On the Prime Spectrum of a Module
over Noncommutative Rings

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Abstract. Let $R$ be an associative ring with identity and $M$ an $R$-module. Let $\text{Spec} (M)$ be the set of all prime submodules of $M$. We topologize $\text{Spec} (M)$ with the Zariski topology and prove some useful results.

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In this paper, we always assume that a ring is associative with identity, and by an ideal we mean a 2-sided ideal. Let $R$ be a associative ring with identity and $M$ be a left $R$-module. By a prime submodule (or a $p$-prime submodule) of $M$, we mean a proper submodule $P$ with $(P : M) = \{ r \in R : rM \subseteq P \} = p$ such that $rRm \subseteq P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in p$. Recall that a proper ideal $P$ of a ring $R$ is called prime if $aRb \subseteq P$ implies that either $a \in P$ or $b \in P$. It is clear that if $N$ is a prime submodule, then $(N : M) = \{ r \in R : rM \subseteq N \}$ is a prime ideal of $R$. The set of all prime submodules of $M$ is called the prime spectrum of $M$ and denoted by $\text{Spec} (M)$ or $X^M$. Several authors have extended the notation of prime ideals to modules (see, for example, [1], [5] - [8], [11]). Note that $\text{Spec} (M)$ may be empty (see, 7). Throughout this paper we assume that $\text{Spec} (M)$ is not empty. We introduce a topology called the Zariski topology on $X^M = \text{Spec} (M)$ for any $R$-module, in which closed sets are varieties $V (N) = \{ P \in X^M : N \subseteq P \}$ where $N$ is a subset of an $R$-module $M$. Clearly,
\( V(E) = V(S) \) where \( S \) is an \( R \)-submodule of \( M \) generated by a subset \( E \) of \( M \). We write \( N \leq M \) to indicate that \( N \) is a submodule of \( M \).

Recall that a ring \( R \) is called prime if \((0)\) is prime ideal of \( R \).

**Definition 1.** Let \( M \) be an \( R \) module. Then a proper submodule \( N \) of \( M \) is reducible if it can be written as the intersection \( N = S_1 \cap S_2 \) of two submodules \( S_1, S_2 \) with \( N \neq S_1 \) and \( N \neq S_2 \), otherwise \( N \) is irreducible.

**Proposition 1.** For any subset \( E \) of \( M \), we consider varieties denoted by \( V(E) \). We define \( V(E) = \{ P \in \text{Spec}(M) : E \subseteq P \} \). Then

\[
(a) \text{ If } N \text{ is a submodule generated by } E, \text{ then } V^R(E) = V^R(N).
(b) \quad V(0_M) = \text{Spec}(M) \text{ and } V(M) = \emptyset.
(c) \quad \bigcap_{i \in J} V(N_i) = V\left(\sum_{i \in J} N_i\right) \text{ for any index set } J.
(d) \quad V(N) \cup V(L) \subseteq V(N \cap L), \text{ where } N, L \leq M.
\]

**Proof.** [see, [2] – [4]].

Our purpose is to study modules for which the inclusion of \((d)\) proposition 1 is always an equality. These modules called Top-module.

An \( R \)-module \( M \) is called a multiplication module if for each \( N \leq M \), there exists an ideal \( I \leq R \) such that \( N = IM \). Then, \( N = (N : M) M \). Indeed, \( N = IM \subseteq (IM : M) M = (N : M) M \subseteq N \). If \( N \) is a prime submodule of a multiplication \( R \)-module \( M \), then \( N_1 \cap N_2 \subseteq N \) implies \( N_1 \subseteq N \) or \( N_2 \subseteq N \) where \( N_1, N_2 \leq M \) [see, for more detail, [9] – [10]].

**Proposition 2.** Every multiplication module is a Top-module.

**Proof.** Let \( N \in V(N_1 \cap N_2) \) and so \( N_1 \cap N_2 \subseteq N \). Then, \( N_1 \subseteq N \) or \( N_2 \subseteq N \). Therefore \( N \in V(N_1) \) or \( N \in V(N_2) \).

