Absolutely Pure Semimodules

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**Abstract.** In the present paper, a semimodule \(M\) over a semiring \(R\) is called absolutely pure if it is pure in every semimodule containing it as a subsemimodule. Some well-known properties of absolutely pure modules are extended to semimodules. We introduce and study two particular subclasses of absolutely pure semimodules, namely strongly absolutely pure (SAP) and finitely injective (\(f\)-injective) semimodules. When the semiring \(R\) is additively idempotent, the SAP \(R\)-semimodules are exactly the \(f\)-injective semimodules. A characterization of Fieldhouse regular semimodules is obtained.

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**Introduction**

The notion of purity in module theory was defined in terms of tensor product. In [3, Thm.2.4], P.M. Cohn proved that a submodule \(M\) of a left module \(N\) (over a ring \(R\)) is pure if every finite system of linear equations \(H\bar{x} = \bar{m}\) with coefficients in \(R\) and parameters from \(M\) is solvable in \(M\) if it is solvable in \(N\). Generally, if \(A\) and \(B\) are \(L\)-structures, where \(L\) is a first-order language, a homomorphism \(f: A \rightarrow B\) is said to be pure if for any positive primitive formula \(\phi\) and any tuple \(\bar{a}\) from \(A\), the validity of \(\phi(f(\bar{a}))\) in \(B\) entails that of \(\phi(\bar{a})\) in \(A\) [12]. This notion of purity was applied to semimodules over an arbitrary semiring and the existence of pure-injective semimodules was proved.
[14, Thm.3] In fact, one can easily show that a subsemimodule $M$ of a left semimodule $N$ (over a semiring $R$) is pure if every finite system of linear equations $H\bar{x} + \bar{m} = K\bar{x} + \bar{m}'$ with coefficients in $R$ and parameters from $M$ and with a solution in $N$ already has a solution in $M$. In the present paper, a semimodule $M$ is called absolutely pure if it is pure in every semimodule containing it as a subsemimodule. Some well-known properties of absolutely pure modules are extended to semimodules. For example, every semimodule has a maximal absolutely pure subsemimodule. We introduce and study two particular subclasses of absolutely pure semimodules, namely strongly absolutely pure (SAP) and finitely injective ($f$-injective) semimodules. A semimodule $M$ is $f$-injective if and only if $M = \lim \left\{ X_i \right\}_{i\in\mathbb{I}}$, where the $X_i$ are injective semimodules and the morphisms of the directed system $\{X_i\}$ are injective. When the semiring $R$ is additively idempotent, the SAP $R$-semimodules are exactly the $f$-injective semimodules. A characterization of Fieldhouse regular semi modules is obtained.

1. Purity in Model theory

In this section, structure means structure for a given finitary similarity type and $L$ is the first-order language of that similarity type. For the basic concepts of model theory we refer to [8]. Let us recall that if $A$ and $B$ are $L$-structures, a homomorphism $f : A \rightarrow B$ is said to be pure if for any positive primitive (p.p. for short) formula and any tuple $\bar{a}$ from $A$, the validity of $\phi(f(\bar{a}))$ in $B$ entails that of $\phi(\bar{a})$ in $A$ [12]. Note that every pure map is an isomorphic embedding, therefore these maps are also called pure embeddings. A substructure $A$ of a structure $B$ is called pure if the inclusion of $A$ in $B$ is pure. Elementary embeddings, that is, embeddings that preserve all first-order formulas, are clearly pure.

**Lemma 1.1** [10]

Let $A$ and $B$ be two $L$-structures. The following conditions are equivalent for any embedding $f : A \rightarrow B$.

(i) $f$ is pure

(ii) There is an elementary embedding $g : A \rightarrow C$ that factors through $f$ (i.e. there is a homomorphism $h : B \rightarrow C$ such that $g = hf$).

**Remark 1.1**

In (ii) above $g$ can be taken to be the diagonal embedding of $A$ into an appropriate ultrapower of $A$ [4, Th.6.4].

