Characterization of Left Quasi-regular and Semisimple Ordered Semigroups in Terms of Fuzzy Sets

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Abstract
We characterize the left (right) quasi-regular and the semisimple ordered semigroups in terms of fuzzy sets.

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1 Introduction
We have seen in [3] that an ordered semigroup $S$ is regular if and only if, for every fuzzy subset $f$ of $S$, we have $f \preceq f \circ 1 \circ f$. It is intra-regular if and only if, for every fuzzy subset $f$ of $S$, we have $f \preceq 1 \circ f^2 \circ 1$ (where $f^2 := f \circ f$). As these characterizations play an essential role in studying the structure of ordered semigroups, it is natural to ask for an analogous characterization in case of left (or right) quasi-regular and semisimple ordered semigroups. Recall that the left (resp. right) quasi-regular ordered semigroups are semisimple. In the present paper we first characterize the ordered semigroups which are left (or right) quasi-regular in terms of their fuzzy subsets and then we characterize the more general class of semiprime ordered semigroups in terms of fuzzy sets. We prove that an ordered semigroup $S$ is left (resp. right) quasi-regular if and only if, for every fuzzy subset $f$ of $S$, we have $f \preceq 1 \circ f \circ 1 \circ f$ (resp. $f \preceq f \circ 1 \circ f \circ 1$). It is semisimple if and only if, for every fuzzy subset $f$ of $S$, we have $f \preceq 1 \circ f \circ 1 \circ f \circ 1$. Left quasi-regular semigroups (without order) using fuzzy sets have been first considered by N. Kuroki in [5].
Following L. Zadeh who introduced the fuzzy sets, if $S$ is an ordered groupoid, a fuzzy subset of $S$ (or a fuzzy set in $S$) is a mapping $f$ of $S$ into the real closed interval $[0,1]$ of real numbers. For each subset $A$ of $S$, the characteristic function $f_A$ is the fuzzy subset of $S$ defined as follows:

$$f_A : S \to [0,1] \mid a \to f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For $a \in S$, we define $A_a = \{(x,y) \in S \times S \mid a \leq xy\}$, for two fuzzy subsets $f, g$ of $S$, we define

$$(f \circ g)(a) := \begin{cases} \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

$a \in S)$, and in the set of all fuzzy subsets of $S$ we define the order relation as follows:

$f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$.

If $f, g$ are fuzzy subsets of $S$ and $f \leq g$ then, for every fuzzy subset $h$ of $S$, we have $f \circ h \leq g \circ h$ and $h \circ f \leq h \circ g$. Denote by $1$ the fuzzy subset of $S$ defined by:

$$1 : S \to [0,1] \mid x \to 1(x) := 1.$$  

This is the greatest element of the set of fuzzy subsets of $S$. In addition, if $S$ is an ordered semigroup, then the multiplication " $\circ$ " on fuzzy subsets of $S$ is associative. A fuzzy subset $f$ of an ordered semigroup $S$ is called a fuzzy left ideal of $S$, if (1) $f(xy) \geq f(y)$ for all $x, y \in S$ and (2) if $x \leq y$, then $f(x) \geq f(y)$. Equivalently, if (1) $1 \circ f \leq f$ for every fuzzy subset $f$ of $S$ and (2) if $x \leq y$, then $f(x) \geq f(y)$. It is called a fuzzy right ideal of $S$, if (1) $f(xy) \geq f(x)$ for all $x, y \in S$ and (2) if $x \leq y$, then $f(x) \geq f(y)$. Equivalently, if (1) $f \circ 1 \leq f$ for every fuzzy subset $f$ of $S$ and (2) if $x \leq y$, then $f(x) \geq f(y)$ [4]. It is called a fuzzy ideal of $S$ if it is both a fuzzy left and a fuzzy right ideal of $S$. Let $(S, \leq)$ be an ordered semigroup. For a subsemigroup $T$ of $S$ and a subset $H$ of $T$ we denote by $(H)_T$ the subset of $T$ defined by

$$(H)_T := \{t \in T \mid t \leq h \text{ for some } h \in H\}.$$ 

For $T = S$, we write $(H)$ instead of $(H)_S$.

A nonempty subset $A$ of $S$ is called a left (resp. right) ideal of $S$ if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$, that is, $A = A$. If $A$ is both a left and right ideal of $S$, then it is called an ideal of $S$. We denote by $L(A), R(A), I(A)$ the left ideal, right ideal, and the ideal of $S$, respectively, generated by $A (A \subseteq S)$. We have $L(A) = (A \cup SA),$. 


