Some Properties of Cayley Graphs on Symmetric Groups $S_n$

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Abstract

Let $G$ be a finite group and $S$ a subset of $G$ such that $S = S^{-1}$ and $1_G \notin S$. Then the Cayley graph $\Gamma = \text{Cay}(G, S)$ relative to $S$ is the graph with vertex set $G$ and edge set $E(\Gamma(G, S)) = \{gh \mid hg^{-1} \in S\}$. Since $S$ is inverse closed and does not contain the identity, this graph is undirected and has no loops. A nonempty subset $S$ of $G$ is called a Cayley subset if $S = S^{-1}$ and $1_G \notin S$. In this paper we determine the number of Cayley graphs on Symmetric group $S_n$ and Alternating group $A_n$ that are undirected. We also show that up to isomorphism there are exactly 8 Cayley graphs of $S_3$ and 4 Cayley graphs of $S_4$ of valency 2.

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1 Introduction

We begin with a short review of some of the basic concepts of graph theory and group theory. One of the most widely known and extensively studied family of vertex transitive graphs are the family of Cayley graphs. The construction of Cayley graphs are depend upon on groups. We restrict our attention only to finite groups.

For a finite group $G$ and a subset $S$ of $G \setminus \{1_G\}$, the Cayley digraph $\Gamma = \text{Cay}(G, S)$ of $G$ with respect to $S$ is defined as the directed graph with vertex set $G$ and arc set $\{(a, b)\mid a, b \in G, ba^{-1} \in S\}$. If $S = S^{-1} := \{s^{-1} \mid s \in S\}$ then $\text{Cay}(G, S)$ may be viewed as an undirected graph and is called the Cayley
graph of \( G \) with respect to \( S \). It easily follows that \( \text{Cay}(G, S) \) has valency \( |S| \) and that \( \text{Cay}(G, S) \) is connected if and only if \( \langle S \rangle = G \). More details about Cayley graphs can be found in [3].

A permutation of a set \( \Omega \) is a bijection \( \alpha : \Omega \rightarrow \Omega \). The family of all the permutations of a set \( \Omega \), denoted by \( S_\Omega \), is called the symmetric group on \( \Omega \). When \( \Omega = \{1, 2, ..., n\} \), \( S_\Omega \) is usually denoted by \( S_n \), and it is called the symmetric group on \( n \) letters.

If \( \alpha \in S_n \) and \( i \in \{1, 2, ..., n\} \), then \( \alpha \) fixes \( i \) if \( \alpha(i) = i \), and \( \alpha \) moves \( i \) if \( \alpha(i) \neq i \).

Let \( i_1, i_2, ..., i_r \) be distinct integers in \( \{1, 2, ..., n\} \). If \( \alpha \in S_n \) fixes the other integers (if any) and if \( \alpha(i_1) = i_2, \alpha(i_2) = i_3, ..., \alpha(i_{r-1}) = i_r, \alpha(i_r) = i_1 \), then \( \alpha \) is called an \( r \)-cycle and is denoted by \( (i_1 i_2 ... i_r) \). One also says that \( \alpha \) is a cycle of length \( r \). A 2-cycle interchanges \( i_1 \) and \( i_2 \) and fixes everything else; 2-cycles are also called transpositions. A 1-cycle is the identity, for it fixes every \( i \); thus, all 1-cycles are equal: \( (i) = (1) \) for all \( i \).

Two permutations \( \alpha, \beta \in S_n \) are disjoint if every \( i \) moved by one is fixed by the other: if \( \alpha(i) \neq i \), then \( \beta(i) = i \), and if \( \beta(j) \neq j \), then \( \alpha(j) = j \). A family \( \beta_1, ..., \beta_t \) of permutations is disjoint if each pair of them is disjoint. Disjoint permutations \( \alpha, \beta \in S_n \) commute. Every permutation \( \alpha \in S_n \) is either a cycle or a product of disjoint cycles. Every permutation is a product of 2-cycles. A permutation \( \alpha \in S_n \) is said to be an even permutation if it can be represented as a product of an even number of transpositions. We call a permutation odd if it is not an even permutation. \( S_n \) has a normal subgroup of index 2 the Alternating group, \( A_n \), consisting of all even permutations. More details about Symmetric groups can be found in [5].

Recently Gholamreza Aghababaei Beny and Zarullo Rakhmonov [1] have derived a formula for number of Cayley graphs on \( \mathbb{Z}_n \) that are undirected. Motivated by this, in the present paper we determine the number of Cayley graphs on Symmetric group \( S_n \) and Alternating group \( A_n \) that are undirected.

