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Characterizations of Left Regular Ordered Abel-Grassmann Groupoids

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Abstract

In this paper,The concept of a $(\in, \in \vee q)$-fuzzy ideal in an ordered Abel-Grassmann groupoid is introduced. In partcicular, left regular ordered Abel-Grassmann groupoids are characterized by using the $(\in, \in \vee q)$-fuzzy ideals.

Keywords: Abel-Grassman groupoids, AG-groupoids, Left invertive law, Medial law, $(\in, \in \vee q)$-fuzzy ideals

1 Introduction

Fuzzy sets were considered with respect to a non-empty set S. The main idea of a fuzzy set S is that each element x of S is assigned a membership grade $f(x)$ in the interval $[0, 1]$ with $f(x) = 0$ corresponding to non-membership and $0 < f(x) < 1$ to partial membership and $f(x) = 1$ to full membership. Mathematically, a fuzzy subset f of a set S is a function from S into a closed interval $[0, 1]$. The concept of fuzzy sets was first introduced by Zadeh [28] in 1965 which has a wide range of applications in various fields such as computer engineering, artificial intelligence, control engineering, operational research, management science,robotics and many more. In recent years, there are a number of papers appeared in fuzzy algebras.

The concept of being "belongingness" of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset was first given by Murali in [20]. In the well known article of Y.M.Liu and P.H.Poh [25], the idea of quasi-coincidence of a fuzzy point with a fuzzy set was mentioned. In partcular,the concept of (α, β)-fuzzy subgroup was considered by Bhakat and Das in [2] and [3] by using the left belongingness to a right relation $(\in)$ and left quasi coincident with a right relation (q) between a fuzzy poit and a fuzzy subgroup where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. is an ideal of the fuzzy semigroup. The idea of (α, β)-fuzzy subgroup is a viable generalization of the Rosenfeld's fuzzy subgroup [26]. The concept of $(\in, \in \vee q)$-fuzzy sub-near-rings of a near-ring was introduced by Davvaz in [5]. Moreover,Y.B. Jun et. al considered the $(\in, \in \vee q)$-fuzzy ordered semigroups and $(\in, \in \vee q)$-fuzzy filters in an Abel-Grassmann groupoid in [9], [18] and [29].

In this paper, the concept of $(\in, \in \vee q)$-fuzzy ideals in a so called ordered Abel-Grassmann groupoid is intoduced. We are going to show that all these $(\in, \in \vee q)$-fuzzy ideals coincide in a left regular ordered Abel-Grassmann groupoid with a left identity. Furthermore, we will characterize the left regular ordered Abel-Grassmann groupoids by using the above special fuzzy ideals.

The idea of generalization of a commutative semigroup (which was called the left almost semigroup) has been first initiated by M. A. Kazim and M. Naseeruddin in 1972 [12]. They first introduced the braces on the left of the ternary commutative law $abc = cba$ to get a new pseudo associative law, that is, $(ab)c = (cb)a$. Since then. This new associative law was called the left invertive law [6]. A groupoid satisfying the left invertive law is called a left almost semigroup and is abbreviated as a LA-semigroup.(see $[21 - 22]$. Similarly, a groupoid satisfying the right invertive law, $a(bc) = c(ba)$ was called a right almost semigroup and is abbreviated as RA-semigroup.Consequently, a groupoid has been called an almost semigroup if it is both an LA-semigroup and an RA-semigroup. A left almost semigroup is also called an Abel-Grassmann groupoid abbreviated as an AG-groupoid in [24] because it is an abelian groupoiid and also satified the invertive law. Henceforth , we shall use the terminology of P.V. Protić and N. Stevanovi ć in [24] and we simply call an LA-semigroup an Abel-Grassmann groupoid.

It is clear that an AG-groupoid,that is ,an Abel-Grassman groupoid, satisfies the medial law $(ab)(cd) = (ac)(bd)$. Basically, an AG-groupoid S is a non-associative algebraic structure between a groupoid and a commutative semigroup. It is important to mention here that if an AG-groupoid contains an identity or a right identity then it becomes a commutative monoid. It is noteworthy that an AG-groupoid is not necessarily to contain a left identity and if it contains a left identity then it is unique [21]. An AG-groupoid S with a left identity satisfies the para medial law that is, $(ab)(cd) = (db)(ca)$ and $a(bc) = b(ac)$, for all a, b, c, d.

An ordered Abel-Grassmann groupoid (AG-groupoid)S is an ordered semigroup $(S, ., \leq)$ in which the following conditions are satisfied

(i) $(S, .)$ is an AG-groupoid.
(ii) $(S, \leq)$ is a poset.
(iii) For all a, b and $x \in S$, $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$.

Let $x \in S$. Then $A_x = \{(y, z) \in S \times S : x \leq yz\}$.

The product of any fuzzy subsets f and g of an ordered AG-groupoid S is defined by

$$(f \circ g)(x) = \begin{cases} (y, z) \in A_x \bigvee \{f(y) \wedge g(z)\}, & \text{if } A_x \neq \emptyset. \\ 0, & \text{if } A_x = \emptyset. \end{cases}$$

The order relation $\subseteq$ between any two fuzzy subsets f and g of S is defined by

$$f \subseteq g \text{ if and only if } f(x) \leq g(x), \text{ for all } x \in S. \tag{1}$$

The symbols $f \cap g$ and $f \cup g$ will mean the following fuzzy subsets of S

$$(f \cap g)(x) = \min\{f(x),\ g(x)\} = f(x) \wedge g(x), \text{ for all } x \text{ in } S, \tag{2}$$

and

$$(f \cup g)(x) = \max\{f(x),\ g(x)\} = f(x) \vee g(x), \text{ for all } x \text{ in } S. \tag{3}$$

For $\emptyset \neq A \subseteq S$, we define

$$(A] = \{t \in S \mid t \leq a,\ for\ some\ a \in A\}. \tag{4}$$

For $A = \{a\}$, we usually written it as $(a]$.

2 Ideals of an AG groupoid

Definition 1. *A subset A of an AG groupoid S is called a semiprime ideal of S if $a^2 \in A$ implies $a \in A$. A fuzzy subset f of S is called a fuzzy semiprime iideal of S if $f(a) \geq f(a^2)$ for all a in S. We note here that prime ideals of S have also been considered in* [13].

Definition 2. *A non-empty subset A of an an AG-groupoid S is called a left (right) ideal of S if the following conditions hold.*

(i) $SA \subseteq A$ *and* $(AS \subseteq A)$.
(ii) *If* $a \in A$ *and* b *is in* S *such that* $b \leq a$, *then* b *is in* A.
A subset A of an AG groupoid S is called a two-sided ideal of S if it is both a left and a right ideal of S.

