On Varieties of \((n, m)\)-Semigroups

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Abstract

We investigate varieties of \((n, m)\)-semigroups. A direct description of the complete system of \((n, m)\)-identities for a variety of \((n, m)\)-semigroups is obtained.

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1 Introduction and preliminaries

Vector valued, i.e. \((m+k, m)\)-groupoids, semigroups and groups were introduced in [9] and [1]. Since then, the theory of \((n, m)\)-structures was developed and interesting results were obtained (see for example [2], [5], [6], [10]). Vector valued algebraic structures are a generalization of the usual binary, i.e. \((2, 1)\) algebraic structures. On the one hand they are similar to the binary structures, but on the other hand they incorporate new ideas and specific properties.

Throughout the sequel we assume that \(Q \neq \emptyset\), \(n, m \in \mathbb{N}\) and \(n-m = k \geq 1\). The set of all positive integers will be denoted by \(\mathbb{N}\) and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). Also, \(\mathbb{N}_t\) and \(\mathbb{N}_{t,0}\) will denote the sets \(\{1, 2, \ldots, t\}\) and \(\{0, 1, 2, \ldots, t\}\), \(t \in \mathbb{N}\). If \(x = (a_1, a_2, \ldots, a_t) \in Q^t\) (where \(Q^t\) is the cartesian product of \(t\) copies of \(Q\)), then we write \(x = a_1\), and we identify \(x\) with the word \(a_1a_2\ldots a_t\). For such an \(x\) we say that its length \(|x|\) is \(t\). The notation \(a_r^t\) where \(r \geq t\) will be identified
with the empty word (denoted by 1). Let \( Q^+ \) be the union of all the cartesian products \( Q^t, t \in \mathbb{N} \), which is, the free semigroup generated by \( Q \).

By \( Q^{m,k} \) we denote the set \( \{x | x \in Q^+, |x| = m + sk, s \in \mathbb{N} \} \).

A map \( f : Q^n \to Q^m \) is called an \((n,m)\)-operation on \( Q \), and \((Q,f)\) is called an \((n,m)\)-groupoid. An \((n,m)\)-groupoid \((Q,f)\) is called an \((n,m)\)-semigroup, if the \((n,m)\)-operation is associative, i.e. if \( f(xf(y)z) = f(uf(v)w) \), for any \( xyz = uvw \in Q^{n+k}, y,v \in Q^n \) (where \( x,z,u,w \in Q^* = Q^+ \cup \{1\} \)).

For \( m = 1 = k \), the above notions are the usual notions of binary groupoids and semigroups, and for \( m = 1, k > 1 \) they are the notions of \( n \)-groupoids and \( n \)-semigroups.

From now on we will assume that \( m \geq 2 \).

An \((n,m)\)-groupoid \((Q,f)\) can be considered as an algebra with \( m \) \( n \)-ary operations \( f_1, f_2, \ldots, f_m : Q^n \to Q \), such that \( f(x) = f_1(x)f_2(x) \ldots f_m(x) \). These operations \( f_j : Q^n \to Q, j \in \mathbb{N}_n \) are called component operations for the \((n,m)\)-operation \( f \). They can be extended to an infinite family of operations \( f_{j,s} : Q^{m+sk} \to Q \) for \( s \in \mathbb{N} \), where for a given \( s \), there are more than one operation \( f_{j,s} \). When \((Q,f)\) is an \((n,m)\)-semigroup, because of the associative law, for each \( s \in \mathbb{N} \), there is only one operation \( f_{j,s} : Q^{m+sk} \to Q \) whose union is a map \( f_j : Q^{m,k} \to Q \). All this leads to the following notions. A map \( g : Q^{m,k} \to Q^m \) is called a poly-\((n,m)\)-operation and the structure \( Q = (Q,g) \) is called a poly-\((n,m)\)-groupoid. A poly-\((n,m)\)-groupoid \( Q = (Q,g) \) is called a poly-\((n,m)\)-semigroup if for each \( xyz \in Q^{m,k} \) (where \( x,z \in Q^*, xz \neq 1 \) and \( y \in Q^{m,k} \), \( g(xg(y)z) = g(xyz) \).