Let $R = (R; +, ·, 0, 1)$ be a semiring, i.e. $(R; +, 0)$ is a commutative monoid with identity $0$, $(R; ·, 1)$ is a monoid with identity $1$, for all $a, b, c ∈ R$, $a.(b+c) = a.b + a.c$ and $(b + c).a = b.a + c.a$. $0.r = 0 = r.0$ for all $r ∈ R$, and $0 ≠ 1$. Let $R$ be a semiring. A left $R$-semimodule is a commutative monoid $(M; +, 0)$ for which we have a function $R × M → M$, denoted by $(r, m) ↦ r.m$ and called scaler multiplication, which satisfies the following conditions for all elements $r$ and $s$ of $R$ and all elements $m$ and $n$ of $M$: (1) $(rs)m = r(sm)$; (2) $r(m + n) = rm + rn$; (3) $(r + s)m = rm + sm$; (4) $1.m = m$; (5) $r.0 = 0 = 0.m$. An element $m$ of $M$ is cancellable if $m + m′ = m + m″$ implies that $m′ = m″$. The semimodule $M$ is cancellative if every element of $M$ is cancellable. If every element $m ∈ M$ has an additive inverse $m′ ∈ M$, the semimodule $M$ is called an $R$-module. For the basic concepts of semirings and semimodules we refer to [7]. Throughout this paper, semimodule means left semimodule over a fixed arbitrary semiring $R$. By ideal we mean a left ideal of $R$. By homomorphism, we mean an $R$-homomorphism. We consider the one-sorted first-order language $L_R$ of left semimodules over a fixed arbitrary semiring $R$. Recall that a p.p. formula $ϕ(¯x)$ is a formula of the form

$$ϕ(¯x) = ϕ(x_1, ..., x_n) = ∃y_1...y_m(\bigwedge_{i=1}^{t}Ψ_i(¯x, ¯y)),$$

where $¯y = (y_1, ..., y_m)$ and $Ψ_i(¯x, ¯y)$ are atomic formulas, $i = 1, ..., t$.

One can easily show that every atomic formula $Ψ(x_1, ..., x_n)$ of $L_R$ is equivalent, modulo the theory of semimodules, to an equation

$$\sum_{i=1}^{n} a_ix_i = \sum_{i=1}^{n} b_ix_i,$$

where $a_i, b_i$ are semiring elements. So, the p.p. formula $ϕ(¯x)$ can be read as saying there are elements $¯y$ such that $A¯x + B¯y = C¯x + D¯y$, where $A, C$ are matrices (over $R$) of size $t × n$, $B, D$ are matrices of size $t × m$, and $¯x, ¯y$ are read as column matrices of semimodule elements. Let $M, N$ be two $R$-semimodules and $f : M → N$ be a pure embedding. This means that $f$ is an injective $R$-homomorphism, and for any p.p. formula $ϕ(¯x)$ and each tuple $¯m$ from $M$, if there is a column matrix $¯b$ (of elements of $N$) such that

$$Af(¯m) + B¯b = Cf(¯m) + D¯b$$

then there is a column matrix $¯c$ (of elements of $M$) such that

$$A ¯m + B ¯c = C ¯m + D ¯c$$

where $f(¯m) = (f(m_1), ..., f(m_k))$.

The following results follow from the definition of purity.

**Lemma 2.1**

Let $ϕ(¯x)$ be a p.p. formula in $L_R$ and $M$ be an $R$-semimodule. Then
(i) $M \models \phi(\overline{0})$.
(ii) If $M \models \phi(\overline{a})$ and $M \models \phi(\overline{b})$, then $M \models \phi(\overline{a} + \overline{b})$.
(iii) If $r \in C(R)$, the center of $R$, and $M \models \phi(\overline{r})$, then $M \models \phi(r\overline{a})$, where $r\overline{a} = (ra_1, ..., ra_n) = (ra_1, ..., ra_n)$.
(iv) $\phi(M) = \{ \overline{a} \in M^n : M \models \phi(\overline{a}) \}$ is a submonoid of $(M^n, +)$.
(v) If $R$ is commutative, $\phi(M)$ is a subsemimodule of $(M^n, +)$.
(vi) If $M$ is an $R$-module, $\phi(M)$ is a subgroup of $(M^n, +)$.

**Lemma 2.2**

Suppose that $E, F$ and $G$ are semimodules over a semiring $R$ such that $E \subset F \subset G$.

(i) If $E$ is pure in $F$ and $F$ is pure in $G$ then $E$ is pure in $G$.
(ii) If $E$ is pure in $G$ then $E$ is pure in $F$.

**3. Absolutely Pure Semimodules**

Let $R$ be a semiring and $M$ be an $R$-semimodule. If $W$ is the subsemimodule of $M \times M$ defined by $W = \{(m, m) | m \in M \}$ then $W$ induces an $R$-congruence relation $\equiv_W$ on $M \times M$, called the Bourne relation, defined by setting $(m, n) \equiv_W (m', n')$ if and only if there exist elements $w$ and $w'$ of $W$ such that $(m, n) + w = (m', n') + w'$.