A fuzzy subset f of an AG groupoid S is called a fuzzy left (right) ideal of S if the following conditions hold.

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* x *and* y *in* S.
(ii) $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$, *for all* x *and* y *in* S.
A fuzzy subset f of an AG groupoid S is called a fuzzy two-sided ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.

A fuzzy subset f of an AG groupoid S is called a fuzzy AG-subgroupoid of S if $f(ab) \geq f(a) \wedge f(b)$, for all a and b in S.

A fuzzy subset f of an AG groupoid S is called a fuzzy generalized bi-ideal of S if the following conditions hold
(i) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* x *and* y *in* S.
(ii) $f((xy)z) \geq f(x) \wedge f(z)$, *for all* x, y *and* z *in* S.
A fuzzy AG-subgroupoid f of an AG groupoid S is called a fuzzy bi-ideal of S if
(i) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* x *and* y *in* S.
(ii) $f((xy)z) \geq f(x) \wedge f(z)$, *for all* x, y *and* z *in* S.

Definition 3. *A fuzzy subset f of an AG groupoid S is called a fuzzy interior ideal of S by Y. B. Jun* [10] *if the following conditions hold*

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* x *and* y *in* S.
(ii) $f((xy)z) \geq f(y)$, *for all* x, y *and* z *in* S.
A fuzzy AG-subgroupoid f of S is called a fuzzy $(1, 2)$*-ideal of S if*
(i) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* x *and* y *in* S.

(ii) $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z)$, *for all* a, x, y *and* z *in* S.

A fuzzy subset f *of an AG-groupoid* S *is called a fuzzy idempotent of* S *if* $f \circ f = f$ *holds.*

For a $\emptyset \neq A \subseteq S$, *the characteristic function* C_A *is defined by*

$$C_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{5}$$

Note that a subset S *can always be considered as a fuzzy subset of itself and we write* $S = \mathbf{C}_S$, *i.e.* $S(x) = 1$, *for all* $x \in S$.

Assume that S *is an ordered AG-groupoid and let* $F(S)$ *denote the set of all fuzzy subsets of* S. *Then,* $(F(S), \circ, \subseteq)$ *is an ordered AG-groupoid*[24].

3 Properties of $(\in, \in \vee q)$-fuzzy left ideals

In this section, we shall discuss the properties of a $(\in, \in \vee q)$*-fuzzy left tdeals of an AG grupoid.*

Lemma 3.1. [23] *Let* f *be a fuzzy subset of an AG-groupoid* S. *Then* f *is a fuzzy left (right) ideal of* S *if and only if* f *satisfies the following conditions.*

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* x *and* y *in* S.
(ii) $S \circ f \subseteq f$ $(f \circ S \subseteq f)$.

Lemma 3.2. [23] *The following properties hold in an AG-groupoid* S.

(i) A non-empty subset A *of* S *is an AG-subgroupoid of* S *if and only if* C_A *is a AG-subgroupoid of* S.

(ii) A non-empty subset A *of* S *is left (right,two-sided) ideal of* S *if and only if* C_A *is a fuzzy left (right,two-sided) ideal of* S.

(iii) For the non-empty subsets A *and* B *of* S, $C_A \cap C_B = C_{A \cap B}$ *and* $C_A \circ C_B = C_{(AB]}$.

From now on, we write $t, r \in (0, 1]$ *unless otherwise specified.*

A fuzzy subset f *of* S *of the form*

$$f(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \tag{6}$$

is said to be a fuzzy point with support x *and value* t *which is denoted by* x_t. *Hereafter, we shall always use* S *to mean a AG-groupoid.*

Note that for a fuzzy point x_t *and a fuzzy subset* f*, The symbols* $x_t \in f$ *means* $f(x) \geq t$ *and* $x_t q f$ *means* $f(x) + t > 1$*. It is clear that* $(\in, \in \vee q)$*-fuzzy ideals are some special type of* (α, β) *-fuzzy ideals, where* $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ *and* $\alpha \neq \in \wedge q$*.This kind of fuzzy ideals and filters in variouus algebraic systems have been considered in* $[8, 9, 11]$ *and* $[17 - 18]$*, respectively.*

Definition 4. *A fuzzy subset* f *of an AG-groupoid* S *is called an* $(\in, \in \vee q)$*-fuzzy left (right) ideal of* S *if the following conditions hold.*

(*i*) *For all* $x, y \in S$, $x \leq y$, $y_t \in f \Longrightarrow x_t \in \vee q f$.
(*ii*) *For all* $x, y \in S$, $y_t \in f \Longrightarrow (xy)_t \in \vee q f$ $(y_t \in f \Longrightarrow (yx)_t \in \vee q f)$.

Theorem 3.3. *Let* f *be a fuzzy subset of an AG-groupoiid* S*. Then* f *is an* $(\in, \in \vee q)$*-fuzzy left (right) ideal of* S *if and only if the following conditions hold*

(*i*) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge 0.5$, *for all* $x, y \in S$.
(*ii*) $f(xy) \geq f(y) \wedge 0.5$ $(f(xy) \geq f(x) \wedge 0.5)$, *for all* $x, y \in S$.

Proof. The proof is trivial and is hence omitted. ∎

Definition 5. *A fuzzy subset* f *of an AG-groupoid* S *is called an* $(\in, \in \vee q)$*-fuzzy AG-subgroupoid of* S*, if for all* $x, y \in S$, $x_t \in f$ *and* $y_r \in f \Longrightarrow (xy)_{t \wedge r} \in \vee q f$.

Theorem 3.4. *Let* f *be a fuzzy subset of an AG-groupoid* S*. Then* f *is an* $(\in, \in \vee q)$*-fuzzy AG-subgroupoid of* S *if and only if* $f(xy) \geq f(x) \wedge f(y) \wedge 0.5$, *for all* $x, y \in S$.

Proof. The proof is easy and is hence omitted. ∎

Definition 6. *A fuzzy subset* f *of an AG-groupoid* S *is called an* $(\in, \in \vee q)$*-fuzzy generalized bi-ideal of* S *if the following conditions hold*

(*i*) *For all* $x, y \in S$, $x \leq y$, $y_t \in f \Longrightarrow x_t \in \vee q f$.
(*ii*) *For all* $x, y, z \in S$, $x_t \in f$ *and* $z_r \in f \Longrightarrow ((xy)z)_{t \wedge r} \in \vee q f$.

