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# Absolutely Pure Semimodules

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**Abstract.** In the present paper, a semimodule $M$ over a semiring $R$ is called absolutely pure if it is pure in every semimodule containing it as a subsemimodule. Some well-known properties of absolutely pure modules are extended to semimodules. We introduce and study two particular subclasses of absolutely pure semimodules, namely strongly absolutely pure (SAP) and finitely injective ($f$ -injective) semimodules. When the semiring $R$ is additively idempotent ,the SAP $R-$semimodules are exactly the $f$ -injective semimodules.A characterization of Fieldhouse regular semimodules is obtained.

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## Introduction

The notion of purity in module theory was defined in terms of tensor product.In [3 , Thm.2.4], P.M. Cohn proved that a submodule $M$ of a left module $N$ (over a ring $R$) is pure if every finite system of linear equations $H\overline{x} = \overline{m}$ with coefficients in $R$ and parameters from $M$ is solvable in $M$ if it is solvable in $N$. Generally, if $A$ and $B$ are $L$-structures, where $L$ is a first-order language, a homomorphism $f : A \to B$ is said to be pure if for any positive primitive formula $\phi$ and any tuple $\bar{a}$ from $A$, the validity of $\phi(f(\bar{a}))$ in $B$ entails that of $\phi(\bar{a})$ in $A$ [12] .This notion of purity was applied to semimodules over an arbitrary semiring and the existence of pure-injective semimodules was proved

[ 14 ,Thm.3].In fact, one can easily show that a subsemimodule $M$ of a left semimodule $N$ (over a semiring $R$) is pure if every finite system of linear equations $H\overline{x} + \overline{m} = K\overline{x} + \overline{m}'$ with coefficents in $R$ and parameters from $M$ and with a solution in $N$ already has a solution in $M$. In the present paper, a semimodule $M$ is called absolutely pure if it is pure in every semimodule containing it as a subsemimodule.Some well-known properties of absolutely pure modules are extended to semimodules .For example , every semimodule has a maximal absolutely pure subsemimodule.We introduce and study two particular subclasses of absolutely pure semimodules, namely strongly absolutely pure (SAP) and finitely injective ($f$ -injective) semimodules.A semimodule $M$ is $f$- injective if and only if $M = \varinjlim X_i$ ,where the $X_i$ are injective semimodules and the morphisms of the directed system $\{X_i\}$are injective.When the semiring $R$ is additively idempotent ,the SAP $R-$semimodules are exactly the $f$ -injective semimodules.A characterization of Fieldhouse regular semimodules is obtained.

# 1. Purity in Model theory

In this section, structure means structure for a given finitary similarity type and $L$ is the first-order language of that similarity type. For the basic concepts of model theory we refer to [8]. Let us recall that if $A$ and $B$ are $L$-structures, a homomorphism $f : A \to B$ is said to be pure if for any positive primitive (p.p. for short) formula and any tuple $\bar{a}$ from $A$, the validity of $\phi(f(\bar{a}))$ in $B$ entails that of $\phi(\bar{a})$ in $A$ [12]. Note that every pure map is an isomorphic embedding, therefore these maps are also called pure embeddings. A substructure $A$ of a structure $B$ is called pure if the inclusion of $A$ in $B$ is pure.Elementary embeddings, that is, embeddings that preserve all first-order formulas, are clearly pure.

**Lemma 1.1** [10]

Let $A$ and $B$ be two $L$-structures. The following conditions are equivalent for any embedding $f : A \to B$.

(i) $f$ is pure

(ii) There is an elementary embedding $g : A \to C$ that factors through $f$ (i.e. there is a homomorphism $h : B \to C$ such that $g = hf$ ).

**Remark 1.1**

In (ii) above $g$ can be taken to be the diagonal embedding of $A$ into an appropriate ultrapower of $A$ [4, Th.6.4].