Left quasi-regular ordered semigroups

Definition 2.1 An ordered semigroup $S$ is called left quasi-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xaya$.

Equivalent Definitions:

1. $a \in (SaSa]$ for every $a \in S$.
2. $A \subseteq (SASA]$ for every $A \subseteq S$.

A subset $A$ of $S$ is called idempotent if $A = (A^2]$. A fuzzy subset $f$ of $S$ is called idempotent if $f^2 := f \circ f = f$.

Theorem 2.1 An ordered semigroup $(S, \leq)$ is left quasi-regular if and only if, for every fuzzy subset $f$ of $S$, we have

$$f \leq 1 \circ f \circ 1 \circ f.$$ 

For its proof we use the Propositions 2.1 and 2.2 below.

Lemma 2.1 [2] Let $(S, \leq)$ be an ordered groupoid. If $A$ is a left (resp. right) ideal of $(S, \leq)$, then the characteristic function $f_A$ is a fuzzy left (resp. fuzzy right) ideal of $(S, \leq)$. "Conversely", if $A$ is a nonempty set and $f_A$ a fuzzy left (resp. right) ideal of $(S, \leq)$, then $A$ is a left (resp. right) ideal of $(S, \leq)$.

Lemma 2.2 [3; Proposition 5] If $S$ is an ordered groupoid, $f, g$ fuzzy subsets of $S$ and $a \in S$, then the following are equivalent:

1. $(f \circ g)(a) \neq 0$.
2. There exists $(x, y) \in A_a$ such that $f(x) \neq 0$ and $g(y) \neq 0$.

Proposition 2.1 An ordered semigroup $S$ is left quasi-regular if and only if the left ideals of $S$ are idempotent.

For this proposition we refer to [6], with the observation that if $I$ is an ideal of $S$ and $a \in I$, then $(IaIa)_I = (IaIa]$ (cf. the proof of Theorem 2.1 in [6]). An independent proof which shows its pointless character, essential for further investigation on the subject, is the following:
\[
(\therefore) \text{Let } A \text{ be a left ideal of } S. \text{ Then we have }
\]
\[
A \subseteq (SASA) \subseteq (A^2) \subseteq (SA) \subseteq (A) = A,
\]
so \((A^2) = A\).

\[
\leftarrow. \text{Let } A \subseteq S. \text{ By hypothesis, we have }
\]
\[
A \subseteq L(A) = (L(A)L(A)) = ((A \cup SA)(A \cup SA))
\]
\[
= ((A \cup SA)(A \cup SA))
\]
\[
= (A^2 \cup SA^2 \cup ASA \cup SASA).
\]
Then
\[
A^2 \subseteq (A^2 \cup SA^2 \cup ASA \cup SASA)(A)
\]
\[
\subseteq (A^3 \cup SA^3 \cup ASA^2 \cup SASA^2)
\]
\[
\subseteq (ASA \cup SASA),
\]
\[
SA^2 \subseteq (S)(ASA \cup SASA) \subseteq (SASA \cup SASA)
\]
\[
= (SASA \cup S^2 ASA) = (SASA),
\]
\[
A \subseteq ((ASA \cup SASA) \cup (SASA) \cup ASA \cup SASA)
\]
\[
= ((ASA \cup SASA)] = (ASA \cup SASA),
\]
\[
ASA \subseteq (ASA \cup SASA)[SA] \subseteq (ASASA \cup SASASA) \subseteq (SASA),
\]
\[
A \subseteq ((SASA) \cup SASA) = ((SASA)] = (SASA),
\]
and \(S\) is left quasi-regular. \(\square\)

**Proposition 2.2** If \(S\) is an ordered semigroup, then the left ideals of \(S\) are idempotent if and only if the fuzzy left ideals of \(S\) are idempotent.