Theorem 3.5. *Let* f *be a fuzzy subset of an AG-groupoid* S*. Then ,*f *is an* $(\in, \in \vee q)$*-fuzzy generalized bi-ideal of* S *if and only if the following conditions hold.*

(*i*) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge 0.5$, *for all* $x, y \in S$.
(*ii*) $f((xy)z) \geq f(x) \wedge f(z) \wedge 0.5$, *for all* $x, y, z \in S$.

Proof. Assume that f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. Let $x, y \in S$ such that $x \leq y$. If $f(y) = 0$, then $f(x) \geq f(y) \wedge 0.5$. Let $f(y) \neq 0$ and assume on contrary that $f(x) < f(y) \wedge 0.5$. Choose an element $t \in (0, 1]$ such that $f(x) < t < f(y) \wedge 0.5$. If $f(y) < 0.5$, then $f(x) < t < f(y)$ so $y_t \in f$ but $f(x) + t < 0.5 + 0.5 = 1$ implies $x_t \overline{q} f$, and therefore, $x_t \overline{\in \vee q} f$, which is a contradiction. Hence, $f(x) \geq f(y) \wedge 0.5$, for all $x, y \in S$.

Suppose on the contrary that there exist $x, y, z \in S$ such that $f((xy)z) < f(x) \wedge f(z) \wedge 0.5$. Choose an element $t \in (0, 1]$ such that $f((xy)z) < t < f(x) \wedge f(z) \wedge 0.5$. Then $f(x) > t$ and $f(z) > t$ implies that $x_t \in f$ and $z_t \in f$ but $f((xy)z) < t$ and $f((xy)z)+t < 0.5+0.5 = 1$ implies that $((xy)z)_{t\wedge r} \overline{q} f$, and so $((xy)z)_{t\wedge r} = ((xy)z)_t \overline{\in \vee q} f$, which is a contradiction. Hence, $f((xy)z) \geq f(x) \wedge f(z) \wedge 0.5$, for all $x, y, z \in S$..

Conversely, let $x, y \in S$ such that $x \leq y$ and $y_t \in f$. Then $f(y) \geq t$ and since $x \leq y$,we have $f(x) \geq f(y) \geq t$ implies that $f(x) \geq t$ which shows that $x_t \in f$ and hence, $x_t \in \vee q f$.

Let $x, y, z \in S$ be such that $x_t \in f$ and $z_r \in f$. Then $f(x) \geq t$ and $f(z) \geq r$. By hypothesis, we have $f((xy)z) \geq f(x) \wedge f(z) \wedge 0.5 \geq t \wedge r \wedge 0.5$. If $t \wedge r \leq 0.5$, then $f((xy)z) \geq t \wedge r$ so $((xy)z)_{t\wedge r} \in f$ and if $t \wedge r > 0.5$, then $f((xy)z) \geq 0.5$ and therefore $f((xy)z) + t \wedge r > 0.5 + 0.5 = 1$. Thus, $((xy)z)_{t\wedge r} q f$ implies $((xy)z)_{t\wedge r} \in \vee q f$. This shows that f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. ∎

Definition 7. *If an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of an AG-groupoid S is also an $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S, then f is called an $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

Theorem 3.6. *An $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if the following conditions hold*

(*i*) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge 0.5$, *for all* $x, y \in S$.
(*ii*) $f((xy)z) \geq f(x) \wedge f(z) \wedge 0.5$, *for all* $x, y, z \in S$.

Proof. The proof follows from Theorems 3.4 and 3.5. ∎

Definition 8. *A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideal of S,*

(*i*) *For all* $x,\ y \in S$, $x \leq y$, $y_t \in f \Longrightarrow x_t \in \vee q f$.

(*ii*) *For all* $a, x,\ y, z \in S$, $x_t \in f$, $y_r \in f$ *and* $z_s \in f \Longrightarrow ((xa)(yz))_{(t\wedge r)\wedge s} \in \vee q f$.

Theorem 3.7. *An $(\in, \in \vee q)$-fuzzy AG-subgroupoid of an AG groupoidS is an $(\in, \in \vee q)$ -fuzzy $(1,2)$-ideal S if and only if the following condiions are satisfied.*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge 0.5$, *for all* $x, y \in S$.
(ii) $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \wedge 0.5$, *for all* $a, x, y, z \in S$.

Proof. The rest of the proof is similar to the proof of Theorem 3.5. ∎

Definition 9. *A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy interior ideal of S by Jun* [10] *if the following conditions hold.*

(i) For all x, $y \in S$, $x \leq y$, $y_t \in f \Longrightarrow x_t \in \vee q f$.
(ii) For all x, $y, z \in S$, $y_t \in f \Longrightarrow ((xy)z)_t \in \vee q f$.

4 $(\in, \in \vee q)$-fuzzy interior ideal of an AG-groupoid

We first characterize the $(\in, \in \vee q)$-fuzzy interior ideal of an AG-groupoid S.

Theorem 4.1. *Let f be a fuzzy subset of S and S an Abel-Grassmann groupoid. Then f is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if the following conditions hold* [10].

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge 0.5$, *for all* $x, y \in S$.
(ii) $f((xy)z) \geq f(y) \wedge 0.5$, *for all* $x, y, z \in S$.

Proof. The proof is similar to the proof of Theorem 3.5. ∎

Definition 10. *A fuzzy subset f is called a $(\in, \in \vee q)$-fuzzy quasi-ideal of an AG-groupoid S if*

$$f(x) \geq (f \circ S)(x) \wedge (S \circ f)(x) \wedge 0.5, \text{ for all } x \in S.$$

Definition 11. *Let f and g be any two fuzzy subsets of an AG-groupoid S. Then the product $f \circ_{0.5} g$ is defined by*

$$(f \circ_{0.5} g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} \{f(y) \wedge g(z) \wedge 0.5\}, & \text{if } A_x \neq \emptyset. \\ 0, & \text{if } A_x = \emptyset. \end{cases}$$

Definition 12. *We will use the symbols $f \cap_{0.5} g$ and $f \cup_{0.5} g$ to mean the following fuzzy subsets of an AG-groupoid S.*

$$\begin{aligned} (f \cap_{0.5} g)(x) &= f(x) \wedge g(x) \wedge 0.5, \text{ for all } x \text{ in } S. \\ (f \cup_{0.5} g)(x) &= f(x) \vee g(x) \vee 0.5, \text{ for all } x \text{ in } S. \end{aligned}$$

Definition 13. *A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy semiprime if for all $x \in S$, $x_t^2 \in f \Longrightarrow x_t \in \vee q f$.*

Definition 14. *An element a of an AG-groupoid S is called a left (right) regular element of S if there exists $x \in S$ such that $a \leq xa^2$ $(a \leq a^2 x)$ and S is called left (right) regular if every element of S is left (right) regular.*

The concepts of left and right regularity coincide in an ordered AG-groupoid with left identity [24].