# 2. Purity in Semimodule Theory.

Let $R = (R; +, ., 0, 1)$ be a semiring, i.e. $(R; +, 0)$ is a commutative monoid with identity 0, $(R; ., 1)$ is a monoid with identity 1, for all $a, b, c \in R$, $a.(b+c) = a.b + a.c$ and $(b + c).a = b.a + c.a$, $0.r = 0 = r.0$ for all $r \in R$, and $0 \neq 1$. Let $R$ be a semiring. A left $R$-semimodule is a commutative monoid $(M; +, 0)$ for which we have a function $R \times M \to M$ , denoted by $(r, m) \longmapsto r\ m$ and called scaler multiplication, which satisfies the following conditions for all elements $r$ and $s$ of $R$ and all elements $m$ and $n$ of $M$: (1) $(rs)m = r(sm)$; (2) $r(m + n) = rm + rn$; (3) $(r + s)m = rm + sm$; (4) $1\ m = m$; (5) $r\ 0 = 0 = 0\ m$. An element $m$ of $M$ is cancellable if $m + m' = m + m''$ implies that $m' = m''$. The semimodule $M$ is cancellative if every element of $M$ is cancellable. If every element $m \in M$ has an additive inverse $m' \in M$, the semimodule $M$ is called an $R$-module.For the basic concepts of semirings and semimodules we refer to [7].Throughout this paper, semimodule means left semimodule over $R$. By ideal we mean a left ideal of $R$. By homomorphism, we mean an $R$-homomorphism.We consider the one-sorted first-order language $L_R$ of left semimodules over a fixed arbitrary semiring $R$ .Recall that a p.p. formula $\phi(\overline{x})$ is a formula of the form

$$\phi(\overline{x}) = \phi(x_1, ..., x_n) = \exists y_1 ... y_m (\bigwedge_{i=1}^{t} \Psi_i(\overline{x}, \overline{y})),$$

where $\overline{y} = (y_1, ..., y_m)$ and $\Psi_i(\overline{x}, \overline{y})$ are atomic formulas, $i = 1, ..., t$.

One can easily show that every atomic formula $\Psi(x_1, ..., x_n)$ of $L_R$ is equivalent, modulo the theory of semimodules, to an equation

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i x_i,$$

where $a_i, b_i$ are semiring elements. So, the p.p. formula $\phi(\overline{x})$ can be read as saying there are elements $\overline{y}$ such that $A\overline{x} + B\overline{y} = C\overline{x} + D\overline{y}$, where $A, C$ are matrices (over $R$) of size $t \times n$, $B, D$ are matrices of size $t \times m$, and $\overline{x}, \overline{y}$ are read as column matrices of semimodule elements. Let $M, N$ be two $R$-semimodules and $f : M \to N$ be a pure embedding.This means that $f$ is an injective $R$-homomorphism, and for any p.p. formula $\phi(\overline{x})$ and each tuple $\overline{m}$ from $M$, if there is a column matrix $\overline{b}$ (of elements of $N$) such that

$Af(\overline{m}) + B\overline{b} = Cf(\overline{m}) + D\overline{b}$

then there is a column matrix $\overline{c}$ (of elements of $M$) such that

$A\ \overline{m} + B\ \overline{c} = C\ \overline{m} + D\ \overline{c}$

where $f(\overline{m}) = (f(m_1), ..., f(m_k))$.

The following results follow from the definition of purity.

**Lemma 2.1**

Let $\phi(\overline{x})$ be a p.p. formula in $L_R$ and $M$ be an $R$-semimodule. Then

(i) $M \vDash \phi(\overline{0})$.

(ii) If $M \vDash \phi(\overline{a})$ and $M \vDash \phi(\overline{b})$, then $M \vDash \phi(\overline{a}+\overline{b})$.

(iiii) If $r \in C(R)$, the center of $R$, and $M \vDash \phi(\overline{a})$, then $M \vDash \phi(r\overline{a})$, where $r\overline{a} = r(a_1, ..., a_n) = (ra_1, ..., ra_n)$.

(iv) $\phi(M) = \{\overline{a} \in M^n : M \vDash \phi(\overline{a})\}$ is a submonoid of $(\mathrm{M}^n, +)$.

(v) If $R$ is commutative, $\phi(M)$ is a subsemimodule of $(M^n, +)$.

(vi) If $M$ is an $R$-module, $\phi(M)$ is a subgroup of $(M^n, +)$.

**Lemma 2.2**

Suppose that $E$,$F$ and $G$ are semimodules over a semiring $R$ such that $E \subset F \subset G$.

(i) If $E$ is pure in $F$ and $F$ is pure in $G$ then$E$ is pure in $G$.

(ii) If $E$ is pure in$G$ then$E$ is pure in $F$.

# 3. Absolutely Pure Semimodules

Let $R$ be a semiring and $M$ be an $R$-semimodule . If $W$ is the subsemimodule of $M \times M$ defined by $W = \{(m, m)|m \in M\}$ then $W$ induces an $R$-congruence relation $\equiv_W$ on $M \times M$, called the Bourne relation, defined by setting $(m, n) \equiv_W (m', n')$ if and only if there exist elements $w$ and $w'$ of $W$ such that $(m, n) + w = (m', n') + w'$.

