On the Variation of Generalized Matrix Function

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Abstract. Let $V$ be an $m$-dimensional Hilbert space. Suppose $H$ is a subgroup of the symmetric group of order $m$, and $\chi : H \to \mathbb{C}$ is a character of degree 1 on $H$. For $A \in M_m$, we define the generalized matrix function $d_\chi(A)$ by

$$d_\chi(A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^m a_{\sigma(i)}.$$

In this paper, we prove that for any $A, B \in M_n$ the inequalities

$$\|d_\chi(A) \pm d_\chi(B)\|_p \leq \left[\|A\|_p^p + \|B\|_p^p\right]^{\frac{1}{p}}$$

and

$$\|d_\chi(A) \pm d_\chi(B)\|_p \leq \left[\|d_\chi(A')A + d_\chi(B')B\|_p + \|2 \text{Re} d_\chi(A'B)\|_p\right]^{\frac{1}{p}}$$

hold, where $\|\cdot\|_p$ is the $L_p$-operator norm ($0 < p \leq 2$).

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1. Background and Notations

Let $V$ be an $m$-dimensional Hilbert space. Suppose $H$ is a subgroup of the symmetric group of order $m$, and $\chi : H \to \mathbb{C}$ is a character of degree 1 on $H.$
Consider the symmetrizer on the tensor space $\otimes^m V$

$$S(V_1 \otimes \ldots \otimes V_m)$$

$$= \frac{1}{h} \sum_{\sigma \in H} \chi(\sigma)V_{\sigma^{-1}(1)} \otimes \ldots \otimes V_{\sigma^{-1}(m)}$$

defined by $H$ and $\chi$, where $h$ is the order of the subgroup $H$.

Obvious $S^2 = S, S^* = S$. The vector space $V^m_\chi(H) = S(\otimes^m V)$ is a subspace of $\otimes^m V$, called the symmetry class of tensors over $V$ associated with $H$ and $\chi$. The elements in $V^m_\chi(H)$ of the form $S(V_1 \otimes \ldots \otimes V_m)$ are called decomposable tensors and denoted by $V_1 \ast \ldots \ast V_m$.

Let $M_m$ be the set of $m \times m$ complex matrices. Define the generalized matrix function

$$d_{\chi} : M_m \rightarrow C$$

associated with $\chi$ by

$$d_{\chi}(A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{m} a_{\sigma(i)}$$

$A = (a_{ij}) \in M_m$.

This general matrix function includes the permanent($H=S_m, (\chi(\sigma))=1$), the determinat ($H=S_m, \chi(\sigma)=\text{sign}\sigma$), and other assorted interesting functions.

Influenced by the general recent interest in general matrix function, the question of the Variation of general matrix function has been studied ([1]). In the paper, we show that the main results is:

**Theorem 1** Let $A, B \in M_m$, then
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\[ |d_x(A) \pm d_x(B)| \leq \left[ \|A\|_p^p + \|B\|_p^p \right]^{\frac{1}{2}}. \]

For \( \| \cdot \| = \| \cdot \|_p \), the \( l_p \)—operator norm (\( 0 < p \leq 2 \)).

**Theorem 2** Let \( A, B \in M_m \),

\[ |d_x(A) \pm d_x(B)| \leq \left[ d_x(A^*A) + d_x(B^*B) + 2 \Re d_x(A^*B) \right]^{\frac{1}{2}}. \]

Here \( 0 < p \leq 2 \) we define as usual the \( l_p \)—norm

\[ \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \]

and the associated operator norm

\[ \|A\|_p = \max \{ \|Ax\|_p : \|x\|_p \leq 1 \}. \]

For any rectangular matrices \( A, B \) we denote \( A \otimes B \) the usual tensor product.

2. Proof of the Theorem

The prove the theorem, we start with four preparatory lemma.

**Lemma 1**\(^{[2]}\) For \( A_i \in M_m \),

\[ i=1,2, \ldots, m, \]

\[ \|A_1 \otimes \cdots \otimes A_m\|_p = \prod_{i=1}^{m} \|A_i\|_p. \] \hspace{1cm} (2.1)

**Lemma 2**\(^{[3]}\) If \( \chi \) is any character of a finite group \( G \), then \( |\chi(\sigma)| > \chi(e) \) for each \( \sigma \in G \).