Example 1. *Let $S = \{a, b, c, d, e\}$ be an ordered AG-groupoid with left identity d defined in the following multiplication table and order below.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

$$\leq := \{(a, a), (a, b), (c, c), (d, d), (e, e), (b, b)\}.$$

Clearly, the groupoid S is left regular. Define a fuzzy subset f of S as follows: $f(a) = 0.9$ and $f(b) = f(c) = f(d) = f(e) = 0.1$. Then it is easy to see that f is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S.

Example 2. *Let $S = \{a, b, c, d, e\}$ be an ordered AG-groupoid with a left identity d. Defined in the following multiplication table with an "order" below.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

$$\leq := \{(a, a), (a, b), (c, c), (a, c), (d, d), (a, e), (e, e), (b, b)\}.$$

Note that the set S is not a left regular AG-groupoid . Indeed, for an element $c \in S$, there does not exist an element $x \in S$ such that $c \leq xc^2$ $(c \leq c^2 x)$.

5 Properties of left regular AG-groupoids

We first prove several lemmas concerning with the left regular AG-groupoids.

Lemma 5.1. *For any fuzzy subset f of a left regular AG-groupoid S, $S \circ_{0.5} f = f_{0.5}$.*

Proof. The proof is straightforward and we omit the proof. ∎

Lemma 5.2. *In a left regular AG-groupoid S, $f \circ_{0.5} S = f_{0.5}$ and $S \circ_{0.5} f = f_{0.5}$ holds for every $(\in, \in \vee q_{0.5})$-fuzzy two-sided ideal f of S.*

Proof. Let S be a left regular AG-groupoid. Now for every $a \in S$, there exists $x \in S$ such that $a \leq a^2 x$. Then we have $a \leq (aa)x = (xa)a$. Thus, $(xa, a) \in A_a$, since $A_a \neq \emptyset$, and therefore, we have

$$\begin{aligned}(f \circ_{0.5} S)(a) &= (f \circ S)(a) \wedge 0.5 = \bigvee_{(xa,a) \in A_a} \{f(xa) \wedge S(a)\} \wedge 0.5 \\ &\geq f(xa) \wedge S(a) \wedge 0.5 \geq f(a) \wedge 1 \wedge 0.5 \geq f(a) \wedge 0.5 = f_{0.5}(a).\end{aligned}$$

It is easy to observe from Lemma 5.1 that $S \circ_{0.5} f = f_{0.5}$ holds for every $(\in, \in \vee q)$-fuzzy two-sided ideal f of S. ∎

Theorem 5.3. *In a left regular AG-groupoid S with left a identity, the following statements are equivalent.*

(*i*) *f is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S.*
(*ii*) *f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

Proof. $(i) \Longrightarrow (ii)$ is trivial .

$(ii) \Longrightarrow (i)$: Let the AG-groupoid S be a left regular with left identity. Then for $b \in S$ there exist $x, y \in S$ such that $b \leq b^2 y$ and $a \leq a^2 x$. Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Then, we can easily deduce the following equalities.

$$\begin{aligned}f(ab) &\geq f(a((bb)y)) = f((bb)(ay)) = f(((ay)b)b) \geq f(((ay)((bb)y))b) \\ &= f(((ay)((yb)b))b) = f(((a(yb))(yb))b) = f(((by)((yb)a))b) \\ &= f(((yb)((by)a))b) = f(((a(by))(by))b) = f((b((a(by))y))b) \\ &\geq f(b) \wedge f(b) \wedge 0.5 = f(b) \wedge 0.5.\end{aligned}$$

This shows that f is a $(\in, \in \vee q)$-fuzzy left ideal of S. Now we have

$$\begin{aligned} f(ab) &\geq f(((aa)x)b) = f(((xa)a)b) = f((ba)(xa)) = f((ax)(ab)) \\ &= f(((ab)x)a) \geq f((((((aa)x)b)x)a) = f(((xb)((aa)x))a) \\ &= f(((aa)((xb)x))a) = f(((a(xb))(ax))a) \\ &= f((a((a(xb))x))a) \geq f(a) \wedge f(a) \wedge 0.5 \\ &= f(a) \wedge 0.5. \end{aligned}$$

Proof. The above condition shows that f is an $(\in, \in \vee q)$-fuzzy right ideal of S and therefore, f is a $(\in, \in \vee q)$-fuzzy two-sided ideal of S. ∎

Lemma 5.4. *A fuzzy subset f of a left regular AG Groupoid S with left identity is an $(\in, \in \vee q)$-fuzzy left ideal of S if and only if it is an $(\in, \in \vee q)$-fuzzy right ideal of S.*

Proof. Assume that S is a left regular AG-groupoid with left identity and let $a, b \in S$. Then there exists $x \in S$ such that $a \leq xa^2$. If f is an $(\in, \in \vee q)$-fuzzy left ideal of S, then we have

$$ab \leq (x(aa))b = (a(xa))b = (b(xa))a \Longrightarrow f(ab) \geq f((b(xa))a) \geq f(a) \wedge 0.5.$$

Similarly, we can show that every $(\in, \in \vee q)$-fuzzy right ideal of S with left identity is an $(\in, \in \vee q)$-fuzzy left ideal of S. ∎

Theorem 5.5. *In a left regular AG-groupoid S with left identity, the following statements are equivalent.*
(i) f is an $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideal of S.
(ii) f is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Assume that f is an $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideal of a left regular AG-groupoid S with left identity and let $a \in S$. Then there exists $y \in S$ such that $a \leq a^2y$. Now , we deduce that

$$\begin{aligned} f(xa) &\geq f(x((aa)y)) = f((aa)(xy)) \geq f((((aa)y)a)(xy)) \\ &= f(((ay)(aa))(xy)) = f(((aa)(ya))(xy)) \\ &= f(((xy)(ya))(aa)) = f(((ay)(yx))a^2) \\ &\geq f(((((aa)y)y)(yx))a^2) = f((((yy)(aa))(yx))a^2) \\ &= f((((aa)y^2)(yx))a^2) = f((((yx)y^2)(aa))a^2) \\ &= f((a(((yx)y^2)a))(aa)) \geq f(a) \wedge f(a) \wedge f(a) \wedge 0.5 \\ &= f(a) \wedge 0.5. \end{aligned}$$

The above properties lead to f is an $(\in, \in \vee q)$-fuzzy left ideal of an AG-groupoid S and by Lemma 5.4, f is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S. $(ii) \Longrightarrow (i)$ is obvious. ∎

Lemma 5.6. *In a left regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

(ii) f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.