If $(m, n) \in M \times M$ then we write $(m, n)/W$ instead of $(m, n)/\equiv_W$.The factor semimodule $M \times M/\equiv_W$ is denoted by $M \times M/W$. Since for all $(m, n) \in M \times M$ we have $(m, n)/W + (n, m)/W = (0, 0)/W$, then $M \times M/W$ is an $R$-module.This left $R$-module, denoted by $M^{\Delta}$, is called the $R$-module of differences of $M$.

**Lemma 3.1** [7]

(i) A subsemimodule of a cancellative semimodule is cancellative .

(ii) Given a semimodule $M$, there is a homomorphism $\xi_M$ of $M$ into $M^{\Delta}$, defined by $\xi_M(m) = (m, 0)/W$ .

(iii) $\xi_M$ is an embedding if and only if $M$ is cancellative .

**Definition**

Let $\Gamma$ be a class of $R$-semimodules. A semimodule $M \in \Gamma$ is said to be absolutely pure ( AP ),in $\Gamma$, if every embedding of $M$ into a semimodule from $\Gamma$, is pure .When $\Gamma$ is the class of all $R$-semimodules, $M$ is said to be AP.

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**Lemma 3.2**

(i) A pure subsemimodule $M$ of a module $N$ is a module.

(ii) $\xi_M$ is a pure embedding if and only if $M$ is a module over the semiring $R$.

(iii) Any cancellative AP semimodule is a module .

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**Proof.**

(i) : Consider the p.p.formula $\phi(x_1) = \exists x_2(x_1 + x_2 = 0)$ ,and $m \in M$.Since $N$ is a module ,then $N \vDash \phi(m)$,and so $M \vDash \phi(m)$.This means that $m$ has an additive inverse in $M$.

(ii) : The "if" part follows from [7, Prop. 14.1] and Lemma 1. The "only if " part follows from (i).

(iii) : It follows from (i) and (ii) .□

**Definition**

Let $M$ be semimodule over a semiring $R$. For two elements $a \in R$ and $m \in M$ , the pair $(a, m)$ is said to be compatible if the equation $ax = m$, has a solution in an extension of $M$.

**Lemma 3.3**

Let $M$ be a cancellative semimodule over a cancellative semiring $R$. The following statements are equivalent for two elements $a \in R$ and $m \in M$:

(i) The pair $(a, m)$ is compatible

(ii) There is a homomorphism $g : Ra \to M$ such that $g(a) = m$.

**Proof.**

(i) $\Rightarrow$ (ii) There is $x_o$ in an extension of $M$ such that $ax_o = m$. So, if $x = ra = ta \in Ra$, then $rm = rax_o = tax_o = tm$. Thus we may define $g : Ra \to M$ by $g(ra) = rm$. Of course, $g$ is a homomorphism and $g(a) = g(1a) = m$ .

(ii) $\Rightarrow$ (i): Let $I = Ra, g : I \to M$ and $g(a) = m$. We prove that there is an extension $V$ of $M$ and there is $x_o \in V$ such that $ax_o = m$ in $V$. Let $i$ be the inclusion mapping of $I$ into $R$ and consider the homomorphism $\alpha = \xi_M g : I \to M \to M^\triangle$ and $\beta = \xi_R i : I \to R \to R^\triangle$. We define $f : I \to M^\triangle \times R^\triangle$ by $f(t) = (\alpha(t), -\beta(t))$ Note that $N = f(I)$ is a subsemimodule of $M^\triangle \times R^\triangle$ and $N$ induces an $R$-congruence relation "Bourne relation" on $M^\triangle \times R^\triangle$. Let $V$ be the factor semimodule $M^\triangle \times R^\triangle/N$. Let $\lambda : M \to M \times R \to M^\triangle \times R^\triangle \to V$; $u \mapsto (u, 0) \mapsto (\xi_M(u), 0) \mapsto (\xi_M(u), o)/N$ and $\mu : R \to M \times R \to M^\triangle \times R^\triangle \to V$;$r \mapsto (0, r) \mapsto (0, \xi_R(r)) \mapsto (0, \xi_R(r))/N$.