**Lemma 3** If \( A_1, A_2 \in M_m, \) \( x_1, x_2 \in V \) and \( \|x_1\| = \|x_2\| = 1 \), then

\[ \|A_1 x_1 * A_2 x_2\|_p \leq \|A_1 x_1\|_p \|A_2 x_2\|_p \leq \|A_1\|_p \|A_2\|_p. \] \hspace{1cm} (0 < p \leq 2) \hspace{1cm} (2.2)

**Proof** By definition
\[ A_{i_1} * A_{i_2} = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) A_{\sigma^{-1}(i)} \otimes A_{\sigma^{-1}(2)} \] which \( H \) is either \{(1)\} or \( H = \{(1), (2)\} \).

If \( H = \{(1)\} \), then
\[
\left\| A_{i_1} \otimes A_{i_2} \right\|_p = \left\| (A_{i_1}) \otimes (A_{i_2}) \right\|_p = \left\| A_{i_1} \right\|_p \left\| A_{i_2} \right\|_p \leq \left\| A_{i_1} \right\|_p \left\| A_{i_2} \right\|_p
\]

(2.3)

If \( H = \{(1), (2)\} \), then
\[
\left\| A_{i_1} \otimes A_{i_2} \right\|_p = \frac{1}{2} \left\| \sum_{\sigma \in H} \chi(\sigma) A_{\sigma^{-1}(i)} \otimes A_{\sigma^{-1}(2)} \right\|_p
\]
\[
\leq \frac{1}{2} \sum_{\sigma \in H} \chi(\sigma) \left\| A_{\sigma^{-1}(i)} \otimes A_{\sigma^{-1}(2)} \right\|_p
\]
\[
\leq \frac{1}{2} \sum_{\sigma \in H} \chi(\sigma) \left\| A_{\sigma^{-1}(i)} \right\|_p \left\| A_{\sigma^{-1}(2)} \right\|_p = \left\| A_{i_1} \right\|_p \left\| A_{i_2} \right\|_p \leq \left\| A_{i_1} \right\|_p \left\| A_{i_2} \right\|_p
\]

(2.4)

and by (2.3) and (2.4) we have (2.2).

An obvious consequence is

**Lemma 4** For \( A_i \in M_m, x_i \in V \)

\[ \|x_i\| = 1 \quad (i = 1, 2, \ldots, s) \]
\[ \left\| A_{i_1} \otimes \cdots \otimes A_{i_s} \right\|_p \leq \prod_{i=1}^s \left\| A_{i_i} \right\|_p \leq \prod_{i=1}^s \left\| A \right\|_p \]

\[ (0 < p \leq 2) \]

**Lemma 5** Let \( A, B \in M_m \), and let \( e_i \) denote the m-tuple with 1 in position, zero else where. Then

\[ (Be_1 * Be_2 * \ldots * Be_m, Ae_1 * \ldots * Ae_m) \]
\[ = d_A (B^* A). \]

(2.5)

In particular

\[ \left\| Ae_1 * \cdots * Ae_m \right\|_p^2 = d_A (A^* A), \]

(2.6)

and

\[ (e_1 * e_2 * \ldots, e_m, Ae_1 * \ldots * Ae_m) \]
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\[ d_\chi(A) = \chi(A). \]  \hspace{1cm} (2.7)

**Proof** In fact

\[ a_{ij} = (A)_{ij} = (Ae_i, e_j) \quad (i, j = 1, 2, \ldots, m). \]

By \( S^2 = S, \quad S^* = S, \) we obtain

\[
(Be_1 \ast Be_2 \ast \ldots \ast Be_m, Ae_1 \ast \ldots \ast Ae_m)
\]

\[ = (Be_1 \otimes \ldots \otimes Be_m, \sum_{\sigma \in \Pi} \chi(\sigma) \quad Ae_{\sigma^{-1}(1)} \]

\[ \otimes \ldots \otimes Ae_{\sigma^{-1}(m)} ) \]

\[ = \sum_{\sigma \in \Pi} \chi(\sigma) (Be_1 \otimes \ldots \otimes B e_m, Ae_{\sigma^{-1}(1)} \]

\[ \otimes \ldots \otimes Ae_{\sigma^{-1}(m)} ) \]

\[ = \sum_{\sigma \in \Pi} \chi(\sigma^{-1}) \prod_{i=1}^m (Be_i, Ae_{\sigma^{-1}(i)} ) \]

\[ = \sum_{\sigma \in \Pi} \chi(\sigma) \prod_{i=1}^m (Be_i, Ae_{\sigma(i)} ) \]

\[ = \sum_{\sigma \in \Pi} \chi(\sigma) \prod_{i=1}^m (A^*Be_i, e_{\sigma(i)} ) \]

\[ = \sum_{\sigma \in \Pi} \chi(\sigma) \prod_{i=1}^m (A^*B)_{\sigma(i)} \]

\[ = d_\chi(B^*A). \]

