\text{Ham}_C(s,t) = \sum_{c \in C} s^{n-W_H(c)} t^{W_H(c)}.\]

can be called as the Hamming weight enumerator of $C$. Furthermore,
\[\text{Swe}_C(s_0, s_1, s_2, s_3, s_4) = \sum_{c \in C} s_0^{n_0(c)} s_1^{n_1(c)} s_2^{n_2(c)} s_3^{n_3(c)} s_4^{n_4(c)}.\]
is the symmetric weight enumerator.

**Theorem 2.1** Let $C$ be a linear code of length $n$ over $R$, then

1. $\text{Lee}_C(s,t) = \text{Swe}_C(s^4, s^3t, s^2t^2, st^3, t^4)$;
2. $\text{Ham}_C(s,t) = \text{Swe}_C(s,t,t,t,t)$.

**Proof** According to
\[n_0(c) + n_1(c) + n_2(c) + n_3(c) + n_4(c) = n,\]
we get
\[4n_0(c) + 3n_1(c) + 2n_2(c) + n_3(c) = 4n - W_L(c),\]
\[n_0(c) = n - W_H(c).\]

(1) $\text{Swe}_C(s^4, s^3t, s^2t^2, st^3, t^4) = \sum_{c \in C} s^{4n_0(c)} (s^3t)^{n_1(c)} (s^2t^2)^{n_2(c)} (st^3)^{n_3(c)} (t^4)^{n_4(c)}$
\[= \sum_{c \in C} s^{4n_0(c) + 3n_1(c) + 2n_2(c) + 3n_3(c) + 4n_4(c)} = \text{Lee}(s,t).\]

(2) $\text{Swe}_C(s,t,t,t,t) = \sum_{c \in C} s^{n_0(c) + n_1(c) + n_2(c) + n_3(c) + n_4(c)}$
\[= \sum_{c \in C} s^{n-W_L(c) + W_H(c)} = \text{Ham}_C(s,t).\]

In order to obtain the weight enumerator of the dual code, we introduce the following lemma.

**Lemma 2.2** Let $C$ be the linear code over $F_2$, $C^\perp$ be the dual code of $C$ over $F_2$. Then we get the MacWilliams identity
\[\text{Ham}_C(s,t) = \frac{1}{|C|} \text{Ham}_C(s+t,s-t).\]

**Theorem 2.3** Let $C$ be a linear code of length $n$ over the ring $R$, $C^\perp$ be the dual code of $C$. Then

1. The Lee weight enumerators of $C$ and $C^\perp$ satisfy the MacWilliams identity.
\[\text{Lee}_C(s,t) = \frac{1}{|C|} \text{Lee}_C(s+t,s-t).\]
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(2) The symmetric weight enumerators of $C$ and $C^\perp$ satisfy the MacWilliams identity.

$$Sw_{C}(s_0, s_1, s_2, s_3, s_4) = \frac{1}{|C|} Sw_{C}(s_0 + 4s_1 + 6s_2 + 4s_4 + s_0 + 2s_3 - 2s_2 + s_4,$$

$$s_0 - 2s_1 + s_4, s_0 - 2s_1 + 2s_4 - s_2 - s_0 - 4s_1 + 6s_2 - 4s_4 + s_4) .$$

(3) The Hamming weight enumerators of $C$ and $C^\perp$ satisfy the MacWilliams identity.

$$Ham_{C^\perp}(s, t) = \frac{1}{|C|} Ham_{C}(s + 15t, s - t) .$$

Proof

(1) since $W_L(c) = W_H(\Phi(c))$, then

$$\text{Lee}_{C}(s, t) = \sum_{c \in C} s^{4n - W_L(s)} W_L(s) = \sum_{\Phi(c) \in \Phi(C)} s^{4n - W_H(\Phi(c))} W_{H}(\Phi(c)) = \text{Ham}_{\Phi(C)}(s, t) .$$