We show that $\lambda$ is injective: Suppose $\lambda(u) = \lambda(v)$. Then $(\xi_M(u), 0)/N = (\xi_M(v), o)/N$ and so there are $n_1, n_2 \in N$ such that $(\xi_M(u), 0)+n_1 = (\xi_M(v), o)+ n_2$.If $n_1 = f(t_1), n_2 = f(t_2)$, then we get $\xi_M(u) + \alpha(t_1) = \xi_M(v) + \alpha(t_2)$ and $-\beta(t_1) = -\beta(t_2)$.

Since $\beta$ is injective and $M$ is cacellative, then $\xi_M(u) = \xi_M(v)$, and so $u = v$. This means that $V$ is an extension of $M$. We prove that $\mu(1)$ is a solution of the equation $ax = m$ in $V$, i.e. $a\mu(1) = \lambda(m)$.

Since $(0, \xi_R(a))+f(a) = (0, \xi_R(a))+(\alpha(a), -\beta(a)) = (\alpha(a), 0) = (\xi_M(g(a)), 0) = (\xi_M(m), 0) = (\xi_M(m), 0) + f(0)$, then $(0, \xi_R(a)/N = (\xi_M(m), o)/N$.Thus, $\mu(a) = \lambda(m)$, and so $a\mu(1) = \lambda(m)$.□

Definition [1]

An $R$-semimodule $M$ is called $P$-injective if for any principal ideal $I$ of $R$ and each homomorphism $g : I \to M$,there exists a homomorphism $f : R \to$

$M$,which extends $g$.

**Corollary 3.4**

Every cancellative AP semimodule $M$ over a cancellative semiring $R$ is $P$-injective.

**Proof**.

Let $I = Ra$ , $a \in R$ ,and $g : I \to M$ be a homomorphism . By the preceding Lemma the equation $ax = g(a)$ has a solution in an extension $N$ of $M$ , say , $\lambda : M \to N$ .Since $M$ is AR , $\lambda$ is pure and so the equation $ax = g(a)$ has a solution $m_o \in M$ .We define a homomorphism $h : R \to M$ , by $h(r) = r\ m_o$.For any $x = ta \in I$ , $g(x) = tg(a) = tam_o = h(x)$.Hence,$h$ extends $g$ and so $M$ is $P$-injective .□

**Corollary 3.5**

Let $M$ be a cancellative semimodule over a cancellative semiring $R$ .The following statements are equivalent:

(i) $M$ is $P$-injective.

(ii)For any compatible pair $(a, m) \in R \times M$ , the equation $\circledast$ "$ax = m$" has a solution in $M$ .

**Proof**.

(i) $\Longrightarrow$ (ii) : Suppose $(a, m)$ is compatible .By Lemma 3.3 , there is a homomorphism $g : Ra \to M$ such that $g(a) = m$. Since $M$ is $P$-injective , there is a homomorphism $f : R \to M$, extends $g$. Observe that $ah(1) = h(a.1) = h(a) = g(a) = m$ ,and so $h(1) \in M$ is a solution of $\circledast$.

(ii) $\Longrightarrow$ (i) :Let $I = Ra$ , $a \in R$ , $g : I \to M$ be any homomorphism and $m_o = g(a)$.By Lemma 3.3, the equation $ax = m_o$ has a solution in an extension of $M$.Under the hypothesis (ii), this equation has a solution $u_o \in M$.We define a homomorphism $h : R \to M$ ,by $h(r) = ru_o$.One easily sees that $h$ extends $g$.□

**Proposition 3.6**

Every pure subsemimodule$M$ of a $P$-injective semimodule $N$ is $P$-injective .

**Proof**.

Let $I = Ra$ , $a \in R$ ,and $g : I \to M$ be a homomorphism.Since $i : M \subset N$ ,and $N$ is $P$-injective , there exists a homomorphism $f : R \to N$,which extends $g$ .Hence , $i\ g(a) = g(a) = f(a) = m_o \in M$.Let $f\ (1) = n_o \in N$ and consider the equation $\circledast$ : $ax = m_o$.Since $an_o = af(1) = f(a.1) = f(a) = m_o$ , then $\circledast$ has a solution in $N$.Observe that $i : M \subset N$ is pure

and so $\circledast$ has a solution $u_o \in M$ ( i.e. $au_o = m_o$ ).Now,define a homomorphism $h : R \to M$ ,by $h(r) = ru_o$.Since $h(ra) = rau_o = rm_o = rg(a) = g(ra)$ ,then $h$ extends $g$ and so $M$ is $P$-injective.□

In [14 , Thm.5],it was proved that the first order theory $T$ of cancellative semimodules over an arbitrary semiring $R$ has the amalgamation property. As an application we have :

**Proposition 3.7**.

Every pure subsemimodule$M$ of a cancellative AP semimodule $N$ is AP in the class of cancllative semimoules.