If $(m, n) \in M \times M$ then we write $(m, n)/W$ instead of $(m, n)/\equiv_W$. The factor semimodule $M \times M/ \equiv_W$ is denoted by $M \times M/W$. Since for all $(m, n) \in M \times M$ we have $(m, n)/W + (n, m)/W = (0, 0)/W$, then $M \times M/W$ is an $R$-module. This left $R$-module, denoted by $M^\Delta$, is called the $R$-module of differences of $M$.

**Lemma 3.1** [7]

(i) A subsemimodule of a cancellative semimodule is cancellative.
(ii) Given a semimodule $M$, there is a homomorphism $\xi_M$ of $M$ into $M^\Delta$, defined by $\xi_M(m) = (m, 0)/W$.
(iii) $\xi_M$ is an embedding if and only if $M$ is cancellative.

**Definition**

Let $\Gamma$ be a class of $R$-semimodules. A semimodule $M \in \Gamma$ is said to be absolutely pure (AP) in $\Gamma$, if every embedding of $M$ into a semimodule from $\Gamma$, is pure. When $\Gamma$ is the class of all $R$-semimodules, $M$ is said to be AP.

**Lemma 3.2**

(i) A pure subsemimodule $M$ of a module $N$ is a module.
(ii) $\xi_M$ is a pure embedding if and only if $M$ is a module over the semiring $R$.
(iii) Any cancellative AP semimodule is a module.

**Proof.**
(i) : Consider the p.p. formula \( \phi(x_1) = \exists x_2(x_1 + x_2 = 0) \), and \( m \in M \). Since \( N \) is a module, then \( N \models \phi(m) \), and so \( M \models \phi(m) \). This means that \( m \) has an additive inverse in \( M \).

(ii) : The "if" part follows from [7, Prop. 14.1] and Lemma 1. The "only if" part follows from (i).

(iii) : It follows from (i) and (ii). \( \square \)

**Definition**

Let \( M \) be semimodule over a semiring \( R \). For two elements \( a \in R \) and \( m \in M \), the pair \( (a, m) \) is said to be compatible if the equation \( ax = m \), has a solution in an extension of \( M \).

**Lemma 3.3**

Let \( M \) be a cancellative semimodule over a cancellative semiring \( R \). The following statements are equivalent for two elements \( a \in R \) and \( m \in M \):

(i) The pair \( (a, m) \) is compatible

(ii) There is a homomorphism \( g : Ra \to M \) such that \( g(a) = m \).

**Proof.**

(i) \( \Rightarrow \) (ii) There is \( x_o \) in an extension of \( M \) such that \( ax_o = m \). So, if \( x = ra = ta \in Ra \), then \( rm = rax_o = tax_o = tm \). Thus we may define \( g : Ra \to M \) by \( g(ra) = rm \). Of course, \( g \) is a homomorphism and \( g(a) = g(1a) = m \).

(ii) \( \Rightarrow \) (i) : Let \( I = Ra, g : I \to M \) and \( g(a) = m \). We prove that there is an extension \( V \) of \( M \) and there is \( x_o \in V \) such that \( ax_o = m \) in \( V \). Let \( i \) be the inclusion mapping of \( I \) into \( R \) and consider the homomorphism \( \alpha = \xi_M g : I \to M \to \overline{M} \) and \( \beta = \xi_R i : I \to R \to \overline{R} \). We define \( f : I \to \overline{M} \times \overline{R} \) by \( f(t) = (\alpha(t), -\beta(t)) \). Note that \( N = f(I) \) is a subsemimodule of \( \overline{M} \times \overline{R} \) and \( N \) induces an \( R \)-congruence relation "Bourne relation" on \( \overline{M} \times \overline{R} \). Let \( V \) be the factor semimodule \( \overline{M} \times \overline{R}/N \). Let \( \lambda : M \to M \times R \to \overline{M} \times \overline{R} \to V; u \mapsto (u, 0) \mapsto (\xi_M(u), 0) \mapsto (\xi_M(u), 0)/N \) and \( \mu : R \to M \times R \to \overline{M} \times \overline{R} \to V; r \mapsto (0, r) \mapsto (0, \xi_R(r))/N \).