Then, by the lemma 1.1 and 2.2, we obtain

$$\text{Lee}_{C^\perp}(s, t) = \text{Ham}_{\Phi(C^\perp)}(s, t) = \text{Ham}_{\Phi(C)}(s + t, s - t) = \frac{1}{|\Phi(C)|} \text{Ham}_{\Phi(C)}(s + t, s - t) = \frac{1}{|C|} \text{Lee}_{C}(s + t, s - t) .$$

(2) According to the theorem 2.1 (1) and the above proof, we can obtain

$$Sw_{C^\perp}(s^4, s^3t, s^2t^2, st^3, t^4) = \text{Lee}_{C^\perp}(s, t) = \frac{1}{|C|} \text{Lee}_{C}(s + t, s - t)$$

$$= \frac{1}{|C|} Sw_{C}((s + t)^4, (s + t)^3(s - t), (s + t)^2(s - t)^2, (s + t)(s - t)^3, (s - t)^4)$$

$$= \frac{1}{|C|} Sw_{C}(s^4 + 4s^3t + 6s^2t^2 + 4st^3 + t^4, s^4 + 2s^3t - 2st^3 - t^4) ,$$
\[ s^4 - 2s^2t^2 + t^4, \ s^4 - 2s^3t + 2st^3 - t^4, \ s^4 - 4s^3t + 6s^2t^2 - 4st^3 + t^4 \]

\[ \text{Swe}_C(s_0, s_1, s_2, s_3, s_4) \]
\[ = \frac{1}{|C|} \text{Swe}_C(s_0 + 4s_1 + 6s_2 + 4s_3 + s_4, s_0 + 2s_1 - 2s_2 - s_4, \]
\[ s_0 - 2s_1 + s_3, s_0 - 2s_1 + 2s_3 - s_4, s_0 - 4s_1 + 6s_2 - 4s_3 + s_4). \]

(3) According to the theorem 2.1 (2) and the above proof, we can obtain

\[ \text{Ham}_C(s, t) = \text{Swe}_C(s, t, t, t, t) \]
\[ = \frac{1}{|C|} \text{Swe}_C(s + 15t, s - t, s - t, s - t, s - t) \]
\[ = \frac{1}{|C|} \text{Ham}_C(s + 15t, s - t). \]

3. Examples

In the following, we show an example, illustrating the function of the theorem 2.

Example 1

Obviously, \( C = \{(0,0), (u,u)\}\) is a linear code of length 2 over \( R \). The Lee weight enumerator is

\[ \text{Lee}_C(s, t) = s^6 + s^2t^4 \]

The symmetric enumerator is

\[ \text{Swe}_C(s_0, s_1, s_2, s_3, s_4) = s_0^2 + s_2^2 \]

The Hamming weight enumerator is \( \text{Ham}_C(s, t) = s^2 + t^2 \)

By theorem 2.3, Lee weight enumerator of \( C^\perp \), the dual code of \( C \), can be written as

\[ \text{Lee}_{C^\perp}(s, t) = \frac{1}{2}[(s + t)^6 + (s + t)^2(s - t)^4] \]
\[ = s^6 + 2s^5t + 7s^4t^2 + 12s^3t^3 + 7s^2t^4 + 2st^5 + t^6 \]

The symmetric
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weight enumerator is

\[ \text{Swe}_{c^*}(s_0,s_1,s_2,s_3,s_4) = \frac{1}{2} \left[ (s_0 + 4s_1 + 6s_2 + 4s_3 + s_4)^2 + (s_0 - 2s_2 + s_4)^2 \right] \]

\[ = s_0^2 + 8s_1^2 + 20s_2^2 + 8s_3^2 + s_4^2 + 4s_0s_1 + 4s_0s_2 + 4s_0s_3 + 2s_0s_4 + 24s_1s_2 + 16s_1s_3 + 4s_1s_4 + 24s_2s_3 + 4s_2s_4 + 4s_3s_4 \]

Hamming weight enumerator is

\[ \text{Ham}_{c^*}(s,t) = \frac{1}{2} \left[ (s + 15t)^2 + (s - t)^2 \right] \]

\[ = s^2 + 14st + 113t^2. \]

References


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