**Proof**.

Consider the following diagram , where $f_1, f_2$ are the identical inclusions ,$f_1$ is pure and $H$ is a cancellative semimodule

$$\begin{tikzcd} & N \\ M \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & H \end{tikzcd} .$$

Let $(\circledast)$ "$A\overline{x} + \overline{m} = B\overline{x} + \overline{m}'$ " be a finite system of linear equations with coefficents in $R$ and parameters from $M$ and with a solution in $H$ .By[14 , Thm.5], there is a cancellative semimodule $F$ and embeddings $g_i$ , $i = 1, 2$,such that

the following diagram is commutative.

$$\begin{tikzcd} & N \arrow[dr, "g_1"] & \\ M \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & F \\ & H \arrow[ur, "g_2"'] & \end{tikzcd}$$

It follows that $(\circledast)$has a solution in $F$. Since $f_1$and $g_1$are pure , $(\circledast)$ has a solution in $M$ and so $M$ is AP in the class of cancellative semimodules.□

**Theorem 3.8.**

(i) If $X_0 \subset X_1 \subset \ ... \subset X_\beta \subset ..., \beta \prec \alpha$ is a chain of AP semimodules , where $\alpha$ is an ordinal , then the union of the chain is AP.

(ii) Every semimodule has a maximal AP subsemimodule .

**Proof.**

(i) Let $M = \cup X_\beta$ and suppose that $M \subset N$.Let $(\circledast)$ : $A\overline{x} + \overline{m} = B\overline{x} + \overline{m}'$ be a finite system of linear equations with coefficents in $R$ and parameters from $M$ and with a solution in $N$.There is an ordinal $\gamma$ such that the elements of the column matrices $\overline{m}$ and $\overline{m}'$ are in $X_\gamma \subset M \subset N$ .Therefore one can consider $\circledast$ as a finite system of linear equations with coefficents in $R$ and parameters from $X_\gamma$ and with a solution in $N$ .Since $X_\gamma \subset N$ , $\circledast$ has a solution in $X_\gamma \subset M$.

(ii) Given a semimodule $E$ , consider the set $\Omega$ of all subsemimodules of $E$ that are AP semimodules.Observe that $\Omega$ is not empty , for the zero semimodule belongs to $\Omega$.Partially order $\Omega$ by inclusion.If $\mathcal{F}$ is a chain in $\Omega$ then $\cup\mathcal{F}$ is AP by (i).Now the result follows by applying Zorn's Lemma. $\square$

# 4. SAP Semimodules

In [2] , Azumaya introduced the notion of locally split homomorphisms to study regular modules. Locally split submodules were introduced by Ramamurthi and Rangaswamy [9], by the name of strongly pure submodules, to study strongly absolutely pure (SAP) and finitely injective ( $f$ -injective in the sense of [13] ) modules. In this section we extend these notions for semimodules over an arbitrary semiring.Let $R$ be a semiring and $M$ be an $R$-semimodule

$M$ is said to be finitely injective ( $f$ -injective for short ) if givn any injective homomorphism $F \to Y$,where $F$ is a finitely generated semimodule, any homomorphism $F \to M$ can be extended to a homomorphism $Y \to M$.Note that every injective semimodule is $f$ -injective. We call a subsemimodule $M$ of a semimodule $N$ strongly pure if to any finite set $\{m_1, ..., m_k\}$ of elements of $M$ there exists a homomorphism $\alpha : N \to M$ such that $\alpha(m_i) = m_i$ ,$i = 1, ..., k$.Finally, a semimodule $M$ is said to be strongly absolutely pure (SAP for short ) if $M$ is stongly pure in every $R$-semimodule containg it as a subsemimodule .

**Proposition 4.1**

Suppose that $E, F$ and $G$ are semimodules over a semiring $R$ such that $E \subset F \subset G$ .

(i) If $E$ is strongly pure in $F$ and $F$ is strongly pure in $G$ then $E$ is strongly pure in $G$ .

(ii) If $E$ is strongly pure in $G$ then $E$ is strongly pure in $F$ .

(iii) If $E$ is strongly pure in $F$ then $E$ is pure in $F$.

(iv) If $E$ is pure in $F$ ,where $F$ is a projective semimodule,then $E$ is strongly pure in $F$.

(v) If $E$ is pure in $F$ ,where $E$ is finitely generated and $F$ is projective,then $E$ is projective.