There is no essential difference between the notions of \((n,m)\)-semigroup and poly-\((n,m)\)-semigroup, due to the General Associative Law which holds in all \((n,m)\)-semigroups (see [4], [2]).

As well as the \((n,m)\)-groupoids, a poly-\((n,m)\)-groupoid \((Q,g)\) can be considered as an algebra with \( m \) \( n \)-ary operations, \( g_1, g_2, \ldots, g_m : Q^{m,k} \to Q \) where \( g(x) = g_1(x)g_2(x) \ldots g_m(x) \). Thus, the usual notions from universal algebra related to (poly-)\((n,m)\)-structures will be considered well known.

We recall the construction of a canonical form of a free poly-\((n,m)\)-groupoid \( F(B) = (F(B),f) \) with a basis \( B \neq \emptyset \) (see [4], [6]).

\[
B_0 = B, \quad B_{p+1} = B_p \cup (\mathbb{N}_m \times B_p^{m,k}), \quad F(B) = \bigcup_{p \geq 0} B_p.
\]

The poly-\((n,m)\)-operation \( f \) on \( F(B) \) is defined by \( f(x) = (1,x)(2,x) \ldots (m,x) \). Hierarchy of the elements of \( F(B) \) is a map \( \chi : F(B) \to \mathbb{N}_0 \) defined by \( \chi(u) = \min\{p | u \in B_p \} \), \( u \in F(B) \). The norm on \( F(B) \) is a map \( \| \| : F(B) \to \mathbb{N} \) defined by induction on the hierarchy:

\[
\|u\| = 1 \text{ for } u \in B_0, \text{ and } \\
\|(i,u_1^{m+sk})\| = \|u_1\| + \ldots + \|u_{m+sk}\| \text{ for } (i,u_1^{m+sk}) \in B_{p+1} \setminus B_p \text{ } (s \geq 1).
\]

Next, we recall the construction of a canonical form of a free \((n,m)\)-semigroup generated by a nonempty set \( B \) ([4]). Define a map \( \psi_0 : F(B) \to \).
$F(B)$ as follows: $\psi_0(b) = b$, $b \in B$; Let $u = (i, u_1^{m+sk}) \in F(B) \setminus B$ ($s \geq 1$), assume that $\psi_0(v) \in F(B)$ is defined for all $v \in F(B)$ with $\|v\| < \|u\|$ and $\psi_0(v) \neq v$ implies $\|\psi_0(v)\| < \|v\|$. Then $\psi_\lambda = \psi_0(u_\lambda)$ is well defined for all $\lambda \in \mathbb{N}_{m+sk}$ and thus $v = (i, v_1^{m+sk}) \in F(B)$. If there exists a $\lambda' \in \mathbb{N}_{m+sk}$ such that $\psi_{\lambda'} \neq u_{\lambda'}$ then $\|v\| < \|u\|$ and consequently define $\psi_0(u) = \psi_0(v)$; If $\psi_\lambda = u_\lambda$ for all $\lambda \in \mathbb{N}_{m+sk}$ and if $u = (i, u_1^1(1, x) \ldots (m, x) u_j^{m+sk+1})$ where $x \in F(B)^{m,k}$ and $j \in \mathbb{N}_{sk,0}$ is the smallest such index, then $\psi_0(u) = \psi_0(i, u_1^1 x u_j^{m+sk+1})$. If $u$ does not satisfy any of the conditions above, then $\psi_0(u) = u$.

Consider the set $\psi_0(F(B)) = \{u \in F(B) \mid \psi_0(u) = u\}$ and define a poly-$(n, m)$-operation $[ ] : (\psi_0(F(B)))^{m,k} \to (\psi_0(F(B)))^m$ by 

$$[u_1^{m+sk}] = u_1^m \iff (\forall i \in \mathbb{N}_m) v_i = \psi_0(i, u_1^{m+sk}).$$

**Theorem 1.1** ([4]). $(\psi_0(F(B)), [ ])$ is a free $(n, m)$-semigroup with a basis $B$. Given an $(n, m)$-semigroup $G = (G, g)$ and a map $\xi : B \to G$, the unique homomorphism $\xi : (\psi_0(F(B)), [ ]) \to (G, g)$ such that $\xi|_B = \xi$, is defined by:

$$\xi(u) = \xi(u), \text{ for } u \in B;$$

$$\xi(i, u_1^{m+sk}) = g_i(\xi(u_1) \ldots \xi(u_{m+sk})), \text{ for } (i, u_1^{m+sk}) \in \psi_0(F(B)) \setminus B \ (s \geq 1),$$

where $g_i$ is the $i$-th component operation of $G$ (induction on hierarchy).