Proof. $(i) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$: Let S be a left regular AG-groupoid with left identity and let $a \in S$. Then there exists $x \in S$ such that $a \leq a^2 x$. Let f be an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. Then, we have

$$\begin{aligned} f(ab) &\geq f(((aa)x)b) = f(((aa)(ex))b) = f(((xe)(aa))b) \\ &= f((a((xe)a))b) \geq f(a) \wedge f(b) \wedge 0.5. \end{aligned}$$

This shows that f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. ∎

Theorem 5.7. *In a left regular AG-groupoid S with left identity, the following statements are equivalent (see* [14] *(i) f is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

(ii) f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Proof. Let S be a left regular AG-groupoid with left identity. Then for any a, b, x and $y \in S$, there exist a', b', $x'$and $y' \in S$ such that $a \leq a^2 a'$, $b \leq b^2 b'$, $x \leq x^2 x'$and $y \leq y^2 y'$.

$(i) \Longrightarrow (ii)$: Let f be an $(\in, \in \vee q)$-fuzzy interior ideal of S with left identity. Then, we deduce the following equalities. ∎

$$\begin{aligned} f((xa)y) &\geq f((((xx)x')a)y) = f((((x'x)x)a)y) = f(((ax)(x'x))y) \\ &= f(((xx')(xa))y) = f((((xa)x')x)y) \geq f(x) \wedge 0.5. \end{aligned}$$

Also, we have the following equalities :

$$f((xa)y) \geq f((xa)((yy)y')) = f((yy)((xa)y')) = f((((xa)y')y)y) \geq f(y) \wedge 0.5.$$

The above equalities imply that $f((xa)y) \geq f(x) \wedge f(y) \wedge 0.5$. Now, we deduce the following equalities:

$$f(ab) \geq f(((aa)a')b) = f(((a'a)a)b) = f((ba)(a'a)) \geq f(a) \wedge 0.5$$

and

$$f(ab) \geq f(a((bb)b')) = f((bb)(ab')) \geq f(b) \wedge 0.5.$$

Thus, f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

$(ii) \Longrightarrow (i)$: Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S with left identity. Then we have

$$\begin{aligned} f((xa)y) &\geq f((x((aa)a^{'}))y) = f(((aa)(xa^{'}))y) = f(((ax)(aa^{'}))y) \\ &= f((y(aa^{'}))(ax)) = f((a(ya^{'}))(ax)) = f(((ax)(ya^{'}))a) \\ &= f(((a^{'}y)(xa))a) \geq f(((a^{'}y)(x((aa)a^{'})))a) \\ &= f(((a^{'}y)((aa)(xa^{'})))a) = f(((a^{'}y)((a^{'}x)(aa)))a) \\ &= f(((a^{'}y)(a((a^{'}x)a)))a) = f((a((a^{'}y)((a^{'}x)a)))a) \\ &\geq f(a) \wedge f(a) \wedge 0.5 = f(a) \wedge 0.5. \end{aligned}$$

This shows that f is an $(\in, \in \vee q)$-fuzzy interior ideal of S. ∎

Theorem 5.8. *In a left regular AG-groupoid S with left identity, the following statements are equivalent.*
(i) f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.
(ii) f is an $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let S be a left regular AG groupoid with a left identity and $x, a, y, z \in S$. Then, there exists $x^{'} \in S$ such that $x \leq x^2x^{'}$. Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Then we have the following equailties:

$$\begin{aligned} f((xa)(yz)) &= f((zy)(ax)) = f(((ax)y)z) \geq f((ax)y) \wedge f(z) \wedge 0.5 \\ &\geq f(ax) \wedge f(y) \wedge f(z) \wedge 0.5 \wedge 0.5 \\ &= f(ax) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &\geq f(a((xx)x^{'})) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &= f((xx)(ax^{'})) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &= f(((ax^{'})x)x) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &\geq f(((ax^{'})((xx)x^{'}))x) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &= f(((ax^{'})((xx)(ex^{'})))x) \wedge f(y) \wedge f(z) \wedge 0.5 \end{aligned}$$

$$\begin{aligned} &= f(((ax^{'})((x^{'}e)(xx)))x) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &= f(((ax^{'})(x((x^{'}e)x)))x) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &= f(x((ax^{'})((x^{'}e)x))x) \wedge f(y) \wedge f(z) \wedge 0.5 \\ &\geq f(x) \wedge f(x) \wedge 0.5 \wedge f(y) \wedge f(z) \wedge 0.5 \\ &= f(x) \wedge f(y) \wedge f(z) \wedge 0.5. \end{aligned}$$

This shows that f is an $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideal of S.

$(ii) \Longrightarrow (i)$: Again if S is a left regular semigroup with a left identity, then for any a, b, x and $y \in S$, there exist $a^{'}$, $b^{'}$, $x^{'}$and $y^{'} \in S$ such that $a \leq a^2a^{'}$,

$b \leq b^2 b'$, $x \leq x^2 x'$ and $y \leq y^2 y'$. Let f be an $(\in, \in \vee q)$-fuzzy $(1,2)$-ideal of S. Then we have

$$\begin{aligned} f((xa)y) &\geq f((xa)((yy)y')) = f((yy)((xa)y')) = f((y'(xa))(yy)) \\ &= f((x(y'a))(yy)) \geq f(x) \wedge f(y) \wedge f(y) \wedge 0.5 \geq f(x) \wedge f(y) \wedge 0.5. \end{aligned}$$

Now , we deduce the following equalities:

$$\begin{aligned} f(ab) &\geq f(a((bb)b')) = f((bb)(ab')) = f((b'a)(bb)) \geq f(b'((aa)a'))(bb) \\ &= f((aa)(b'a'))(bb) = f((a'b')(aa))(bb) \geq f(a((a'b')a))(bb) \wedge 0.5 \\ &= f(a) \wedge f(b) \wedge f(b) \wedge 0.5 = f(a) \wedge f(b) \wedge 0.5. \end{aligned}$$

This shows that f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. ∎

Theorem 5.9. *In a left regular AG-groupoid S with left identity, the following statements are equivalent*

(*i*) f *is an* $(\in, \in \vee q)$*-fuzzy* $(1,2)$*-ideal of* S.

(*ii*) f *is an* $(\in, \in \vee q)$*-fuzzy interior ideal of* S.