**Proof.**

(i) and (ii) are obvious. (iii) : Let $(\circledast)$ "$A\overline{x} + \overline{u} = B\overline{x} + \overline{v}$ " be a finite system of linear equations with coefficents in $R$ and parameters from $E$ and with a solution $\overline{c}$ in $F$.Let $\{u_1, ..., u_t, v_1, ..., v_t\} \subset E$ be the elements of

the column matrices $\overline{u}$ , $\overline{v}$ .Under the hypothesis,there exists a homomorphism $\alpha : F \to E$ , such that

$\alpha(u_i) = u_i$ , $\alpha(v_i) = v_i$ . It follows that$A\ \alpha(\overline{c}) + \overline{u} = B\ \alpha(\overline{c}) + \overline{v}$ ,and so $\alpha(\overline{c})$ is solution of $\circledast$ in $E$ .(iv): If $f : E \subset F$ is pure, then by Lemma 1,there is an ultrafilter $u$ over an infinite set $I$ and a homomrphism $h : F \to E^I/u$ such that $h\ f = \delta$ ,where $\delta : E \to E^I/u$ is the diagonal embedding of $E$ into an ultrapower of $E$.Let $\phi : E^I \to E^I/u$ be the canonical homomorpism.Since $F$ is projective ,there exists a homomorphism $g : F \to E^I$ such that $\phi g = h$.Let $\{e_1, ..., e_n\}$ be a finite

set of elements of $E$. For each $e_k$ , $1 \leq k \leq n$ , $= \delta(e_k) = h(e_k) = \phi g\ (e_k) = g\ (e_k)/u$.Hence there exists a set $\Omega_k \in u$ suchs that $(g\ (e_k))(i) = e_k$ for all $i \in \Omega_k$.Let $\Omega = \cap\ \Omega_k\ \in u$ , and define a homomorphism $\alpha = p_i\ g : F \to E^I \to E$ , where $p_i$ is the canonicl projection , $i \in \Omega$.It follows that $\alpha(e_k) = e_k$ , $1 \leq k \leq n$ ,and so $E$ is strongly pure in $F$ .

(v) : Let $\{e_1, ..., e_n\}$ be a finite set of generators of $E$.Since $E \subset F$ is strongly pure , there exists a homomorphism $\alpha : F \to E$ such that $\alpha(e_i) = e_i$ , $i = 1, ..., n$.It follows that $E$ is a retract of $F$ and so $E$ is projective.□

**Corollary 4.2**

Every SAP semimodule is AP

**Proposition 4.3**

Let $R$ be a semiring ,$N$ be an $R$- module and $M$ be a subsemimodule of $N$ .The following statements are equivalent :

(i) $M \subset N$ is strongly pure.

(ii) $M \subset N$ is pure and for any element $x_o \in M$ there exists a homomorphism

$\alpha : N \to M$ such that $\alpha(x_o) = x_o$.

**Proof.**

(i)$\Longrightarrow$ (ii) follows from Proposition 4.1.

(ii)$\Longrightarrow$ (i) :By Lemma 3.2(i) $M$ is a module . Let $\{x_1, ..., x_n\}$ be any finite set of elements of $M$ .We prove by induction, suppose $n \geq 2$ and our statement is true for $n - 1$.This means that there is a homomorphism $\alpha : N \to M$ such that $\alpha(x_k) = x_k$ for $k = 1, 2, ..., n - 1$.Since $(x_n - \alpha(x_n)\ ) \in M$ , there is a $\beta : N \to M$ such that $\beta(x_n - \alpha(x_n)) = x_n - \alpha(x_n)$ .Let $\delta = \alpha + \beta - \beta i\ \alpha : N \to M$ ,where $i : M \subset N$ is the inclusion map.Then for any $k, k = 1, 2, ..., n-1, \delta(x_k) = \alpha(x_k) + \beta(x_k) - \beta i\ \alpha(x_k) = x_k + \beta(x_k) - \beta(x_k) = x_k$.And $\delta(x_k) = \alpha(x_n) + \beta(x_n) - \beta i\ \alpha(x_n) = \alpha(x_n) + \beta(x_n - \alpha(x_n)) = \alpha(x_n) + x_n - \alpha(x_n) = x_n$.Thus $M$ is strongly pure in $N$ .□

The following result connects finite injectivity with strong purity.

**Proposition 4.4**

Every $f$ -injective semimodule is SAP.