From now on, the free $(n, m)$-semigroup with a basis $B$ will be denoted by $\psi_0(F(B))$, i.e. $\psi_0(F(B)) = (\psi_0(F(B)), [ ])$.

In this work we explore varieties of $(n, m)$-semigroups. A class of $(n, m)$-semigroups is said to be a variety of $(n, m)$-semigroups if it has an axiom system which is a set of identities. We recall the precise definition of an identity in the class of poly-$(n, m)$-groupoids ([3]).

Consider the free poly-$(n, m)$-groupoid $F(\mathbb{N})$ with a basis $\mathbb{N}$.

Let $Q = (Q, h)$ be a poly-$(n, m)$-groupoid. For each $\tau \in F(\mathbb{N})$ there exists a smallest $t \in \mathbb{N}$ such that $\tau \in F(\mathbb{N}_t)$ and the element $\tau$ defines a $t$-ary operation $\tau^Q$ on $Q$ as follows:

i) If $\tau = j \in \mathbb{N}_t$ and $a = a_1^t \in Q^t$ then $\tau^Q(a) = a_j$;

ii) If $\tau = (i, \tau_1^{m+sk}) \ (s \geq 1)$, $a = a_1^t \in Q^t$ and $\tau^Q(a) = b_\nu$, $\nu \in \mathbb{N}_{m+sk}$, then $\tau^Q(a) = h_i(b_1^{m+sk})$ i.e. $\tau^Q(a) = h_i(\tau_1^Q(a) \ldots \tau_{m+sk}(a))$ (induction on $\chi$).

Moreover, for $q \in \mathbb{N}$ and $q > t$ we have that $\tau \in F(\mathbb{N}_q)$ and thus $\tau$ defines a $q$-ary operation $\tau^Q$ on $Q$ as well. It is easy to check that $\tau^Q(a_1^q) = \tau^Q(a_1^t)$ for all $a_1^q \in Q$.

Further on, we will omit the superscript and write $\tau(a)$ instead of $\tau^Q(a)$, i.e. we write $\tau(a_1^t)$ instead of $\tau^Q(a_1^t)$.

Let $\tau, \omega \in F(\mathbb{N})$. Then $\tau, \omega \in F(\mathbb{N}_t)$ for some $t \in \mathbb{N}$.

We say that the poly-$(n, m)$-groupoid $Q$ satisfies the $(n, m)$-identity $(\tau, \omega)$, and write $Q \models (\tau, \omega)$, if $\tau(a) = \omega(a)$ for any $a = a_1^t \in Q^t$.
Proposition 1.2 ([3]). If \( G \) is an \((n, m)\)-semigroup and \( \tau \in F(\mathbb{N}) \) then \( G \models (\tau, \psi_0(\tau)) \). \( \square \)

An \((n, m)\)-identity \((\tau, \omega)\) is said to be reduced, if \( \psi_0(\tau) = \tau \) and \( \psi_0(\omega) = \omega \).

Proposition 1.3 ([3]). If \((\tau, \omega)\) is a reduced \((n, m)\)-identity and \( G \models (\tau, \omega) \) for every \((n, m)\)-semigroup \( G \), then \( \tau = \omega \). \( \square \)

Let \( \Theta \subseteq F(\mathbb{N}) \times F(\mathbb{N}) \). Then we say that \( \Theta \) is a set of \((n, m)\)-identities.

By \( \text{Var} \Theta \) we denote the class of all \((n, m)\)-semigroups \( G \) such that \( G \models \Theta \), where \( G \models \Theta \iff G \models (\tau, \omega) \) for every \((\tau, \omega) \in \Theta \).

We say that \( \text{Var} \Theta \) is a variety of \((n, m)\)-semigroups generated by \( \Theta \). A class of \((n, m)\)-semigroups \( \mathcal{V} \) is a variety iff there exists a set of \((n, m)\)-identities \( \Theta \) such that \( G \in \mathcal{V} \iff G \models \Theta \). (In this case, \( \mathcal{V} = \text{Var} \Theta \)).