## 2 Main results

A fundamental problem in graph theory is the so-called isomorphism problem, that is, to decide whether two given graphs are isomorphic or not. In this section we investigate the isomorphism problem for finite Cayley graphs on symmetric group \( S_n \). The complement \( \bar{S} \) of Cayley subset \( S \) with respect to \( G^* = G \setminus \{1_G\} \) is also a Cayley subset. Because if \( x \in \bar{S} \) then \( x \notin S \) and since \( S \) is a Cayley subset \( x^{-1} \notin S \). Hence \( x^{-1} \in \bar{S} \) i.e. \( \bar{S} \) is Cayley subset. It is clear that \( \bar{\Gamma} = \text{Cay}(G, \bar{S}) \) and \( \Gamma = \text{Cay}(G, S) \) have the same vertex set as \( G \), where vertices \( g \) and \( h \) are adjacent in \( \bar{\Gamma} = \text{Cay}(G, \bar{S}) \) if and only if they are...
not adjacent in $\Gamma = Cay(G,S)$. So the automorphism group of $\Gamma = Cay(G,S)$ is equal to the automorphism group of $\bar{\Gamma} = Cay(G,\bar{S})$.

Here we proceed towards some theorems about the number of Cayley graphs.

**Theorem 2.1.** Suppose $|S_k|$ denotes the number of Cayley graphs $\Gamma = Cay(S_n, S)$ with $|S| = k$. Then

$$|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\gamma(S_n)}{k-2m} \left( \frac{n! - \gamma(S_n)}{2m} \right)$$

where $\gamma(S_n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!2^k(n-2k)!}$

and $|x|$ denotes the greatest integer $\leq x$.

*Proof.* Suppose $S_n^* = S_n \setminus \{1, s_n\}$. Then $S_n^* = A \cup \bar{A}$, where $A = \{\rho_1\rho_2...\rho_t \mid \rho_i \in S_n\}$ and $\bar{A} = S_n^* \setminus A$. Since in the symmetric group $S_n$, every $k$-cycle has order $k$, and if $\sigma$ is $k$-cycle and $\tau$ is an is a $l$-cycle and these two cycles are disjoint then $\sigma \tau = \tau \sigma$ and this element has order $lcm(k,l)$. Therefore all of elements in $A$ has order 2. Hence for every element of $A$ its inverse is itself. Also if $1 < r \leq n$ there are $\frac{1}{r}[n(n-1)...(n-r+1)]$ $r$-cycles in $S_n$ [5]. And if $kr \leq n$ then the number of $\rho \in S_n$ where $\rho$ is a product of $k$ disjoint $r$-cycles, is $\frac{1}{kr!x}[n(n-1)...(n-kr+1)]$ [5]. So $|A| = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!2^k(n-2k)!}$. We denote $|A|$ by $\gamma(S_n)$. We have $\left[\frac{n! - \gamma(S_n)}{2} \right]$ pairs of the type $(\rho, \rho^{-1})$ with $\rho \neq \rho^{-1}$ in $\bar{A}$. We can construct a Cayley subset with $k$ elements by choosing $m$ $(m \leq \lfloor \frac{k}{2} \rfloor)$ pairs of $A$ and $k - 2m$ elements of $A$. This implies that the number of Cayley graphs $\Gamma = Cay(S_n, S)$ that undirected and having $k$ elements is

$$|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\gamma(S_n)}{k-2m} \left( \frac{n! - \gamma(S_n)}{2m} \right).$$

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**Theorem 2.2.** Let $N(S_n)$ denotes the number of Cayley graphs on a symmetric group $S_n$. Then

$$N(S_n) = 2 + 2 \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\gamma(S_n)}{k-2m} \left( \frac{n! - \gamma(S_n)}{2m} \right)$$

where

$$\gamma(S_n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!2^k(n-2k)!}.$$
Proof. If $S$ is a Cayley subset of $S_n$ with $k$ elements ($0 \leq k \leq n! - 1$), then its complement $S$ with respect to $S_n^* = S_n \setminus \{1_{S_n}\}$ is also a Cayley subset of $S_n$ with $(n! - 1 - k)$ elements. Thus $|S_k| = |S_{n!-1-k}|$, $0 \leq k \leq n! - 1$. So

$$N(S_n) = \sum_{k=0}^{n!-1} |S_k| = |S_0| + |S_{n!-1}| + \sum_{k=1}^{n!-2} |S_k| = 2 + 2 \sum_{k=1}^{n!-2} |S_k|$$

$$= 2 + 2 \sum_{k=1}^{n!-2} \frac{\lfloor \frac{k}{2} \rfloor}{2} \left(\frac{\gamma(S_n)}{2} \right) \left(\frac{\lfloor \frac{n!-1-\gamma(S_n)}{2} \rfloor}{m}\right), \text{ using Theorem 2.1.} \tag{□}$$