We show that \( \lambda \) is injective: Suppose \( \lambda(u) = \lambda(v) \). Then \( (\xi_M(u), 0)/N = (\xi_M(v), 0)/N \) and so there are \( n_1, n_2 \in N \) such that \( (\xi_M(u), 0) + n_1 = (\xi_M(v), 0) + n_2 \). If \( n_1 = f(t_1), n_2 = f(t_2) \), then we get \( \xi_M(u) + \alpha(t_1) = \xi_M(v) + \alpha(t_2) \) and \( -\beta(t_1) = -\beta(t_2) \).

Since \( \beta \) is injective and \( M \) is cancellative, then \( \xi_M(u) = \xi_M(v) \), and so \( u = v \). This means that \( V \) is an extension of \( M \). We prove that \( \mu(1) \) is a solution of the equation \( ax = m \) in \( V \), i.e. \( a\mu(1) = \lambda(m) \).

Since \( (0, \xi_R(a)) + f(a) = (0, \xi_R(a)) + (\alpha(a), -\beta(a)) = (\alpha(a), 0) = (\xi_M(g(a)), 0) = (\xi_M(m), 0) + f(0) \), then \( (0, \xi_R(a))/N = (\xi_M(m), 0)/N \). Thus, \( \mu(a) = \lambda(m) \), and so \( a\mu(1) = \lambda(m) \). \( \square \)

**Definition [1]**

An \( R \)-semimodule \( M \) is called \( P \)-injective if for any principal ideal \( I \) of \( R \) and each homomorphism \( g : I \to M \), there exists a homomorphism \( f : R \to \)
Corollary 3.4
Every cancellative AP semimodule $M$ over a cancellative semiring $R$ is $P$-injective.

Proof.
Let $I = Ra, a \in R$, and $g : I \to M$ be a homomorphism. By the preceding Lemma the equation $ax = g(a)$ has a solution in an extension $N$ of $M$, say, $\lambda : M \to N$. Since $M$ is AR, $\lambda$ is pure and so the equation $ax = g(a)$ has a solution $m_o \in M$. We define a homomorphism $h : R \to M$, by $h(r) = r m_o$. For any $x = ta \in I$, $g(x) = tg(a) = tam_o = h(x)$. Hence, $h$ extends $g$ and so $M$ is $P$-injective. $\square$

Corollary 3.5
Let $M$ be a cancellative semimodule over a cancellative semiring $R$. The following statements are equivalent:

(i) $M$ is $P$-injective.

(ii) For any compatible pair $(a, m) \in R \times M$, the equation $ax = m$ has a solution in $M$.

Proof.
(i) $\Rightarrow$ (ii): Suppose $(a, m)$ is compatible. By Lemma 3.3, there is a homomorphism $g : Ra \to M$ such that $g(a) = m$. Since $M$ is $P$-injective, there is a homomorphism $f : R \to M$, extends $g$. Observe that $ah(1) = h(a.1) = h(a) = g(a) = m$, and so $h(1) \in M$ is a solution of $\otimes$.

(ii) $\Rightarrow$ (i): Let $I = Ra, a \in R$, $g : I \to M$ be any homomorphism and $m_o = g(a)$. By Lemma 3.3, the equation $ax = m_o$ has a solution in an extension of $M$. Under the hypothesis (ii), this equation has a solution $u_o \in M$. We define a homomorphism $h : R \to M$, by $h(r) = ru_o$. One easily sees that $h$ extends $g$. $\square$

Proposition 3.6
Every pure subsemimodule $M$ of a $P$-injective semimodule $N$ is $P$-injective.

Proof.
Let $I = Ra, a \in R$, and $g : I \to M$ be a homomorphism. Since $i : M \subseteq N$, and $N$ is $P$-injective, there exists a homomorphism $f : R \to N$, which extends $g$. Hence, $i g(a) = g(a) = f(a) = m_o \in M$. Let $f(1) = n_o \in N$ and consider the equation $\otimes : ax = m_o$. Since $an_o = af(1) = f(a.1) = f(a) = m_o$, then $\otimes$ has a solution in $N$. Observe that $i : M \subseteq N$ is pure.
and so $\oplus$ has a solution $u_o \in M$ (i.e. $au_o = m_o$). Now, define a homomorphism $h : R \to M$ by $h(r) = ru_o$. Since $h(ra) = rau_o = rm_o = rg(a) = g(ra)$, then $h$ extends $g$ and so $M$ is $P$-injective.

In [14, Thm.5], it was proved that the first order theory $T$ of cancellative semimodules over an arbitrary semiring $R$ has the amalgamation property. As an application we have:

**Proposition 3.7.**

Every pure subsemimodule $M$ of a cancellative AP semimodule $N$ is AP in the class of cancellative semimodules.