2 Main Results

We give a description of the set which consists of all \((n, m)\)-identities satisfied by all \((n, m)\)-semigroups in a given variety of \((n, m)\)-semigroups \( \text{Var} \Theta \), that is the complete system of \((n, m)\)-identities for \( \text{Var} \Theta \). Since we are interested only in \((n, m)\)-identities satisfied by poly-\((n, m)\)-groupoids which are \((n, m)\)-semigroups, given an \((n, m)\)-identity \((\tau, \omega)\) we can always assume that \((\tau, \omega) \in \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N})) \) (according to Proposition 1.2, Proposition 1.3). Thus, from now on by \((n, m)\)-identity we will mean reduced \((n, m)\)-identity.

Let \( H = (H, h) \) be an \((n, m)\)-semigroup.

Let \( \tau \in \psi_0(F(\mathbb{N})) \). Then \( \tau \in \psi_0(F(\mathbb{N}_t)) \) for some \( t \in \mathbb{N} \). Let \( a = a'_1 \in H^t \), let \( \xi : \mathbb{N} \to H \) be a map such that \( a_j = \xi(j), j \in \mathbb{N}_t \), and let \( \overline{\xi} : \psi_0(F(\mathbb{N})) \to H \) be the unique homomorphism which is an extension of \( \xi \). Then, concerning the \( t \)-ary operation on \( H \) defined by \( \tau \), we will show that

\[
\tau(a) = \overline{\xi}(\tau).
\]

The proof is by induction on hierarchy: If \( \tau = l \in \mathbb{N}_t \) then \( \tau(a) = a_e = \xi(l) = \xi(\tau) = \overline{\xi}(\tau) \). If \( \tau = (i, \tau_1^{m+sk}) \in \psi_0(F(\mathbb{N}_t)) \setminus \mathbb{N}_t \) (where \( s \geq 1 \)), the hypothesis will imply that \( \tau_\nu(a) = \overline{\xi}(\tau_\nu) \) for all \( \nu \in \mathbb{N}_{m+sk} \), and thus \( (i, \tau_1^{m+sk})(a) = h_i(\overline{\xi}(\tau_1) \ldots \overline{\xi}(\tau_{m+sk})) = \overline{\xi}(i, \tau_1^{m+sk}) \). \( \square \)

Let \( (\tau, \omega) \in \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N})) \). Then \( \tau, \omega \in \psi_0(F(\mathbb{N}_t)) \) for some \( t \in \mathbb{N} \).

Lemma 2.1. If \( a = a'_1 \in H^t \) and \( \overline{\xi} : \psi_0(F(\mathbb{N})) \to H \) is the (unique) homomorphic extension of a map \( \xi : \mathbb{N} \to H \) such that \( a_j = \xi(j), j \in \mathbb{N}_t \), then \( \tau(a) = \omega(a) \iff \overline{\xi}(\tau) = \overline{\xi}(\omega) \).

Proof. Applying the equality above. \( \square \)

Corollary 2.2. Let \( H \models (\tau, \omega) \) and let \( \xi : \mathbb{N} \to H \). Then \( (\tau, \omega) \in \ker \overline{\xi} \).
Proof. Since $\tau, \omega \in \psi_0(F(N_t))$ for some $t \in \mathbb{N}$, given a map $\xi : \mathbb{N} \to H$, for $j = 1, \ldots, t$ we have that $\xi(j) = a_j$ for some sequence $a_j^t \in H^t$. Now, $H \models (\tau, \omega)$ implies $\tau(a_j^t) = \omega(a_j^t)$ and thus $\xi(\tau) = \xi(\omega)$ (by Lemma 2.1). \qed

Let $\Theta$ be a set of $(n, m)$-identities ($\Theta \subseteq \psi_0(F(N)) \times \psi_0(F(N))$).

As a consequence of Lemma 2.1, we get the following

**Proposition 2.3.** If $H \models \Theta$ and $\xi : \mathbb{N} \to H$ is a map, then $\Theta \subseteq \ker \xi$, where $\xi : \psi_0(F(N)) \to H$ is the (unique) homomorphic extension of $\xi$.