Proof. $(i) \Longrightarrow (ii)$: Let S be a left regular AG-groupoid with left identity and let $x, a, y, z \in S$, then there exists $a' \in S$ such that $a \leq a^2 a'$. Let f be an $(\in, \in \vee q)$-fuzzy $(1,2)$-ideal of S, then we have

$$\begin{aligned} f((xa)(yz)) &\geq f((x((aa)a'))(yz)) = f(((aa)(xa'))(yz)) = f((((xa')a)a)(yz)) \\ &\geq f((((xa')((aa)a'))a)(yz)) = f((((aa)((xa')a'))a)(yz)) \\ &= f(((yz)a)((aa)((xa')a'))) = f((aa)(((yz)a)((xa')a'))) \\ &= f(((aa)(a'(xa')))(a(yz))) = f(((a(yz))(a'(xa')))(aa)) \\ &\geq f(((((aa)a')(yz))(a'(xa')))(aa)) \\ &= f(((((a'a)a)(yz))(a'(xa')))(aa)) \\ &= f((((xa')a')((yz)((a'a)(ea))))(aa)) \\ &= f((((xa')a')((yz)((ae)(aa'))))(aa)) \\ &= f((((xa')a')((yz)(a((ae)a'))))(aa)) \end{aligned}$$

$$\begin{aligned} &= f((((xa')a')(a((yz)((ae)a'))))(aa)) \\ &= f((a(((xa')a')((yz)((ae)a'))))(aa)) \\ &\geq f(a) \wedge f(a) \wedge f(a) = f(a) \wedge 0.5. \end{aligned}$$

This shows that f is an $(\in, \in \vee q)$-fuzzy interior ideal of S.

$(ii) \Longrightarrow (i)$: Again let S be a left regular with left identity and let $x, a, y, z \in S$, then there exist x' and $z' \in S$ such that $x \leq x^2 x'$ and $z \leq z^2 z$. Now we have

$$f((xa)(yz)) = f((zy)(ax)) \geq f(y) \wedge 0.5.$$

Now.

$$\begin{aligned} f((xa)(yz)) &\geq f((((xx)x^{'})a)(yz)) = f(((ax^{'})(xx))(yz)) = f(((xx)(x^{'}a))(yz)) \\ &= f((((x^{'}a)x)x)(yz)) \geq f(x) \wedge 0.5. \end{aligned}$$

Now we deduce that

$$\begin{aligned} f((xa)(yz)) &\geq f((xa)(y(((zz)z^{'})))) = f((xa)((zz)(yz^{'}))) \\ &= f((zz)((xa)(yz^{'}))) \geq f(z) \wedge 0.5. \end{aligned}$$

Thus, we get $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \wedge 0.5$.

Let a and $b \in S$ then there exist $a^{'}$ and $b^{'} \in S$ such that $a \leq a^2a^{'}$ and $b \leq b^2b^{'}$. Now, we deduce that

$$f(ab) \geq f(((aa)a^{'})b) = f((ba^{'})(aa)) = f((aa)(a^{'}b)) \geq f(a) \wedge 0.5$$

and

$$f(ab) = f(a((bb)b^{'})) = f((bb)(ab^{'})) \geq f(b) \wedge 0.5.$$

Thus, f is an $(\in, \in \vee q)$-fuzzy $(1,2)$-ideal of S. ∎

Note that $(\in, \in \vee q)$-fuzzy two-sided ideals, $(\in, \in \vee q)$-fuzzy bi-ideals, $(\in, \in \vee q)$-fuzzy generalized bi-ideals, $(\in, \in \vee q)$-fuzzy $(1,2)$-ideals, $(\in, \in \vee q)$-fuzzy interior ideals and $(\in, \in \vee q)$-fuzzy quasi-ideals coincide in a left regular ordered AG-groupoid with a left identity.

Lemma 5.10. *If f is an $(\in, \in \vee q)$-fuzzy quasi ideal of a left regular AG groupoid S, then $(f \circ S) \cap_{0.5} (S \circ f) = f_{0.5}$.*

Proof. The proof follows from Lemma 5.2. ∎

Theorem 5.11. *Let S be a left regular AG-groupoid S with left identity. Then the following statements are equivalent.*

(i) f is an $(\in, \in \vee q)$-fuzzy left ideal of S.
(ii) f is an $(\in, \in \vee q)$-fuzzy right ideal of S.
(iii) f is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S.
(iv) f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.
(v) f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.
(vi) f is an $(\in, \in \vee q)$-fuzzy $(1,2)$-ideal of S.
(vii) f is an $(\in, \in \vee q)$-fuzzy interior ideal of S.
(viii) f is an $(\in, \in \vee q)$-fuzzy quasi ideal of S.
(ix) $f \circ_{0.5} S = f_{0.5}$ and $S \circ_{0.5} f = f_{0.5}$.

Proof. $(i) \Longrightarrow (ix)$: Let f be an $(\in, \in \vee q)$-fuzzy left ideal of a left regular AG groupoid S with left identity. Let $a \in S$ then there exists $a^{'} \in S$ such that $a \leq a^2 a^{'}$. Now

$$a \leq (aa)a^{'} = (a^{'}a)a \text{ and } a \leq (aa)a^{'} = (aa)(ea^{'}) = (a^{'}e)(aa).$$

Thus, $(a^{'}a, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}(f \circ_{0.5} S)(a) &= \bigvee_{(a^{'}a,a)\in A_a} \left\{ f(a^{'}a) \wedge S(a) \wedge 0.5 \right\} \geq f(a^{'}a) \wedge 1 \wedge 0.5 \\ &\geq f(a) \wedge 0.5 = f_{0.5}(a)\end{aligned}$$

and also $(a^{'}e, aa) \in A_a$, since $A_a \neq \emptyset$, we have

$$\begin{aligned}(S \circ_{0.5} f)(a) &= \bigvee_{(a^{'}e,aa)\in A_a} \left\{ S(a^{'}e) \wedge f(aa) \wedge 0.5 \right\} \geq 1 \wedge f(aa) \wedge 0.5 \\ &\geq f(a) \wedge 0.5 = f_{0.5}(a).\end{aligned}$$

Now by using Lemmas 5.4 , we get the required $f \circ_{0.5} S = f_{0.5}$ and $S \circ_{0.5} f = f_{0.5}$.

$(ix) \Longrightarrow (viii)$ is obvious.