**Theorem 2.3.** Suppose $|S_k|$ denotes the number of Cayley graphs $\Gamma = Cay(A_n, S)$ with $|S| = k$. Then

$$|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left(\frac{\gamma(A_n)}{k-2m}\right) \left(\frac{\lfloor \frac{n!-\gamma(A_n)-1}{2} \rfloor}{m}\right)$$

where

$$\gamma(A_n) = \sum_{k \in U} \frac{n!}{k!2^k(n-2k)!}; U = \{2i \mid 1 \leq i \leq \lfloor \frac{n}{4} \rfloor\}.$$ 

Proof. Suppose $A_n^* = A_n \setminus \{1_{A_n}\}$. Then $A_n^* = A \cup \tilde{A}$, where $A = \{\rho_1, \rho_2, ..., \rho_s \mid \rho_i \in S_n\}$ are disjoint, 2-cycle and $t$ is even, $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$ and $\tilde{A} = S_n^* \setminus A$. As in the previous theorem all of elements in $A$ has order 2. Hence for every element of $A$ its inverse is itself. If $kr \leq n$ then the number of $\rho \in S_n$ where $\rho$ is a product of $k$ disjoint $r$-cycles, is $\frac{1}{k!r^r}[n(n-1)...(n-kr+1)]$. If $r = 2$ and $k$ is even then $\rho \in A \subseteq A_n$. So $|A| = \sum_{k \in U} \frac{n!}{k!2^k(n-2k)!}; U = \{2i \mid 1 \leq i \leq \lfloor \frac{n}{4} \rfloor\}$. We denotes $|A|$ by $\gamma(A_n)$. We have $\lfloor \frac{n!-\gamma(A_n)-1}{2} \rfloor$ pairs of the type $(\rho, \rho^{-1})$ with $\rho \neq \rho^{-1}$ in $\tilde{A}$. We can construct a Cayley subset with $k$ elements by choosing $m(o \leq m \leq \lfloor \frac{k}{2} \rfloor)$ pairs of $\tilde{A}$ and $k - 2m$ elements of $A$. This implies that the number of Cayley graphs $\Gamma = Cay(A_n, S)$ that undirected and having $k$ elements is

$$|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left(\frac{\gamma(A_n)}{k-2m}\right) \left(\frac{\lfloor \frac{n!-\gamma(A_n)-1}{2} \rfloor}{m}\right). \tag{□}$$

**Theorem 2.4.** Let $N(A_n)$ denotes the number of Cayley graphs on a Alternating group $A_n$. Then
Proof. If its complement \( \overline{C} \) is a Cayley graph of \( A \) and \( N \), where \( t(n) = n \) except when \( n = 1 \).

The following Theorems are basic for Cayley graphs and Cayley subsets.

3 Non-Isomorphic Cayley Graphs of Symmetric Groups

Some properties of Cayley graphs on symmetric groups

Theorem 3.1. \[2\] Let \( S_n \) be a symmetric group and \( n > 2 \) then \( \text{Aut}(S_n) = S_n \) except when \( n = 6 \). In the exceptional case \( \text{Aut}(S_6) = S_6 \rtimes C_2 \).

If \( \delta = (i_1 i_2 \ldots i_r) \in S_n \), \( \alpha(j_1, j_2, \ldots, j_k) \in \text{Aut}(S_n) \). Then \( (i_1 i_2 \ldots i_r)\alpha(j_1, j_2, \ldots, j_k)(t_1 t_2 \ldots t_r) \) where,

\[
t_m = \begin{cases} 
  i_m & \text{if } i_m \notin \{j_1, j_2, \ldots, j_k\}, \\
  j_{i+1} & \text{if } i_m = j_1, \text{ for some } 1 \leq l < k, \\
  j_k & \text{if } i_m = j_k.
\end{cases}
\]

Theorem 3.2. \[3\] If \( \alpha \) is an automorphism of group \( G \), then \( \text{Cay}(G, S) \) and \( \text{Cay}(G, \alpha(S)) \) are isomorphic.

The converse of the Theorem 3.2 is not true. Two Cayley graphs for a group \( G \) can be isomorphic even if there is no automorphism of \( G \) relating their connection sets. The converse of Theorem 3.2 about Cayley Graphs of Alternating group \( A_4 \) [7] and all disconnected Cayley graphs of \( A_5 \) and all Cayley graphs of \( A_5 \) of valency 4 is true[6] which was a conjecture posed by Li and Praegar [4].
In general it is difficult to compute the number of Cayley graphs on a group $G$ that are undirected up to isomorphism. In this section we determine the number of non-isomorphic Cayley graphs of symmetric group $S_n$ and Alternating group $A_n$ for $n = 3, 4$.