Proof. Applying Corollary 2.2: $H \models (\tau, \omega)$ for all $(\tau, \omega) \in \Theta$ and therefore $(\tau, \omega) \in \ker \xi$ for all $(\tau, \omega) \in \Theta$. \qed

**Lemma 2.4.** Let $H = (H, h)$ and $H' = (H', h')$ be $(n, m)$-semigroups and let $\varphi : H \to H'$ be a homomorphism. Then for every $\tau \in \psi_0(F(N_t)) \subseteq \psi_0(F(N))$ (where $t \in \mathbb{N}$) and $a_t^t \in H^t$

$$\varphi(\tau(a_1 \ldots a_t)) = \tau(\varphi(a_1) \ldots \varphi(a_t)).$$

Proof. Straightforward, by induction on the norm (or hierarchy). \qed

Consider the free $(n, m)$-semigroup $\psi_0(F(N)) = (\psi_0(F(N)), [ ])$ with a basis $N$ and let $\tau = \psi_0(F(N_t)) \subseteq \psi_0(F(N))$, $t \in \mathbb{N}$. Let $u = u_1^t \in (\psi_0(F(N)))^t$.

Then $\tau(u) \in \psi_0(F(N))$, because $\tau$ induces a $t$-ary operation on $\psi_0(F(N))$. Thus, every element $(\tau, \omega) \in \psi_0(F(N_t)) \subseteq \psi_0(F(N)) \times \psi_0(F(N))$ (where $t \in \mathbb{N}$), induces a set $(\tau, \omega) \in \psi_0(F(N)) \times \psi_0(F(N))$ defined by $(\tau, \omega)(\psi_0(F(N))) = \{ (\tau(u), \omega(u)) | u \in (\psi_0(F(N)))^t \}$.

Consequently, given a set of $(n, m)$-identities $\Theta \subseteq \psi_0(F(N)) \times \psi_0(F(N))$, it induces a set $\Theta(\psi_0(F(N)))$ defined by $\Theta(\psi_0(F(N))) = \bigcup_{(\tau, \omega) \in \Theta} (\tau, \omega)(\psi_0(F(N))) = \{ (\tau(u), \omega(u)) | (\tau, \omega) \in \Theta, \tau, \omega \in \psi_0(F(N_t)), u = u_1^t \in (\psi_0(F(N)))^t, t \in \mathbb{N} \}$.

It is clear that $\Theta(\psi_0(F(N))) \subseteq \psi_0(F(N)) \times \psi_0(F(N))$ i.e. $\Theta(\psi_0(F(N)))$ is a set of $(n, m)$-identities as well.