$(viii) \Longrightarrow (vii)$: Let f be an $(\in, \in \vee q)$-fuzzy quasi ideal of a left regular S. Now for $a \in S$ there exists $a^{'} \in S$ such that $a \leq a^2 a^{'}$ and therefore

$$(xa)y = (xa)(ey) = (ye)(ax) = a((ye)x)$$

and

$$(xa)y \leq (x((aa)a^{'}))y = ((aa)(xa^{'}))y = ((a^{'}x)(aa))y = (a((a^{'}x)a))y = (y((a^{'}x)a))a.$$

Since f is an $(\in, \in \vee q)$-fuzzy quasi ideal of S, by Lemma 5.10, we deduce the following equalities:

$$f_{0.5}((xa)y) = ((f \circ S) \cap_{0.5} (S \circ f))((xa)y) = (f \circ S)\,((xa)y) \wedge (S \circ f)\,((xa)y) \wedge 0.5.$$

Since $(a, (ye)x) \in A_a$, and $A_a \neq \emptyset$, we can easily verify the following:

$$(f \circ S)\,((xa)y) = \bigvee_{(a,(ye)x)} \{f(a) \wedge S((ye)x)\} \geq f(a)$$

and also $(y((a^{'}x)a), a) \in A_a$, since $A_a \neq \emptyset$, we have

$$(S \circ f)\,((xa)y) = \bigvee_{(y((a^{'}x)a),a)\in A_a} \left\{ S(y((a^{'}x)a)) \wedge f(a) \right\} \geq f(a).$$

This shows that $f_{0.5}((xa)y) \geq f(a) \wedge 0.5 \Longrightarrow f((xa)y) \geq f(a) \wedge 0.5$. Thus,$f$ is an $(\in, \in \vee q)$-fuzzy interior ideal of S.

$(vii) \Longrightarrow (vi)$ can be followed by Theorem 5.9.

$(vi) \Longrightarrow (v)$ can be followed by Theorem 5.8.

$(v) \Longrightarrow (iv)$ can be followed by Lemma 5.6.

$(iv) \Longrightarrow (iii)$ can be followed by Theorem 5.3.

$(iii) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (i)$ are an easy consequences of Lemma 5.4. ∎

6 Characterizations of left regular ordered Abel-Grassmann groupoids

In this section, our aim is to give a characterization theorem of a left regular ordered AG-groupoid. We first prove the following crucial lemma.

Lemma 6.1. *Every $(\in, \in \vee q)$-fuzzy left ideal of an ordered AG-groupoid S with left identity becomes an $(\in, \in \vee q)$-fuzzy left ideal of S.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy right ideal of an ordered AG-groupoid S with left identity. Now we have the following equalities:

$$f(ab) = f((ea)b) = f((ba)e) \geq f(b) \wedge 0.5.$$

Therefore, f is an $(\in, \in \vee q)$-fuzzy left ideal of S. ∎

The converse of the above lemma is in general not true. To verify the above claim, we first define a fuzzy subset f of S in Example 2 as follows: $f(a) = 0.8$, $f(b) = 0.5$, $f(c) = 0$, $f(d) = 0.3$ and $f(e) = 0.6$. Then it is easy to observe that f is an $(\in, \in \vee q)$-fuzzy left ideal of S but it is not an $(\in, \in \vee q)$-fuzzy right ideal of S, because $f(bd) \neq f(b) \wedge 0.5$.

Lemma 6.2. *The following properties hold in an ordered AG-groupoid S.*

(i) A is an Abel Grasmann subgroupoid of S if and only if $(C_A)_{0.5}$ is an $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S.

(ii) A is a left (right, two-sided) ideal of S if and only if $(C_A)_{0.5}$ is an $(\in, \in \vee q)$-fuzzy left (right, two-sided) ideal of S.

(iii) For any non-empty subsets A and B of S, $C_A \circ_{0.5} C_B = (C_{(AB]})_{0.5}$ and $C_A \cap_{0.5} C_B = (C_{A\cap B})_{0.5}$.

Proof. The proof is straightforward. ∎

Lemma 6.3. *The following conditions are equivalent for an AG-groupoid S with left identity.*

(i) S is left regular.

(ii) $f \circ_{0.5} f = f_{0.5}$ *for every* $(\in, \in \vee q)$*-fuzzy left (right, two-sided) ideal of* S.

Proof. $(i) \Longrightarrow (ii)$: Let f be an $(\in, \in \vee q)$-fuzzy left ideal of S, then clearly $f \circ_{0.5} f \leq f_{0.5}$. Let $a \in S$. Then since S is left regular so, $a \leq (aa)x = (xa)a$. Thus $(xa, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}(f \circ_{0.5} f)(a) &= \bigvee_{(xa,a)\in A_a} \{f(xa) \wedge f(a) \wedge 0.5\} \geq f(a) \wedge f(a) \wedge 0.5 \wedge 0.5 \\ &= f(a) \wedge 0.5 = f_{0.5}(a) \Longrightarrow f \circ_{0.5} f = f_{0.5}.\end{aligned}$$

$(ii) \Longrightarrow (i)$: Assume that $f \circ_{0.5} f = f_{0.5}$ holds for $(\in, \in \vee q)$-fuzzy left ideal of S with left identity. Since $(Sa]$ is a left ideal of S ,by Lemma 6.2, we have $C_{(Sa]})_{0.5}$ is an $(\in, \in \vee q)$-fuzzy left ideal of S. Since $a \in (Sa]$, $(C_{Sa})_{0.5}(a) = 0.5$. Now by using the given assumption and Lemma 6.2, we deduce that

$$(C_{(Sa]})_{0.5} \circ_{0.5} (C_{(Sa]})_{0.5} = (C_{(Sa]})_{0.5} \text{ and } (C_{(Sa]})_{0.5} \circ_{0.5} (C_{(Sa]})_{0.5} = \left(C_{((Sa](Sa]]}\right)_{0.5}.$$

Thus, we obtain the following equality $\left(C_{((Sa](Sa]]}\right)_{0.5}(a) = (C_{(Sa]})_{0.5}(a) = 0.5$, which implies that $a \in ((Sa](Sa]]$. Now , we have the following equality

$$a \in ((Sa](Sa]] \subseteq (((Sa)(Sa)]] = ((Sa)(Sa)] = ((aS)(aS)] = (a^2S]. \text{ (see [13])}$$

This shows that the AG-groupoid S is left regular. ∎

We now characterize an AG-groupoid S with a left identity by using its $(\in, \in \vee q)$*-fuzzy left (right, two-sided) ideals.*

Theorem 6.4. *The following conditions are equivalent for an AG-groupoid S with a left identity.*

(i) S is a left regular AG-groupoid.