**Proof.** \(\Rightarrow\). Let \(g\) be a fuzzy left ideal of \(S\). Since \(g \preceq 1\) and \(g \preceq g\), we have \(g \circ g \preceq 1 \circ g \preceq g\). Let now \(a \in S\). By hypothesis and Proposition 2.1, \(S\) is left quasi-regular. Then there exist \(x, y \in S\) such that \(a \leq xaya\). Since \((xa, ya) \in A_a\), we have \(A_a \neq \emptyset\) and
\[
(g \circ g)(a) : = \bigvee_{(u, v) \in A_a} \min\{g(u), g(v)\}
\]
\[
\geq \min\{g(xa), g(ya)\}.
\]
Since \(g\) is a fuzzy left ideal of \(S\), we have \(g(xa) \geq g(a)\), \(g(ya) \geq g(a)\). Thus we have \((g \circ g)(a) \geq \min\{g(a), g(a)\} = g(a)\). Then \(g \preceq g \circ g\), and \(g \circ g = g\).
\[\Leftarrow\]. Let \(L\) be a left ideal of \(S\) and \(a \in L\). Since \(f_L\) is a fuzzy left ideal of \(S\), by hypothesis, we have \(f_L \circ f_L = f_L\). Then we have \((f_L \circ f_L)(a) = f_L(a) = 1\). Since \((f_L \circ f_L)(a) \neq 0\), by Lemma 2.2, there exists \((b, c) \in A_a\) such that \(f_L(b) \neq 0\) and \(f_L(c) \neq 0\). Then we have \(f_L(b) = f_L(c) = 1\), then \(b, c \in L, a \leq bc \in L^2\), and \(a \in (L^2)\). Then \(L \subseteq (L^2) \subseteq (SL) \subseteq (L) = L, \) and \((L^2) = L\). \(\square\)

**Proof of the theorem**

\[\Rightarrow\]. Let \(f\) be a fuzzy subset of \(S\) and \(a \in S\). Since \(S\) is left quasi-regular, there exist \(x, y \in S\) such that \(a \leq xaya\). Since \((xay, a) \in A_{xay}\), we have \(A_{xay} \neq \emptyset\) and

\[(1 \circ f \circ 1 \circ f)(a) : = \bigvee_{(u,v) \in A_a} \min\{(1 \circ f \circ 1)(u), f(v)\} \geq \min\{(1 \circ f \circ 1)(xay), f(a)\}\]

Since \((x, ay) \in A_{xay}, we have \(A_{xay} \neq \emptyset, and\)

\[(1 \circ f \circ 1)(xay) : = \bigvee_{(w,t) \in A_{xay}} \min\{1(w), (f \circ 1)(t)\} \geq \min\{1(x), (f \circ 1)(ay)\} = (f \circ 1)(ay)\]

Since \((a, y) \in A_{ay}, we have \(A_{ay} \neq \emptyset, and\)

\[(f \circ 1)(ay) : = \bigvee_{(k,h) \in A_{ay}} \min\{f(k), 1(h)\} \geq \min\{f(a), 1(y)\} = f(a)\]

Hence we obtain

\[(1 \circ f \circ 1 \circ f)(a) \geq \min\{(1 \circ f \circ 1)(xay), f(a)\} \geq \min\{(f \circ 1)(ay), f(a)\} \geq \min\{f(a), f(a)\} = f(a)\]

and thus \(f \leq 1 \circ f \circ 1 \circ f\).

\[\Leftarrow\]. Let \(f\) be a fuzzy left ideal of \(S\). By hypothesis, we have

\[f \leq (1 \circ f) \circ (1 \circ f) \leq f \circ f \leq 1 \circ f \leq f,\]

so \(f \circ f = f\). By Proposition 2.2, the left ideals of \(S\) are idempotent. Then, by Proposition 2.1, \(S\) is left quasi-regular. \(\square\)
Definition 2.2 An ordered semigroup $S$ is called right quasi-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq axay$.

Equivalent Definitions:
(1) $a \in (aSaS]$ for every $a \in S$.
(2) $A \subseteq (ASAS]$ for every $A \subseteq S$.

The right analogue of the above results also hold, and we have

Theorem 2.2 An ordered semigroup $S$ is right quasi-regular if and only if, for every fuzzy subset $f$ of $S$, we have

\[ f \leq f \circ 1 \circ f \circ 1. \]

3 Semisimple ordered semigroups

In the previous section we characterized the left quasi-regular, and the right quasi-regular ordered semigroups in terms of fuzzy sets. Each left (or right) quasi-regular ordered semigroup is semisimple. In this section we characterize the more general class of semisimple ordered semigroups using fuzzy sets.

Definition 3.1 An ordered semigroup $S$ is called semisimple if for every $a \in S$ there exist $x, y, z \in S$ such that $a \leq xayaz$.