**Proof.**

Consider the following diagram, where $f_1, f_2$ are the identical inclusions, $f_1$ is pure and $H$ is a cancellative semimodule

\[ \begin{array}{ccc}
N & \xrightarrow{f_1} & M \\
\downarrow & & \downarrow \\
H & \xleftarrow{f_2} & \end{array} \]

Let $(\oplus)"A\bar{x} + \overline{m} = B\bar{x} + \overline{m}'"$ be a finite system of linear equations with coefficients in $R$ and parameters from $M$ and with a solution in $H$. By [14, Thm.5], there is a cancellative semimodule $F$ and embeddings $g_i, i = 1, 2$, such that

the following diagram is commutative.

\[ \begin{array}{ccc}
N & \xrightarrow{f_1} & M \\
\downarrow & & \downarrow \\
F & \xleftarrow{g_i} & H \\
\downarrow & & \\
H & \xleftarrow{g_2} & \end{array} \]

It follows that $(\oplus)$ has a solution in $F$. Since $f_1$ and $g_1$ are pure, $(\oplus)$ has a solution in $M$ and so $M$ is AP in the class of cancellative semimodules.

**Theorem 3.8.**

(i) If $X_0 \subset X_1 \subset \ldots \subset X_\beta \subset \ldots, \beta < \alpha$ is a chain of AP semimodules, where $\alpha$ is an ordinal, then the union of the chain is AP.

(ii) Every semimodule has a maximal AP subsemimodule.

**Proof.**
(i) Let $M = \cup X_\beta$ and suppose that $M \subset N$. Let $(\otimes) : A\overline{x} + \overline{m} = B\overline{x} + \overline{m'}$ be a finite system of linear equations with coefficients in $R$ and parameters from $M$ and with a solution in $N$. There is an ordinal $\gamma$ such that the elements of the column matrices $\overline{m}$ and $\overline{m'}$ are in $X_\gamma \subset M \subset N$. Therefore one can consider $\otimes$ as a finite system of linear equations with coefficients in $R$ and parameters from $X_\gamma$ and with a solution in $N$. Since $X_\gamma \subset N$, $\otimes$ has a solution in $X_\gamma \subset M$.

(ii) Given a semimodule $E$, consider the set $\Omega$ of all subsemimodules of $E$ that are AP semimodules. Observe that $\Omega$ is not empty, for the zero semimodule belongs to $\Omega$. Partially order $\Omega$ by inclusion. If $\mathcal{F}$ is a chain in $\Omega$ then $\cup \mathcal{F}$ is AP by (i). Now the result follows by applying Zorn’s Lemma. □

4. SAP Semimodules

In [2], Azumaya introduced the notion of locally split homomorphisms to study regular modules. Locally split submodules were introduced by Ramamurthi and Rangaswamy [9], by the name of strongly pure submodules, to study strongly absolutely pure (SAP) and finitely injective ($f$-injective in the sense of [13]) modules. In this section we extend these notions for semimodules over an arbitrary semiring. Let $R$ be a semiring and $M$ be an $R$-semimodule.

$M$ is said to be finitely injective ($f$-injective for short) if given any injective homomorphism $F \to Y$, where $F$ is a finitely generated semimodule, any homomorphism $F \to M$ can be extended to a homomorphism $Y \to M$. Note that every injective semimodule is $f$-injective. We call a subsemimodule $M$ of a semimodule $N$ strongly pure if to any finite set $\{m_1, ..., m_k\}$ of elements of $M$ there exists a homomorphism $\alpha : N \to M$ such that $\alpha(m_i) = m_i$, $i = 1, ..., k$. Finally, a semimodule $M$ is said to be strongly absolutely pure (SAP for short) if $M$ is strongly pure in every $R$-semimodule containing it as a subsemimodule.

**Proposition 4.1**

Suppose that $E, F$ and $G$ are semimodules over a semiring $R$ such that $E \subset F \subset G$.

(i) If $E$ is strongly pure in $F$ and $F$ is strongly pure in $G$ then $E$ is strongly pure in $G$.

(ii) If $E$ is strongly pure in $G$ then $E$ is strongly pure in $F$.

(iii) If $E$ is strongly pure in $F$ then $E$ is pure in $F$.

(iv) If $E$ is pure in $F$, where $F$ is a projective semimodule, then $E$ is strongly pure in $F$.