**Definition 2.** Let \( M \) be an \( R \)-module. For every subset \( Y \) of \( X^M \), let us denote by \( J(Y) \) the intersection of all prime submodules of \( M \) which belong to \( Y \).

**Definition 3.** Let \( M \) be an \( R \) module. \( M \) is distributive if it satisfy the following condition \((S_1 + S_2) \cap N = (S_1 \cap N) + (S_2 \cap N)\) for all submodules \( S_1, S_2 \) and \( N \) of \( M \).

For any submodule \( N \) of an \( R \)-module \( M \), the radical, \( \text{rad}N \), of \( N \) is defined to be the intersection of all prime submodules of \( M \) containing \( N \), and in case \( N \) is not contained in any prime submodule then \( \text{rad}N \) is defined to be \( M \). The radical of the module \( M \) is defined to be \( \text{rad}(0) \).
Theorem 1. \( M \) is a Top-module and \( \text{rad} S = S \) for each submodule \( S \) of \( M \). Then \( M \) is a distributive module.

Proof. Let \( S_1, S_2 \) and \( N \) be any submodules of \( M \). Then,
\[
(S_1 + S_2) \cap N = \text{rad}((S_1 + S_2) \cap N)
\]
\[
= J(V((S_1 + S_2) \cap N))
\]
\[
= J(V(S_1 + S_2) \cup V(N))
\]
\[
= J(V(S_1 \cup S_2) \cup V(N))
\]
\[
= J(V(S_1) \cap V(S_2)) \cup V(N))
\]
\[
= J(V(S_1) \cup V(N)) \cap (V(S_2) \cup V(N))
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\[
= J(V(S_1 \cap N) \cap V(S_2 \cap N))
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\[
= J(V(S_1 \cap N) \cap V(N))
\]
\[
= J(V(S_1 \cap N) \cap (S_2 \cap N))
\]
\[
= J(V(S_1 \cap N) \cap (S_2 \cap N))
\]
\[
= \text{rad}((S_1 \cap N) \cap (S_2 \cap N))
\]
\[
= (S_1 \cap N) \cap (S_2 \cap N)
\]

\[\blacksquare\]

Lemma 1. Let \( M \) be a Top-module and \( N_1, N_2 \) be two submodules of \( M \). Then the equalities \( \text{rad}(N_1 \cap N_2) = \text{rad}N_1 \cap \text{rad}N_2 \) holds.

Proof. \( \text{rad}(N_1 \cap N_2) = J(V(N_1 \cap N_2)) \)
\[
= J(V(N_1) \cup V(N_2))
\]
\[
= J(V(N_1) \cap J(V(N_2))
\]
\[
= \text{rad}N_1 \cap \text{rad}N_2 \]

Note that the closure of a subset \( Y \) of \( \text{Spec}(M) \) denoted by \( \overline{Y} \).

Theorem 2. Let \( M \) be a Top-module. Then \( \overline{Y} = V((J(Y)) \).
Proof. Let $V(S)$ be a closed set containing $Y$. Then $S \subseteq N$ for every prime submodule $N$ in $Y$, so $S \subseteq J(Y)$ and consequently $V(J(Y)) \subseteq V(S)$. Since $Y \subseteq V(J(Y))$, then $V(J(Y))$ is the smallest closed subset of $X^M$ containing $Y$. Thus $\overline{Y} = V((J(Y))$.

A topological space $X$ is $T_1$-space if and only if given any two distinct points $x$ and $y$ in $X$, each lies in an open sets which does not contain the other.

**Theorem 3.** $X^M$ is $T_1$-space if and only if each prime submodule is maximal in the family of all prime submodule of $M$.

**Proof.** Suppose $N$ is maximal in $\text{Spec}(M)$. Then $\{N\} = V(J(N)) = V(N)$ and since $N$ is maximal submodule so $\{N\} = \{N\}$, this means $\{N\}$ is closed. Then $X^M$ is a $T_1$-space, and vice versa.