Take \( B = A \) or \( B = I \), this implies

\[ \|Ae_1 \ast \ldots \ast Ae_m\|_2^2 = d_\chi(A^*A). \]

and

\[
( e_1 \ast e_2 \ast \ldots, e_m, Ae_1 \ast \ldots \ast Ae_m )
\]

\[ = d_\chi(A). \]
Proof of the Theorem 1. From (2.7) and the Schwarz inequality we have
\[
\left| d_x (A) \pm d_x (B) \right|
\]
\[
= \left| \left( e_1 \ast \cdots \ast e_m, Ae_1 \ast \cdots \ast Ae_m \right) \right|
\]
\[
= \left| \left( e_1 \ast \cdots \ast e_m, Be_1 \ast \cdots \ast Be_m \right) \right|
\]
\[
\leq \left\| A e_1 \ast \cdots \ast Ae_m \pm Be_1 \ast \cdots \ast Be_m \right\|
\]
\[
\leq \left\| A \right\|_p \left\| e_1 \ast \cdots \ast e_m \right\|_p \sum_{i=1}^{m} \left( A e_i \ast \cdots \ast Ae_m \right)
\]
\[
\leq \left[ \left\| A \right\|_p \right]^2 \left\| e_1 \ast \cdots \ast e_m \right\|_p \sum_{i=1}^{m} \left( A e_i \ast \cdots \ast Ae_m \right)
\]
\[
\leq \left[ \left\| A \right\|_p + \left\| B \right\|_p \right]^2.
\]
Further, we recall that \( \left\| A \right\|_p \) is a monotone decreasing function of \( p \),
\[
0 \leq p \leq \infty.
\]
Hence
\[
\left\| A \right\|_p \geq \left\| A \right\|_2, \left\| B \right\|_p \geq \left\| B \right\|_2, \text{ for all}
\]
\[
0 \leq p \leq 2.
\]
Hence
\[
\left| d_x (A) \pm d_x (B) \right| \leq \left[ \left\| A \right\|_p + \left\| B \right\|_p \right]^2
\]
holds for all \( 0 \leq p \leq 2 \).

Proof of the Theorem 2. By (2.5), (2.6), (2.7), we have
\[
\left| d_x (A) \pm d_x (B) \right|
\]
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\[ \pm \left( e_1 \ast \ldots \ast e_m, Ae_1 \ast \ldots \ast Ae_m \right) \]

\[ \pm \left( e_1 \ast e_2 \ast \ldots \ast e_m, Be_1 \ast \ldots \ast Be_m \right) \]

\[ \pm \left( e_1 \ast \ldots \ast e_m, Ae_1 \ast \ldots \ast Ae_m \pm Be_1 \ast \ldots \ast Be_m \right) \]

\[ \leq \left\| A e_1 \ast \ldots \ast Ae_m \pm Be_1 \ast \ldots \ast Be_m \right\| \]

\[ \leq \| A e_1 \ast \ldots \ast Ae_m \pm Be_1 \ast \ldots \ast Be_m \| \frac{1}{2} \| e_1 \ast \ldots \ast e_m \| \frac{1}{2} \]

\[ = (Ae_1 \ast \ldots \ast Ae_m \pm Be_1 \ast \ldots \ast Be_m) \frac{1}{2} \]

\[ = \left[ \| Ae_1 \ast \ldots \ast Ae_m \| \frac{1}{2} + \| Be_1 \ast \ldots \ast Be_m \| \frac{1}{2} \right] \]

\[ \pm (Be_1 \ast \ldots \ast Be_m, Ae_1 \ast \ldots \ast Ae_m) \]

\[ \pm (Ae_1 \ast \ldots \ast Ae_m, Be_1 \ast \ldots \ast Be_m) \| \frac{1}{2} \]

\[ = \left[ d_x(A^*A) + d_x(B^*B) \pm d_x(A^*B) \pm d_x(B^*A) \right] \frac{1}{2} \]

\[ \leq \left[ d_x(A^*A) + d_x(B^*B) \pm 2 Re d_x(A^*B) \right] \frac{1}{2} \]

References


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