Equivalent Definitions:
(1) $a \in (SaSaS]$ for every $a \in S$ and
(2) $A \subseteq (SASAS]$ for every $A \subseteq S$.

An element $a$ of an ordered semigroup $S$ is called left (resp. right) quasi-regular if $a \in (SaSa]$ (resp. $a \in (aSaS]$), it is called semisimple if $a \in (SaSaS]$, and intra-regular if $a \in (Sa^2S]$.

Proposition 3.1 An ordered semigroup $S$ has a semisimple element if and only if $S$ has an intra-regular element.

Proof. $\Longrightarrow$. Let $a$ be a semisimple element of $S$. Then there exist $x, y, z \in S$ such that $a \leq xayaz$. Then we have

\[ yaz \leq y(xayaz)z \leq (yx)(xayaz)yaz^2 = yx^2a(yaz)^2z, \]

where $yx^2a, z \in S$, so the element $yaz$ is an intra-regular element of $S$.

$\Longleftarrow$. Let $a$ be an intra-regular element of $S$. Then there exist $x, y \in S$ such that $a \leq xa^2y$. Then we have $a \leq x(xa^2y)ay = x^2a(ay)ay \in SaSaS$, so $a \in (SaSaS]$, and $a$ is a semisimple element of $S$. \qed

Proposition 3.2 Let $S$ be an ordered semigroup. If an element $a$ of $S$ is left (or right) quasi-regular, then it is semisimple.
Proof. Left a be left quasi-regular element of $S$. Then we have
\[
    a \in (SaSa) \subseteq (S(SaSa)Sa) \subseteq ((S)(SaS)(S)a] \\
    \subseteq (S^2aSaSa) \subseteq (SaSaS),
\]
and $a$ is semisimple. $\square$

Proposition 3.3 The ordered semigroup which are left (or right) quasi-
regular are semisimple.

This is an immediate consequence of Proposition 3.2. An independent proof
which shows its pointless character can be obtained by putting "$A$" ($A \subseteq S$)
instead of "$a$" in the proof of Proposition 3.2.

Proposition 3.4 (cf. [1; Lemma 2]) An ordered semigroup $S$ is semisimple
if and only if the ideals of $S$ are idempotent.

Exactly as in Proposition 2.1, in the proof of Proposition 3.4 points do not
play any essential role, but the sets. In fact:
$\longrightarrow$. Let $A$ be an ideal of $S$. By hypothesis, we have
\[
    A \subseteq ((SA)(AS)] \subseteq (A(SA]) \subseteq (A^2] \subseteq (SA] \subseteq (A] = A,
\]
so $(A^2] = A$.

$\Longleftarrow$. Let $A \subseteq S$. For the ideal $I(A)$ of $S$ generated by $A$, by hypothesis, we
have $I(A) = (I(A)^2]$. By putting "$A$" instead of "$a$" in the proof of 4) $\Rightarrow$ 5)
in [1; Lemma 2], we get $A \subseteq (SASAS)$.

$\square$

Proposition 3.5 Let $S$ be an ordered semigroup. The ideals of $S$ are idem-
potent if and only if the fuzzy ideals of $S$ are idempotent.

Proof. $\Longrightarrow$. Let $f$ be a fuzzy ideal of $S$. Since $f \preceq 1$ and $f \preceq f$, we
have $f \circ f \preceq 1 \circ f \preceq f$. Let now $a \in S$. By hypothesis and Proposition
3.4, $S$ is semisimple. So there exist $x, y, z \in S$ such that $a \preceq xayaz$. Since
$(xay, az) \in A_a$, we have $A_a \neq \emptyset$ and
\[
    (f \circ f)(a) : = \bigvee_{(u,v) \in A_a} \min\{f(u), f(v)\} \geq \min\{f(xay), f(az)\}.
\]
Since $f$ is a fuzzy ideal of $S$, we have $f(xay) \geq f(ay) \geq f(a)$, $f(az) \geq f(a)$.
So $(f \circ f)(a) \geq \min\{f(a), f(a)\} = f(a)$. Hence we have $f \leq f \circ f$, and so
$f \circ f = f$.