**Lemma 2.5.** $\Theta \subseteq \Theta(\psi_0(F(N)))$.

Proof. For $\tau \in \psi_0(F(N_t))$ ($t \in \mathbb{N}$) and a sequence $u = u_1^t \in (\psi_0(F(N)))^t$ such that $u_j = j$ for all $j \in \mathbb{N}_t$ we will show that $\tau(u) = \tau$. If $t = l \in N_t$ then $\tau(u) = u_l = l = \tau$. Let $\tau' = (i, \tau_{m+sk}^t) \in \psi_0(F(N_t)) \setminus \psi_0(F(N_t) \setminus (s \geq 1)$, and assume that $\tau'(u) = \tau'$ for all $\tau' \in \psi_0(F(N))$ with $\chi(\tau') < \chi(\tau)$. Since $\chi(\tau_0) < \chi(\tau)$ for all $\nu \in N_{m+sk}$, and since $\tau$ is reduced, we get: $\tau(u) = \tau_1(u) \ldots \tau_{m+sk}^t(u) = \psi_0(i, \tau_{m+sk}^t(u)) = \psi_0(i, \tau_{m+sk}^t(u)) = \tau_1(u) \ldots \tau_{m+sk}^t(u)$. Now, if $(\tau, \omega) \in \Theta$ then $(\tau, \omega) \in \psi_0(F(N_t)) \times \psi_0(F(N_t))$ for some $t \in \mathbb{N}$ and $(\tau, \omega) = (\tau(u), \omega(u))$ for a sequence $u = u_1^t \in (\psi_0(F(N)))^t$ such that $u_j = j$ for all $j \in \mathbb{N}_t$. Thus, $(\tau, \omega) \in \Theta(\psi_0(F(N)))$. \qed
Denote by $\widehat{\Theta}$ the smallest congruence on $\psi_0(F(N))$ which contains the set $\Theta(\psi_0(F(N)))$. We give its description. First define relations $\vdash$ and $\sim$ on $\psi_0(F(N))$ by induction as follows: $u \vdash^0 v \iff (u, v) \in \Theta(\psi_0(F(N)))$; assume that $\vdash^\alpha$ is defined on $\psi_0(F(N))$ and define a relation $\vdash^\alpha \cup \vdash^\alpha_1 \subseteq \psi_0(F(N)) \times \psi_0(F(N))$ by $u \vdash^{\alpha+1} v \iff u = \psi_0(i, xu'_{y})$, $v = \psi_0(i, xv'_{y})$ and $u' \vdash^\alpha v'$, $x, y \in F(N)^+ \cup \{1\} (|xy| \geq m)$, $i \in \mathbb{N}_m$. Next, $u \vdash v \iff (\exists \alpha \in \mathbb{N}_0)(u \vdash^\alpha v)$ and let $u \sim v \iff u \vdash v \lor v \vdash u$. Then $\widehat{\Theta}$ is the reflexive and transitive closure of $\sim$. i.e. $u \widehat{\Theta} v \iff u = v$ or there exists a sequence $u_0, u_1, \ldots, u_{r-1}, u_r$ from $\psi_0(F(N))$ such that $u = u_0$, $v = u_r$ and $u_j \sim u_{j+1}$ for all $j \in \mathbb{N}_r (r \geq 1)$.

**Proposition 2.6.** $\text{Var}\Theta = \text{Var}\widehat{\Theta}$.

**Proof.** (⊇). Since $\Theta \subseteq \widehat{\Theta}$, it is obvious that $\text{Var}\Theta \subseteq \text{Var}\widehat{\Theta}$.

(⊇). Let $H = (H, h) \in \text{Var}\Theta$. For $\tau \in \psi_0(F(N)) \subseteq \psi_0(F(N)) \ (t \in \mathbb{N})$ and a sequence $u^t_1 \in (\psi_0(F(N)))^t$, there exist $r \in \mathbb{N}$ such that $\tau(u^t_1) \in \psi_0(F(N))$. (This $r$ depends on the elements $u_\mu$ and clearly $u_\mu \in \psi_0(F(N))$, $\mu \in \mathbb{N}_t$.) Moreover, we will show that $\tau(u^t_1)(z) = (\tau(u^1_1(z)) \ldots u^t_1(z)), z \in H^r$. By induction on $||\ ||$ or $\chi$. For $\tau = l \in \mathbb{N}_r$, $\tau(u^1_1(z)) = u_l(z) = u_l(z) \ldots u_l(z)$. Then $\tau(u^t_1), \omega(u^t_1) \in \psi_0(F(N))$, $\tau(u^t_1), \omega(u^t_1) \in \psi_0(F(N))$ for some $r \in \mathbb{N}$ and let $z \in H^r$. Applying the equality above and the fact that $H \models \tau$, $\omega$ we get that $\tau(u^t_1)(z) = \tau(u^t_1(z)) \ldots u^t_1(z) = \omega(u^t_1(z)) \ldots u^t_1(z)$. Hence, $H \models \tau(u^t_1)(z)$.

Assume that $H \models \vdash^\alpha$ and let $(u, v) \in \vdash^\alpha_1$ (induction on $\alpha$). Then $u = \psi_0(i, x_{j-1}^{-1}u'_{xj+1}^{m+sk})$, $v = \psi_0(i, x_{j-1}^{-1}v'_{xj+1}^{m+sk})$ where $u' \vdash^\alpha v'$, $x_\eta \in F(N), j \in \mathbb{N}_{m+sk}$ and $i \in \mathbb{N}_m$. Clearly, $u, v \in \psi_0(F(N))$ for some $r \in \mathbb{N}$. Now, for any $z \in H^r$, by the hypothesis and Proposition 1.2 we get the following:

$u(z) = \psi_0(i, x_{j-1}^{-1}u'_{xj+1}^{m+sk})(z) = (i, x_{j-1}^{-1}u'_{xj+1}^{m+sk})(z) = h_i(x_1(z) \ldots x_{j-1}(z) u'(z)x_{j+1}(z) \ldots x_{m+sk}(z)) = h_i(x_1(z) \ldots x_{j-1}(z) v'(z)x_{j+1}(z) \ldots x_{m+sk}(z)) = (i, x_{j-1}^{-1}v'_{xj+1}^{m+sk})(z) = \psi_0(i, x_{j-1}^{-1}v'_{xj+1}^{m+sk})(z) = v(z)$. Thus, $H \models (u, v)$.