(v) If $E$ is pure in $F$, where $E$ is finitely generated and $F$ is projective, then $E$ is projective.

**Proof.**

(i) and (ii) are obvious. (iii) : Let $(\otimes) \ " A\overline{x} + \overline{u} = B\overline{x} + \overline{v} "$ be a finite system of linear equations with coefficients in $R$ and parameters from $E$ and with a solution $\overline{v}$ in $F$. Let $\{u_1, ..., u_t, v_1, ..., v_t\} \subset E$ be the elements of...
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the column matrices \( \overline{u}, \overline{v} \). Under the hypothesis, there exists a homomorphism \( \alpha : F \to E \), such that

\[
\alpha(u_i) = u_i, \quad \alpha(v_i) = v_i.
\]

It follows that \( A \alpha(\overline{v}) + B = B \alpha(\overline{v}) + \overline{v} \), and so \( \alpha(\overline{v}) \) is solution of \( \mathcal{R} \) in \( E \). (iv): If \( f : E \subseteq F \) is pure, then by Lemma 1, there is an ultrafilter \( u \) over an infinite set \( I \) and a homomorphism \( h : F \to E^I/u \) such that \( h f = \delta \), where \( \delta : E \to E^I/u \) is the diagonal embedding of \( E \) into an ultrapower of \( E \). Let \( \phi : E^I \to E^I/u \) be the canonical homomorphism. Since \( F \) is projective, there exists a homomorphism \( g : F \to E^I \) such that \( \phi g = h \). Let \( \{e_1, \ldots, e_n\} \) be a finite set of elements of \( E \). For each \( e_k \), \( 1 \leq k \leq n \), \( \delta(e_k) = h(e_k) = \phi g(e_k) = g(e_k)/u \). Hence there exists a set \( \Omega_k \subseteq u \) such that \( g(e_k)(i) = e_k \) for all \( i \in \Omega_k \). Let \( \Omega = \cap \Omega_k \subseteq u \), and define a homomorphism \( \alpha = p_i g : F \to E^I \to E \), where \( p_i \) is the canonical projection, \( i \in \Omega \). It follows that \( \alpha(e_k) = e_k, 1 \leq k \leq n \), and so \( E \) is strongly pure in \( F \).

(v): Let \( \{e_1, \ldots, e_n\} \) be a finite set of generators of \( E \). Since \( E \subseteq F \) is strongly pure, there exists a homomorphism \( \alpha : F \to E \) such that \( \alpha(e_i) = e_i, i = 1, \ldots, n \). It follows that \( E \) is a retract of \( F \) and so \( E \) is projective. □

**Corollary 4.2**

Every SAP semimodule is AP

**Proposition 4.3**

Let \( R \) be a semiring, \( N \) be an \( R \)-module and \( M \) be a subsemimodule of \( N \). The following statements are equivalent:

(i) \( M \subset N \) is strongly pure.

(ii) \( M \subset N \) is pure and for any element \( x_o \in M \) there exists a homomorphism \( \alpha : N \to M \) such that \( \alpha(x_o) = x_o \).

**Proof.**

(i) \( \implies \) (ii): By Lemma 3.2(i) \( M \) is a module. Let \( \{x_1, \ldots, x_n\} \) be any finite set of elements of \( M \). We prove by induction, suppose \( n \geq 2 \) and our statement is true for \( n - 1 \). This means that there is a homomorphism \( \alpha : N \to M \) such that \( \alpha(x_k) = x_k \) for \( k = 1, 2, \ldots, n - 1 \). Since \( (x_n - \alpha(x_n)) \subseteq M \), there is a \( \beta : N \to M \) such that \( \beta(x_n - \alpha(x_n)) = x_n - \alpha(x_n) \). Let \( \delta = \alpha + \beta - \beta i \alpha : N \to M \), where \( i : M \subset N \) is the inclusion map. Then for any \( k, k = 1, 2, \ldots, n - 1 \), \( \delta(x_k) = \alpha(x_k) + \beta(x_k) - \beta i \alpha(x_k) = x_k + \beta(x_k) - \beta(x_k) = x_k \). And \( \delta(x_n) = \alpha(x_n) + \beta(x_n) - \beta i \alpha(x_n) = \alpha(x_n) + \beta(x_n - \alpha(x_n)) = \alpha(x_n) + x_n - \alpha(x_n) = x_n \). Thus \( M \) is strongly pure in \( N \). □

The following result connects finite injectivity with strong purity.

**Proposition 4.4**

Every \( f \)-injective semimodule is SAP.