**Proof.**

Let $M \subset N$ , where $M$ is an $f$ -injective semimodule.For any finite set $T = \{m_1, ..., m_k\}$ of elements of $M$ , let $F$ be the semimodule generated by $T$.Consider the inclusion maps $f : F \subset N$ and $j : F \to M$.There exists a homomorphism $\alpha : N \to M$ ,such that $\alpha f = j$ .Observe that $\alpha(x) = x$,for all $x \in F$ , and so $M$ is SAP.□

**Proposition** 4.5

Every $f$ -injective semimodule$M$ contains an injective hull of each of its finitely generated subsemimodule.

**Proof.**

Let $F \subset M$ be a finitely generated subsemimodule of $M$ with an injective hull $E$. Consider the inclusion maps $i : F \subset M$ and $j : F \subset E$.Since $M$ is $f$ -injective semimodule,there exists a homomorphism $h : E \to M$ such that $hj = i$. Observe that $h$ is injective since $i$ is injective and $j$ is essential.□

**Corollary 4.6**

Every finitely generated $f$ -injective semimodule $M$ is injective.

**Theorem 4.7**

(i) If $X_0 \subset X_1 \subset \ ... \subset X_\beta \subset ..., \beta \prec \alpha$ is a chain of $f$ -injective semimodules , where $\alpha$ is an ordinal , then the union of the chain is $f$ -injective.

(ii) Every semimodule has a maximal $f$ -injective subsemimodule

(iii) A semimodule $M$ is $f$ -injective if and only if $M = \lim\limits_{\rightarrow} X_i$ ,where the $X_i$ are injective semimodules and the morphisms of the directed system $\{X_i$ }are injective.

**Proof.**

We prove only (iii). Note that $M = \lim\limits_{\rightarrow}\{ F_i , \alpha_{ij} \}, i, j \in I$ ,where $\{ F_i \}$ is the family of all finitely generated subsemimodules of $M$ and the morphisms $\{ \alpha_{ij} \}$are the inclusion maps.Since $M$ is $f$ -injective,it contains an injective hull $\widehat{F_i}$ of each $F_i$ . One can easily check that $M = \cup \widehat{F_i}$ and the $\alpha_{ij} : F_i \to F_j$ induce injective homomorphisms $\widehat{\alpha_{ij}} : \widehat{F_i} \to \widehat{F_j}$ ,such that $\{\widehat{F_i}, \alpha_{ij}\}$ is a directed system and $M = \lim\limits_{\rightarrow}\{ \widehat{F_i} \}$.□

**Remark. 4.1**

If $R$ is a ring ,then it is well-known that every $R$ -module is contained in an injective $R$- module. However ,for arbitraty semirings $R$ this not the case ; e.g. there are no nonzere injective $\mathbb{N}$ -semimodules. In [11],H.Wang proved that every $R$ -semimodule has an injective hull ,in the case that $R$ is additvely idempotent (i.e. a semiring satisfying $r + r = r$ for all $r \in R$).For these semirings we prove the convrse of Proposition 4.4.

**Theorem 4.8**

Let $R$ be an additvely idempotent semiring.Then an $R$-semimodule $M$ is $f$ -injective if and only if $M$ is SAP.

**Proof.**

Suppose $M$ is SAP. Let $i : E \subset Y$,with $E$ finitely generated by $\{e_1, ..., e_n\}$

and $g : E \to M$ .Let $\widehat{M}$ be the injetive hull of $M$ and $j : M \to \widehat{M}$ .Since $\widehat{M}$ is injective , there is $h : Y \to \widehat{M}$ , such that $hi = jg$ .Note that $\{g(e_k) : 1 \leq k \leq n\} \subset M$ and $M$ is SAP . Hence there exists a homomorphism $\alpha : \widehat{M} \to M$ such that $\alpha(g(e_k)) = g(e_k)$ , $1 \leq k \leq n$.One can easily show

that $\beta = \alpha h$ extends $g$ and so $M$ is $f$ -injective.□

**5. Regular Semimodules**

A semiring $R$ is said to be von Neumann regular if for each $a \in R$, there is some $b \in R$ such that $a = aba$. In [6], Fieldhouse generalized the concept of Von Neumann's regular rings to the module case : a module $M$ ( over a ring ) is said to be regular if every submodule of $M$ is pure in $M$ . We extend this concept to semimodules ove an arbitrary semiring $R$.An $R$- semimodule $M$ is said to be Fieldhouse regular if every subsemimodule of $M$ is pure in $M$ .