Therefore $H \models \vdash^\alpha_1$ and we conclude that $H \models \vdash$, which implies that $H \models \sim$.

Let $(\rho, \varpi) \in \widehat{\Theta}$, $\rho \neq \varpi$. Then $\rho = \rho_0 \sim \rho_1 \sim \ldots \sim \rho_{c-1} \sim \rho_c = \varpi$ where $\rho_1, \ldots, \rho_{c-1} (c \geq 1)$ is a sequence from $\psi_0(F(N))$ and $\rho, \rho_1, \ldots, \rho_{c-1}, \varpi \in \psi_0(F(N))$ for some $r' \in \mathbb{N}$. Since $H \models \sim$, we have that $\rho(z') = \rho_1(z') = \ldots = \rho_{c-1}(z') = \varpi(z')$, for all $z' \in H^{r'}$. This means that $H \models (\rho, \varpi)$. Consequently, $H \in \text{Var}\widehat{\Theta}$. □
Proposition 2.7. The \((n,m)\)-semigroup \(\psi_0(F(N))/\hat{\Theta}\) is a free object in \(\text{Var}\Theta\) with a basis \(N\).

Proof. For \(\tau \in \psi_0(F(N_t)) \subseteq \psi_0(F(N))\) \((t \in \mathbb{N})\) and a sequence \(u_1^\Theta, \ldots, u_t^\Theta \in \psi_0(F(N))/\hat{\Theta}\) we have \(\tau(u_1^\Theta \ldots u_t^\Theta) = (\tau(u_1^\Theta))^\Theta\). This follows by Lemma 2.4, since \(\text{nat}\Theta : \psi_0(F(N)) \to \psi_0(F(N))/\hat{\Theta}\) is epimorphism. Let \((\tau, \omega) \in \Theta\). Then \(\tau, \omega \in \psi_0(F(N_t))\) for some \(t \in \mathbb{N}\) and let \(u_1^\Theta, \ldots, u_t^\Theta \in \psi_0(F(N))/\hat{\Theta}\). Since \((\tau(u_1^\Theta), \omega(u_1^\Theta)) \in \Theta(\psi_0(F(N))) \subseteq \hat{\Theta}\), we get \(\tau(u_1^\Theta \ldots u_t^\Theta) = (\tau(u_1^\Theta))^\Theta = (\omega(u_1^\Theta))^\Theta = \omega(u_1^\Theta \ldots u_t^\Theta)\). Thus, \(\psi_0(F(N))/\hat{\Theta} \models (\tau, \omega)\) and \(\psi_0(F(N))/\Theta \models \Theta\).

Also, \(\psi_0(F(N))/\hat{\Theta} = < \text{nat}\hat{\Theta}_\mathbb{N}(N)>\) where \(\text{nat}\hat{\Theta}_\mathbb{N} : N \to \psi_0(F(N))/\hat{\Theta}\). (This is quite easy to verify, since \(\psi_0(F(N)) = < N >\). Let \(H = (H, h) \in \text{Var}\Theta\) and let \(\xi : N \to H\) be a map. Consider its unique homomorphic extension \(\hat{\xi} : \psi_0(F(N)) \to H\). Proposition 2.6 and Proposition 2.3 imply that \(\Theta \subseteq \ker\hat{\xi}\). Thus, we can define a map \(\eta : \psi_0(F(N))/\hat{\Theta} \to H\) by \(\eta(u^\Theta) = \hat{\xi}(u)\), and \(\eta\) is a homomorphism from \(\psi_0(F(N))/\hat{\Theta}\) to \(H\), since \(\hat{\xi}\) is a homomorphism from \(\psi_0(F(N))\) to \(H\). Also, \(\eta\text{nat}\hat{\Theta} = \hat{\xi}\) implies that \(\eta\text{nat}\hat{\Theta}|_N = \hat{\xi}|_N = \xi\).