**Proof.**
Let \( M \subset N \), where \( M \) is an \( f \)-injective semimodule. For any finite set \( T = \{m_1, \ldots, m_k\} \) of elements of \( M \), let \( F \) be the semimodule generated by \( T \). Consider the inclusion maps \( f : F \subset N \) and \( j : F \rightarrow M \). There exists a homomorphism \( \alpha : N \rightarrow M \), such that \( \alpha f = j \). Observe that \( \alpha(x) = x \), for all \( x \in F \), and so \( M \) is SAP.

**Proposition 4.5**

Every \( f \)-injective semimodule \( M \) contains an injective hull of each of its finitely generated subsemimodule.

**Proof.**

Let \( F \subset M \) be a finitely generated subsemimodule of \( M \) with an injective hull \( E \). Consider the inclusion maps \( i : F \subset M \) and \( j : F \subset E \). Since \( M \) is \( f \)-injective semimodule, there exists a homomorphism \( h : E \rightarrow M \) such that \( h j = i \). Observe that \( h \) is injective since \( i \) is injective and \( j \) is essential.

**Corollary 4.6**

Every finitely generated \( f \)-injective semimodule \( M \) is injective.

**Theorem 4.7**

(i) If \( X_0 \subset X_1 \subset \ldots \subset X_\beta \subset \ldots, \beta \prec \alpha \), is a chain of \( f \)-injective semimodules, where \( \alpha \) is an ordinal, then the union of the chain is \( f \)-injective.

(ii) Every semimodule has a maximal \( f \)-injective subsemimodule

(iii) A semimodule \( M \) is \( f \)-injective if and only if \( M = \lim \{ X_i \} \), where the \( X_i \) are injective semimodules and the morphisms of the directed system \( \{ X_i \} \) are injective.

**Proof.**

We prove only (iii). Note that \( M = \lim \{ F_i, \alpha_{ij} \} \), \( i, j \in I \), where \( \{ F_i \} \) is the family of all finitely generated subsemimodules of \( M \) and the morphisms \( \{ \alpha_{ij} \} \) are the inclusion maps. Since \( M \) is \( f \)-injective, it contains an injective hull \( \hat{F}_i \) of each \( F_i \). One can easily check that \( M = \hat{\bigcup \{ F_i \}} \) and the \( \alpha_{ij} : F_i \rightarrow F_j \) induce injective homomorphisms \( \hat{\alpha}_{ij} : \hat{F}_i \rightarrow \hat{F}_j \), such that \( \{ \hat{F}_i, \hat{\alpha}_{ij} \} \) is a directed system and \( M = \lim \{ \hat{F}_i \} \).

**Remark. 4.1**

If \( R \) is a ring, then it is well-known that every \( R \)-module is contained in an injective \( R \)-module. However, for arbitrary semirings \( R \) this is not the case; e.g. there are no nonzero injective \( \mathbb{N} \)-semimodules. In [11], H. Wang proved that every \( R \)-semimodule has an injective hull, in the case that \( R \) is additively idempotent (i.e. a semiring satisfying \( r + r = r \) for all \( r \in R \)). For these semirings we prove the converse of Proposition 4.4.

**Theorem 4.8**

Let \( R \) be an additively idempotent semiring. Then an \( R \)-semimodule \( M \) is \( f \)-injective if and only if \( M \) is SAP.

**Proof.**

Suppose \( M \) is SAP. Let \( i : E \subset Y \), with \( E \) finitely generated by \( \{ e_1, \ldots, e_n \} \).
Absolutely pure semimodules

Let $M$ be the injective hull of $M$ and $j : M \to \tilde{M}$. Since $\tilde{M}$ is injective, there is $h : Y \to \tilde{M}$ such that $hi = jg$. Note that $\{g(e_k) : 1 \leq k \leq n\} \subset M$ and $M$ is SAP. Hence there exists a homomorphism $\alpha : \tilde{M} \to M$ such that $\alpha(g(e_k)) = g(e_k), 1 \leq k \leq n$. One can easily show that $\beta = \alpha h$ extends $g$ and so $M$ is $f$-injective. □

5. Regular Semimodules

A semiring $R$ is said to be von Neumann regular if for each $a \in R$, there is some $b \in R$ such that $a = aba$. In [6], Fieldhouse generalized the concept of Von Neumann’s regular rings to the module case: a module $M$ (over a ring) is said to be regular if every submodule of $M$ is pure in $M$. We extend this concept to semimodules over an arbitrary semiring $R$. An $R$-semimodule $M$ is said to be Fieldhouse regular if every subsemimodule of $M$ is pure in $M$.