**Theorem 5.1**

For any $R$- semimodule $M$ the following statements are equivalent :

(i) $M$ is Fieldhouse regular.

(ii) Every finitely generated subsemimodule of $M$ is pure in $M$.

If $M$ is projective one can add :

(iii) Every finitely generated subsemimodule of $M$ is a retract of $M$.

**Proof.**

(i)$\Longrightarrow$ (ii) is trivial.

(ii)$\Longrightarrow$ (i) : Let $E \subset M$ . Note that $E = \varinjlim\{ F_i , \alpha_{ij} \}, i, j \in I$ ,where $\{ F_i \}$ is the family of all finitely generated subsemimodules of $M$ and the morphisms $\{ \alpha_{ij} \}$are the inclusion maps.To show that $E$ is pure in $M$ , let $(\circledast)$ "$A\overline{x} + \overline{u} = B\overline{x} + \overline{v}$ " be a finite system of linear equations with coefficents in $R$ and parameters from $E$ and with a solution $\overline{m}$ in $M$. Let $\{u_1, ..., u_t, v_1, ..., v_t\} \subset E$ be the elements of the column matrices $\overline{u}$ , $\overline{v}$.

There is $k \in I$ such that$\{u_1, ..., u_t, v_1, ..., v_t\} \subset F_k \subset E \subset M$ .Therefore one can consider $\circledast$ as a finite system of linear equations with coefficents in $R$ and parameters from$F_k$ and with a solution in $M$.Since$F_k$ is pure $M$ , $\circledast$ has a solution in

$F_k$ . Thus $E$ is pure in $M$ and so $M$ is Fieldhouse regular. Now suppose $M$ is projective Fieldhouse regular and $E$ is a finitely generated subsemimodule of $M$ .Let $\{e_1, ..., e_n\}$ be a finite set of generators of $E$.By Proposition 4.1,

$E \subset M$ is strongly pure , thus there exists a homomorphism $\alpha : M \to E$ such that $\alpha(e_i) = e_i$ , $i = 1, ..., n$.It follows that $E$ is a retract of $M$ .□

**Corollary 5.2**

For any semiring $R$ consider the following statements :

(i) Every ideal of $R$ is stongly pure in ${}_RR$

(ii) ${}_RR$ is Fieldhouse regular.

(iii) Every principal ideal of $R$ is pure in $R$.

(iv) $R$ is Von Neumann regular.

Then (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv).

**Proof.**

(i)$\Longrightarrow$(ii) and (ii) $\Longrightarrow$ (iii) are tivial . (ii) $\Longrightarrow$ (i ) follows from Proposition 4.1.

(iii) $\Longrightarrow$ (iv) : For each $a \in R$ , $Ra$ is a pure subsemimodule of the $R$-semimodule ${}_RR$.The equation $ax + 0 = 0x + a$, with parameters from $Ra$, has a solution $(x = 1)$ in ${}_RR$. So, it has a solution in $Ra$. This means that there is $x_o = ra \in Ra$, for some $r \in R$, such that $ax_o = a$ .Thus $ara = a$, and so $R$ is von Neumann regular

(iv) $\Rightarrow$ (iii): Suppose $R$ is von Neumann regular and $I = Ra$. There is $b \in R$ such that $a = aba$. Let $e = ba$ and note that $e^2 = baba = ba = e$. Hence $I = \mathrm{Re}$.If $\theta : \mathrm{Re} \to {}_RR$ is the inclusion map and $\alpha :_R R \to \mathrm{Re}$, $\alpha(r) = re$, then $\alpha\theta = 1_{\mathrm{Re}}$

This means that $I = \mathrm{Re}$ is a retract of ${}_RR$, and, in particular, $I$ is pure in $R$.$\square$

**Corollary 5.3**

If every $R$-semimodule is AP then $R$ is von Neumann regular.

**Remark 5.1**

For any ring $R$, the converse of the precding Corollary is true [5].On the other hand, the semiring $R = \mathbf{Q}^+$ is von Neumann regular and $M = {}_RR$ is not AP.

**Remark 5.2**

For any cancellative semiring $R$ the following statements are equivalent :

(i) Every $R$- semimodule is AP.

(ii) Every cancellative $R-$semimodule is AP.

(iii) $R$ is a regular ring.

Semimodules over rins are modules, so (iii) $\Longrightarrow$ (i) follows . (i) $\Longrightarrow$(ii) is obvious.

Now suppose (ii) , then ${}_RR$ is a module ,i.e. $R$ is a ring.Indeed, $R$ is a regular ring.

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