**Definition 4.** A topological space $X$ is called irreducible if every finite intersection of non-empty open sets of $X$ is non-empty.

**Proposition 3.** Let $M$ be a Top-module and $Y$ a subset of $X^M$. If $J(Y)$ is prime submodule, then $Y$ is an irreducible space.

**Proof.** Suppose $N = J(Y)$ prime submodule. $\overline{Y} = V(J(Y)) = V(N)$ by Theorem 2. $\overline{Y} = V(N) = V(J(N)) = \{N\}$. As a set consisting of a single element is irreducible, then $\{N\}$ is irreducible, that is $\overline{Y}$ is irreducible. Then $Y$ is irreducible.

**Corollary 1.** Let $M$ be a Top-module. Then $V(N)$ is an irreducible space for every prime submodule $N$.

**Proof.** Since $J(V(N)) = \bigcap_{N \subseteq P} P = \text{rad}N = N$, $V(N)$ is irreducible space by proposition 3.

We denote the complement of $V(N)$ by $D(N)$. Note that $D(m) = D(Rm)$ for every $m \in M$.

**Theorem 4.** Let $M$ be a Top-module. Then the sets $D(m_i) (i \in I)$ form a base of $X^M$. 

Proof. Let \( D(S) \) be an open set, where \( S \) is a submodule of \( M \) which is in the form \( S = \bigcup_{i \in I} \{m_i\}, m_i \in S \), then \( D(S) = D(\bigcup_{i \in I} \{m_i\}) = \bigcup_{i \in I} D(m_i) \).

**Theorem 5.** Let \( M \) be a Noetherian Top-module. Then each open set of \( X^M \) is compact.

**Proof.** Suppose \( D(S) \) is an open set of \( X \). Let \( \{D(m_i)\}_{i \in I} \) be a basic open cover, \( m_i \in M \), for each \( i \in I \).

\[
D(S) \subseteq \bigcup_{i \in I} D(m_i) = D(\bigcup_{i \in I} m_i) \quad \text{where} \quad K \text{ is the submodule of } M \text{ generated by } A = \{m_i\}_{i \in I}.
\]

Since \( M \) is Noetherian, \( K \) is finitely generated. Let \( K = \langle b_1, \ldots, b_r \rangle \).

Thus, \( b_i = \sum_{j=1}^{r} r_{ij}m_{ij} \) where \( m_{ij} \in A \).

That is there exists \( \{m_{i1}, \ldots, m_{in}\} \subseteq A \) such that \( K = \langle m_{i1}, \ldots, m_{in} \rangle \).

Thus \( D(S) \subseteq D(\langle m_{i1}, \ldots, m_{in} \rangle) \) so \( D(S) \subseteq \bigcup_{i=1}^{n} D(m_i) \).

Thus \( D(S) \) is compact.

**Proposition 4.** Let \( M \) be a Top-module such that every open set of \( \text{spec}(M) \) is compact and \( \text{rad}S = S \) for each submodule \( S \) of \( M \). Then \( M \) is Noetherian module.

**Proof.** Let \( S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq \ldots \) be an ascending chain of submodules of \( M \).

Then \( D(S_1) \subseteq D(S_2) \subseteq \ldots \subseteq D(S_n) \subseteq \ldots \). Let \( K = \bigcup_{i \in I} S_i \).

Then \( D(K) = D(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} D(S_i) \).

Thus by Theorem 5, \( D(K) = \bigcup_{i=1}^{n} D(S_i) \) and \( V(K) = V(\bigcup_{i=1}^{n} S_i) \).

Hence, \( J(V(K)) = J(V(\bigcup_{i=1}^{n} S_i)) \).

Therefore \( \text{rad}K = \text{rad}(\bigcup_{i=1}^{n} S_i) \) and by hypothesis \( K = \bigcup_{i=1}^{n} S_i \).

Hence \( K = S_j \) for some \( j \in I \).

Thus \( M \) is Noetherian.

**REFERENCES**


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