Corollary 2.8. \(\hat{\Theta} = \hat{\Sigma}\).

Proof. \((\supseteq)\). Since \(\Theta \subseteq \hat{\Theta}\), it is straightforward to check that \(\hat{\Theta} \subseteq \hat{\Sigma}\). (Recall the description of \(\hat{\Theta}\)). \((\subseteq)\). Let \((\tau, \omega) \in \hat{\Theta}\). By Proposition 2.6 we have that \(\text{Var}\Theta = \text{Var}\hat{\Theta} = \text{Var}\hat{\Sigma}\). Thus, and by Proposition 2.7, \(\psi_0(F(N))/\hat{\Theta}\) is a free object in \(\text{Var}\hat{\Theta}\) and therefore \(\psi_0(F(N))/\hat{\Theta} \models (\tau, \omega)\). Clearly, \(\tau\) and \(\omega\) are in \(\psi_0(F(N_t))\) for some \(t \in \mathbb{N}\). Choose a sequence \(u_1, \ldots, u_t\) from \(\psi_0(F(N))\) such that \(u_j = j\) for all \(j \in N_t\). Since \(\tau(u_1^\Theta \ldots u_t^\Theta) = \omega(u_1^\Theta \ldots u_t^\Theta)\) we get that \((\tau(u_1^\Theta))^\Theta = (\omega(u_1^\Theta))^\Theta\). But, \(\tau(u_1^\Theta) = \tau\) and \(\omega(u_1^\Theta) = \omega\) (see the proof of Lemma 2.5), and thus we obtain that \(\tau^\Theta = \omega^\Theta\) i.e. \((\tau, \omega) \in \hat{\Theta}\).

Corollary 2.9. Let \(\Theta\) and \(\Sigma\) are sets of \((n,m)\)-identities.

\[\text{Var}\Theta = \text{Var}\Sigma \iff \hat{\Theta} = \hat{\Sigma}\]

Proof. \((\iff)\). If \(\hat{\Theta} = \hat{\Sigma}\) we immediately get that \(\text{Var}\Theta = \text{Var}\hat{\Theta} = \text{Var}\hat{\Sigma} = \text{Var}\Sigma\) (by Proposition 2.6). \((\Rightarrow)\). Let \(\text{Var}\Theta = \text{Var}\Sigma\). We will show that \(\hat{\Sigma} \subseteq \hat{\Theta}\). (The proof that \(\hat{\Theta} \subseteq \hat{\Sigma}\) is analogous). Let \((\tau, \omega) \in \hat{\Sigma}\). Consider the free \((n,m)\)-semigroup \(\psi_0(F(N))/\hat{\Theta}\) in \(\text{Var}\Theta = \text{Var}\Sigma\) and note that \(\psi_0(F(N))/\hat{\Theta} \models (\tau, \omega)\). Since \(\tau, \omega \in \psi_0(F(N_t))\) for some \(t \in \mathbb{N}\), we choose a sequence \(u_1, \ldots, u_t \in \psi_0(F(N))\) such that \(u_j = j\) for all \(j \in N_t\). Then \(\tau(u_1^\Theta) = \tau\) and \(\omega(u_1^\Theta) = \omega\). Now, for the corresponding sequence \(u_1^\Theta, \ldots, u_t^\Theta \in \psi_0(F(N))/\hat{\Theta}\), \(\tau(u_1^\Theta \ldots u_t^\Theta) = \omega(u_1^\Theta \ldots u_t^\Theta)\), which implies that \((\tau(u_1^\Theta))^\Theta = (\omega(u_1^\Theta))^\Theta\). Thus, \(\tau^\Theta = \omega^\Theta\) i.e. \((\tau, \omega) \in \hat{\Theta}\).
As a consequence of all the statements above, we obtain the following

**Theorem 2.10.** The set $\hat{\Theta}$ is a complete system of $(n,m)$-identities for $\text{Var}\Theta$ and it consists of all $(n,m)$-identities satisfied by all $(n,m)$-semigroups in $\text{Var}\Theta$. □

**References**


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