Theorem 5.1

For any $R$-semimodule $M$ the following statements are equivalent:

(i) $M$ is Fieldhouse regular.

(ii) Every finitely generated subsemimodule of $M$ is pure in $M$.

If $M$ is projective one can add:

(iii) Every finitely generated subsemimodule of $M$ is a retract of $M$.

Proof.

(i) $\implies$ (ii) is trivial.

(ii) $\implies$ (i): Let $E \subset M$. Note that $E = \lim_{\to} \{ F_i, \alpha_{ij} \}, i, j \in I$, where $\{ F_i \}$ is the family of all finitely generated subsemimodules of $M$ and the morphisms $\{ \alpha_{ij} \}$ are the inclusion maps. To show that $E$ is pure in $M$, let $(\otimes) "A\overline{\pi} + \overline{\eta} = B\overline{\pi} + \overline{\eta}"$ be a finite system of linear equations with coefficients in $R$ and parameters from $E$ and with a solution $\overline{\eta}$ in $M$. Let $\{ u_1, \ldots, u_t, v_1, \ldots, v_t \} \subset E$ be the elements of the column matrices $\overline{\eta}, \overline{\eta}$.

There is $k \in I$ such that $\{ u_1, \ldots, u_t, v_1, \ldots, v_t \} \subset F_k \subset E \subset M$. Therefore one can consider $\otimes$ as a finite system of linear equations with coefficients in $R$ and parameters from $F_k$ and with a solution in $M$. Since $F_k$ is pure $M$, $\otimes$ has a solution in $F_k$. Thus $E$ is pure in $M$ and so $M$ is Fieldhouse regular. Now suppose $M$ is projective Fieldhouse regular and $E$ is a finitely generated subsemimodule of $M$. Let $\{ e_1, \ldots, e_n \}$ be a finite set of generators of $E$. By Proposition 4.1, $E \subset M$ is strongly pure, thus there exists a homomorphism $\alpha : M \to E$ such that $\alpha(e_i) = e_i, i = 1, \ldots, n$. It follows that $E$ is a retract of $M$. □

Corollary 5.2

For any semiring $R$ consider the following statements:

(i) Every ideal of $R$ is strongly pure in $R R$.

(ii) $R R$ is strongly regular.

(iii) Every principal ideal of $R$ is pure in $R$.
(iv) $R$ is Von Neumann regular.

Then (i) $\iff$ (ii) $\implies$ (iii) $\iff$ (iv).

**Proof.**

(i)$\implies$(ii) and (ii) $\implies$ (iii) are trivial. (ii) $\implies$ (i) follows from Proposition 4.1.

(iii) $\implies$ (iv): For each $a \in R$, $Ra$ is a pure subsemimodule of the $R$-semimodule $R$. The equation $ax + 0 = 0 + a$, with parameters from $Ra$, has a solution $(x = 1)$ in $R$. So, it has a solution in $Ra$. This means that there is $x_o = ra \in Ra$, for some $r \in R$, such that $ax_o = a$. Thus $ara = a$, and so $R$ is von Neumann regular.

(iv) $\implies$ (iii): Suppose $R$ is von Neumann regular and $I = Ra$. There is $b \in R$ such that $a = aba$. Let $e = ba$ and note that $e^2 = baba = ba = e$. Hence $I = Re$. If $\theta : Re \rightarrow R$ is the inclusion map and $\alpha : R \rightarrow Re$, $\alpha(r) = re$, then $\alpha \theta = 1_{Re}$.

This means that $I = Re$ is a retract of $R$, and, in particular, $I$ is pure in $R$. $\Box$

**Corollary 5.3**

If every $R$-semimodule is AP then $R$ is von Neumann regular.

**Remark 5.1**

For any ring $R$, the converse of the preceding Corollary is true [5]. On the other hand, the semiring $R = Q^+$ is von Neumann regular and $M = R$ is not AP.

**Remark 5.2**

For any cancellative semiring $R$ the following statements are equivalent:

(i) Every $R$- semimodule is AP.

(ii) Every cancellative $R$-semimodule is AP.

(iii) $R$ is a regular ring.

Semimodules over rins are modules, so (iii) $\implies$ (i) follows. (i) $\implies$ (ii) is obvious.

Now suppose (ii), then $R$ is a module, i.e. $R$ is a ring. Indeed, $R$ is a regular ring.

**References**


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