**Lemma 3.3.** Let $S_n$ be Symmetric group. Then $\text{Cay}(S_n, \{ij\})$ $i, j \in \{1, 2, ..., n\}$ are all isomorphic to each other. i.e up to isomorphism, there is exactly one Cayley graph on Symmetric group $S_n$ of valency 1.

**Proof.** Follows easily from Theorem 3.2.

**Lemma 3.4.** Let $S_n$ ($n \geq 4$) be Symmetric group. Then $\text{Cay}(S_n, \{(12), (13)\})$, $\text{Cay}(S_n, \{(12), (34)\})$, $\text{Cay}(S_n, \{(12), (13), (24)\})$, $\text{Cay}(S_n, \{(123), (132)\})$ are all Non-Isomorphic to each other. i.e up to isomorphism, there are at least 4 Cayley graph on Symmetric group $S_n$ of valency 2.

**Proof.** Let $H = \langle (12), (13) \rangle$ be a subgroup of $S_n$. Let gh be an edge of $\text{Cay}(S_n, \{(12), (13)\})$, then $gh^{-1}$ is an element of $H \setminus \{1_{S_n}\}$, which implies that cosets $Hg$ and $Hh$ are equal. So two vertices are adjacent if and only if they are in the same coset. Thus there are $|S_n : H|$ components; each of which is of size $|H|$. But $H = \langle (12), (13) \rangle = \{(), (12), (13), (23), (123), (132)\}$

$K = \langle (12), (34) \rangle = \{(), (12), (34), (12)(34)\}$

$M = \langle (12), (13)(24) \rangle = \{(), (12), (34), (12)(34), (13)(24), (1423), (1324), (14)(23)\}$

$N = \langle (123), (132) \rangle = \{(), (123), (132)\}$. And $|S_n : H| = n! / 6$, $|S_n : K| = n! / 4$, $|S_n : M| = n! / 8$, $|S_n : N| = n! / 3$. This implies that Cayley graphs of $S_n$ of valency 2 as above are all Non-Isomorphic to each other.

**Theorem 3.5.** Up to isomorphism, there are exactly 8 Cayley graphs of $S_3$.

**Proof.** From the definition of Cayley subset it easily follows that a subset $S$ of $S_n^* = S_n \setminus \{1_{S_n}\}$ is a Cayley subset of $S_n$ if and only if $S_n^* \setminus S$ is a Cayley subset of $S_n$. So we shall always assume that $|S| \leq 2$. Since $\langle (12), (13) \rangle \overset{\alpha(1,2,3)}{\rightarrow} \{12, (23)\}$ and $\langle (12), (13) \rangle = S_3$, and $\text{Cay}(S_3, \{(12), (13)\})$ is connected. Also $\langle (123), (132) \rangle = A_3$ so $\text{Cay}(S_3, \{(123), (132)\})$ is disconnected. Now by Lemma 3.3 there are exactly 1, 1, 2 non-isomorphic Cayley graphs of $S_3$ of valency 0, 1, 2 respectively. Thus there are exactly 1, 1, 2 non-isomorphic Cayley graphs of $S_3$ of valency 5, 4, 3 respectively. Therefore up to isomorphism, there are exactly 8 Cayley graphs of $S_3$.

**Theorem 3.6.** [7] Up to isomorphism, there are exactly 22 Cayley graphs of $A_4$.

**Theorem 3.7.** Up to isomorphism, there are exactly 4 Cayley graphs of $S_4$ of valency 2.
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Proof. By Lemma 3.4 and Sylow Theorem every Cayley subsets of Symmetric group $S_4$ can be expressed as $T_1 = \{(12), (13)\}$, $T_2 = \{(12), (34)\}$, $T_3 = \{(12), (13)(24)\}$, $T_4 = \{(123), (132)\}$. And $|\langle T_1 \rangle| = 6$, $|\langle T_2 \rangle| = 4$, $|\langle T_3 \rangle| = 8$, $|\langle T_4 \rangle| = 3$. Since $\text{Cay}(S_4, T_1) \cong 4C_6$, $\text{Cay}(S_4, T_2) \cong 6C_4$, $\text{Cay}(S_4, T_3) \cong 3C_8$, $\text{Cay}(S_4, T_4) \cong 8C_3$. So there are exactly 4 Cayley graphs of $S_4$ of valency 2. \qed

4 Open problems

We conclude this paper with two open problems.

(1) Determine the Number of Cayley Graphs of Symmetric Groups $S_n$ up to isomorphism?

(2) Determine the Number of Cayley Graphs of Alternating Groups $A_n$ up to isomorphism?

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