Some Sufficient Conditions for Solvability
of Finite Groups

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Abstract

Let $G$ be a finite group. Suppose that none of degrees of irreducible
characters (lengths of conjugacy classes, respectively) are divisible by
3 or 5. In this note, we prove that $G$ is solvable, and the odd prime
number pair $(3, 5)$ is unique in order to guarantee the solvability of $G$.

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1 Introduction

The celebrated Odd Order Theorem of Feit-Thompson [4] shows that if the
order of the finite group $G$ is not divisible by prime 2, then $G$ is solvable. It is
a natural question whether there exists another distinct prime $p$ such that if $p$
does not divide the order $|G|$, then $G$ is solvable. D. Gorenstein pointed out
that a finite group $G$ is solvable if its order $|G|$ is coprime to 15. It is indeed
proved in [5] that there is nonexistence of a single prime $p(\neq 2)$ such that if $p \mid
|G|$ then $G$ is solvable. Furthermore, if allowing for two primes (excluding 2),
then the only choice is $pq = 15$ such that if $(pq, |G|) = 1$ then $G$ is solvable.
Here $p, q$ are two different primes. In this note we generalize this result in
terms of character degrees and conjugacy class lengths of a finite group. In
particular, we prove the following results.

**Theorem A.** Assume that none of irreducible character degrees of the
finite group $G$ are divisible by 3 or 5. Then $G$ is solvable.

**Theorem B.** Assume that none of conjugacy class lengths of the finite
group $G$ are divisible by 3 or 5. Then $G$ is solvable.

We also prove that, in the results above, the odd prime pair $(3, 5)$ is unique.
2 Proofs of results

The following result is the famous Ito-Micheler theorem, which can be found in [6, Theorem 13.1, Remarks 13.13].

**Theorem 2.1.** Let $G$ be a finite group and $p$ be a prime divisor of $|G|$. And let $cd(G)$ denote the set of degrees of irreducible characters of $G$. Then $p$ does not divide any member of $cd(G)$ if and only the Sylow $p$-subgroup of $G$ is normal and abelian.

**Theorem 2.2.** If every irreducible character degree of the finite group $G$ is relatively prime to 15, then $G$ is solvable.

**Proof.** If $|G|$ is coprime to 15, then $G$ is solvable by Gorenstein’s result. Otherwise, either 3 or 5 divides $|G|$, the application of the above Ito-Micheler’s theorem on $G$ yields that $G$ has an abelian and normal Sylow subgroup $N$. The inductive hypothesis applies, we get that $G/N$ is solvable, and so $G$ is solvable since $N$ is solvable. \[\square\]

Observe that the set of the irreducible character degrees of $A_5$ is $cd(A_5) = \{1, 3, 4, 5\}$, the pair $(3, 5)$ is the only “odd” prime pair choice for theorem 2.2. We call $n$ is a character solvable number if $n$ is coprime to every degree of irreducible character of $G$, then $G$ is solvable. By the theorem above, 15 is a character solvable number. We further obtain the following result.

**Theorem 2.3.** Assume that $n$ is an odd number. Then $n$ is a character solvable number if and only if $n$ is divisible by 15.

**Proof.** By the above theorem, the “if” part is immediate. Now we deal with the “only if” part. The alternating group $A_5$ implies that $3 \mid n$ or $5 \mid n$. It is known that $A_6$ has two irreducible characters whose degrees are 9 and 10 respectively. Note that $3 \mid 9$ but $5 \nmid 9$, and $3 \nmid 10$ but $5 \mid 10$. Thus the only possibility choosing $n$ is that 15 divides $n$, as desired. \[\square\]

Let $n$ be a positive integer, by $n_p$, we denote the biggest $p$-power divisor of $n$. If $p \nmid n$, set $n_p = 1$. The following consequence is the main result of [1], which is proved by using the method of prime graph. The result is independent of interest. It can also be proved by using the technique of $p$-block of defect 0. Note that both of these proofs depend heavily on the classification theorem of finite simple groups.

**Theorem 2.4.** Let $G$ be a finite nonabelian simple group and $p$ a prime number. Then there exists $x \in G$ such that $|x^G|_p = |G|_p$.

**Theorem 2.5.** If every conjugacy class length of finite nonabelian group $G$ is relatively prime to 15, then $G$ is solvable.
Proof. If $|G|$ is coprime to 15, then $G$ is solvable by Gorenstein’s result. Otherwise, either 3 or 5 divides $|G|$. If $G$ is a simple group, then theorem 2.5 implies that $G$ has a conjugacy class whose length equals $|G|_p$ for $p = 3$ or 5. This contradicts our hypothesis. Thus $G$ is not a simple group.

Let $N$ be a proper normal subgroup of $G$. Without loss of generality, we may assume that both $N$ and $G/N$ are nonabelian. Note that the lengths of conjugacy classes of $N$ and $G/N$ are relatively prime to 15. Using the inductive argument, we get that $G/N$ and $N$ are solvable, then $G$ is solvable, as desired.

We mention that the nontrivial conjugacy class lengths of $A_5$ are $\{12, 15, 20\}$, thus the pair $(3, 5)$ is also the only “odd” prime pair choice for theorem 2.6. We further obtain the following result. We call $n$ a conjugacy class solvable number if $n$ is coprime to every nontrivial conjugacy class length of $G$, then $G$ is solvable.

**Theorem 2.6.** Suppose that $n$ is an odd number. Then $n$ is a conjugacy class solvable number if and only if $n$ is divisible by 15.

**Proof.** The “if” part is immediate from theorem 2.5. Now we deal with “only if” part. Considering the nontrivial conjugacy class lengths of $A_5$ are $\{12, 15, 20\}$, we get that $n$ is divisible by 3 or 5. Suppose that $3 \mid n$ but $5 \nmid n$. Because the classical group $PSL(3, 3)$ is of order $2^4 \cdot 3^3 \cdot 13$, we conclude via theorem 2.4 that this case does not occurs. If $5 \mid n$ but $3 \nmid n$, then the exceptional group $Sz(2^3)$ is a counterexample since its order is $2^6 \cdot 5 \cdot 13 \cdot 7 \cdot 13$. Therefore $n$ must be divided by 15. The proof is finished.

We observe that it may be proved by results of [3] that if all conjgacy class lengths or all irreducible character degrees of $G$ are of odd number, then $G$ is solvable.

**References**


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