$\Longleftarrow$. Let $I$ be an ideal of $S$ and $a \in I$. Since $f_I$ is a fuzzy ideal of $S$, by
hypothesis, we have $f_I \circ f_I = f_I$, then $(f_I \circ f_I)(a) = f_I(a) = 1$. By Lemma
2.2, there exists $(b, c) \in A_a$ such that $f_I(b) = f_I(c) = 1$, then $b, c \in I$. Hence
we have $a \leq bc \in I^2$, and $a \in (I^2]$. Besides, $(I^2] \subseteq (IS] \subseteq (I] = I$. So $I$ is
idempotent. $\square$
Theorem 3.1  An ordered semigroup \((S,\cdot,\leq)\) is semisimple if and only if, for every fuzzy subset \(f\) of \(S\), we have

\[ f \preceq 1 \circ f \circ 1 \circ f \circ 1. \]

Proof. \(\implies\). Let \(f\) be a fuzzy subset of \(S\) and \(a \in S\). Since \(S\) is semisimple, there exist \(x, y, z \in S\) such that \(a \leq xayaz\). Since \(xayaz \in A_a\), we have \(A_a \neq \emptyset\) and

\[
(1 \circ f \circ 1 \circ f \circ 1)(a) := \bigvee_{(u,v) \in A_a} \min\{(1 \circ f \circ 1)(u), (f \circ 1)(v)\}
\]

\[
\geq \min\{(1 \circ f \circ 1)(xay), (f \circ 1)(az)\}.
\]

Since \((xa, y) \in A_{xay}\), we have \(A_{xay} \neq \emptyset\), and

\[
(1 \circ f \circ 1)(xay) := \bigvee_{(w,t) \in A_{xay}} \min\{(1 \circ f)(w), 1(t)\}
\]

\[
\geq \min\{(1 \circ f)(xa), 1(y)\}
\]

\[
= (1 \circ f)(xa).
\]

Since \((x, a) \in A_{xa}\), we have \(A_{xa} \neq \emptyset\), and

\[
(1 \circ f)(xa) := \bigvee_{(k,h) \in A_{xa}} \min\{1(k), f(h)\}
\]

\[
\geq \min\{1(x), f(a)\}
\]

\[
= f(a).
\]

Since \((a, z) \in A_{az}\), we have \(A_{az} \neq \emptyset\), and

\[
(f \circ 1)(az) := \bigvee_{(s,p) \in A_{az}} \min\{f(s), 1(p)\}
\]

\[
\geq \min\{f(a), 1(z)\}
\]

\[
= f(a).
\]

Hence we obtain

\[
(1 \circ f \circ 1 \circ f \circ 1)(a) \geq \min\{(1 \circ f \circ 1)(xay), (f \circ 1)(az)\}
\]

\[
\geq \min\{(1 \circ f)(xa), f(a)\}
\]

\[
\geq \min\{f(a), f(a)\}
\]

\[
= f(a),
\]

and thus \(f \preceq 1 \circ f \circ 1 \circ f \circ 1\).
\text{Left quasi-regular and semisimple fuzzy ordered semigroups}

\[\iff\quad \text{Let } f \text{ be a fuzzy ideal of } S. \text{ By hypothesis, we have}
\begin{align*}
f \preceq (1 \circ f \circ 1) \circ f \circ 1 & \preceq f \circ f \preceq 1 \circ f \preceq f,
\end{align*}
\]
so \(f \circ f = f\). By Proposition 3.5, the ideals of \(S\) are idempotent. Then, by Proposition 3.4, \(S\) is semisimple.

\textbf{Remark 3.1} The "\(\iff\)" part of Theorem 2.1 can be also proved as follows:
Let \(a \in S\). Consider the characteristic function \(f_{(a)}\) denoted by \(f_a\). This is the mapping of \(S\) into \([0,1]\) defined by \(f_a(x) = 1\) if \(x = a\), \(f_a(x) = 0\) if \(x \neq a\). By hypothesis, we have \((1 \circ f_a \circ 1 \circ f_a)(a) = 1\). By Lemma 2.2, there exists \((x, y) \in A_a\) such that \((1 \circ f_a)(x) \neq 0\) and \((1 \circ f_a)(y) \neq 0\). Again by Lemma 2.2, there exist \((z, t) \in A_x\) and \((h, k) \in A_y\) such that \(f_a(t) \neq 0\) and \(f_a(k) \neq 0\). Then we have \(t = k = a\) and
\[a \leq xy \leq zthk \in SaSa,
\]
so \(a \in (SaSa)\). The "\(\iff\)" part of Theorem 3.1 can be also proved in the same way.

\textbf{References}


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