(ii) $f_{0.5} = (S \circ f) \circ_{0.5} (S \circ f)$, *where f is any* $(\in, \in \vee q)$*-fuzzy left (right, two-sided) ideal of S.*

Proof. $(i) \Longrightarrow (ii)$: Let S be a left regular AG-groupoid and f any $(\in, \in \vee q)$-fuzzy left ideal of S. Then,it is clear that $S \circ f$ is also an $(\in, \in \vee q)$-fuzzy left ideal of S. Now, by Lemma 6.3, we have

$$(S \circ f) \circ_{0.5} (S \circ f) = (S \circ f) \wedge 0.5 \subseteq f \wedge 0.5 = f_{0.5}.$$

Now let $a \in S$, since S is a left regular groupoid, there exists $x \in S$ such that $a \leq a^2 x$ and

$$a \leq (aa)x = (xa)a \leq (xa)((aa)x) = (xa)((xa)a).$$

Thus, $(xa, (xa)a) \in A_a$, since $A_a \neq \emptyset$. Therefore,

$$\begin{aligned} ((S \circ f) \circ_{0.5} (S \circ f))(a) &= \bigvee_{(xa,(xa)a) \in A_a} \{(S \circ f)(xa) \wedge (S \circ f)((xa)a) \wedge 0.5\} \\ &\geq (S \circ f)(xa) \wedge (S \circ f)((xa)a) \wedge 0.5 \end{aligned}$$

$$\begin{aligned} &= \bigvee_{(x,a) \in A_{xa}} \{S(x) \wedge f(a)\} \wedge \bigvee_{(xa,a) \in A_{(xa)a}} \{S(xa) \wedge f(a)\} \wedge 0.5 \\ &\geq S(x) \wedge f(a) \wedge S(xa) \wedge f(a) \wedge 0.5 \\ &= f(a) \wedge 0.5 = f_{0.5}(a). \end{aligned}$$

Thus, we obtain our required equality $f_{0.5} = (S \circ f) \circ_{0.5} (S \circ f)$.

Soppose that the equality $f = (\mathcal{S} \circ f)^2$ holds for any fuzzy left ideal f of $\mathcal{S}$. Then, by the given assumption, we have

$$f = (\mathcal{S} \circ f)^2 \subseteq f^2 = f \circ f \subseteq \mathcal{S} \circ f \subseteq f.$$

$(ii) \Longrightarrow (i)$: Let $f_{0.5} = (S \circ f) \circ_{0.5} (S \circ f)$ holds for every $(\in, \in \vee q)$-fuzzy left ideal f of S. Then by the given assumption, we deduce the following equality.

$$f_{0.5} = (S \circ f) \circ_{0.5} (S \circ f) \subseteq f \circ_{0.5} f \subseteq S \circ_{0.5} f \leq f_{0.5}.$$

Thus ,by Lemma 6.3, S is a left regular AG-groupoid. ∎

We now define a fuzzy subset f of S in Example 2 as follows: $f(a) = 0.2$, $f(b) = 0.5$, $f(c) = 0.6$, $f(d) = 0.1$ and $f(e) = 0.4$, then f is an $(\in, \in \vee q)$-fuzzy semiprime.

Lemma 6.5. *Let f be a fuzzy subset of an AG-groupoid S. Then f is an $(\in, \in \vee q)$-fuzzy semiprime if and only if $f(x) \geq \{f(x^2) \wedge 0.5\}$, for all $x \in S$.*

Proof. The proof is trivial and is hence omitted. ∎

Lemma 6.6. *For a left regular AG-groupoid S, the following conditions holds.*

(i) Every $(\in, \in \vee q)$-fuzzy right ideal of S is an $(\in, \in \vee q)$-fuzzy semiprime.

(ii) Every $(\in, \in \vee q)$-fuzzy left ideal of S is an $(\in, \in \vee q)$-fuzzy semiprime if S has a left identity.

Proof. (i) : The proof is straightforward.

(ii) : Let f be an $(\in, \in \vee q)$-fuzzy left ideal of a left regular AG groupoiid S with left identity and let $a \in S$. Then, there exists $x \in S$ such that $a \leq a^2x$. Now , we have

$$f(a) \geq f((aa)(ex)) = f((xe)a^2) \geq f(a^2) \wedge 0.5.$$

The above equality shows that f is an $(\in, \in \vee q)$-fuzzy semiprime. ∎

Lemma 6.7. *A right (left, two-sided) ideal R of an AG-groupoid S is semiprime if and only if $(C_R)_{0.5}$ is $(\in, \in \vee q)$-fuzzy semiprime.*

Proof. Let R be any right ideal of an AG groupoid S. Then, by Lemma 6.2, $(C_R)_{0.5}$ is a $(\in, \in \vee q)$-fuzzy right ideal of S. Now let $a \in S$ then by given assumption $(C_R)_{0.5}(a) \geq (C_R)_{0.5}(a^2)$. Let $a^2 \in R$. Then $(C_R)_{0.5}(a^2) = 0.5 \Longrightarrow (C_R)_{0.5}(a) = 0.5$, which implies that $a \in R$. Thus, every right ideal of S is semiprime. The converse of the above lemma is simple.

Similarly, every left and two-sided ideal of S is semiprime if and only if their characteristic functions are $(\in, \in \vee q)$-fuzzy semiprime. ∎

Lemma 6.8. *Every right (left, two-sided) ideal of an AG groupoid S is semiprime if every fuzzy right (left, two-sided) ideal of S is an $(\in, \in \vee q)$-fuzzy semiprime (see* [13].

Proof. The direct part easily follows by Lemma 6.7. ∎

The converse of the above Lemma is in general not true.To prove our claim, we first consider the groupoid S in Example 2. It is easy to observe that the only left ideals of S are $\{a, b, e\}$, $\{a, c, e\}$, $\{a, b, c, e\}$ and $\{a, e\}$ which are semiprime. Clearly, the right and two-sided ideals of S are $\{a, b, c, e\}$ and $\{a, e\}$ which are also semiprime. Now on the other hand, if we define a fuzzy subset f of S as follows: $f(a) = f(b) = f(c) = 0.2$, $f(d) = 0.1$ and $f(e) = 0.3$, then f is an $(\in, \in \vee q)$-fuzzy right (left, two-sided) ideal of S but f is not an $(\in, \in \vee q)$-fuzzy semiprime. Indeed, $f(c) \neq f(c^2) \wedge 0.5$.

Lemma 6.9. *The following statements are equivalent for an AG-groupoid S with a left identity.*

(i) *It is clear that S is a left regular AG-groupoid.*

(ii) *Every $(\in, \in \vee q)$-fuzzy right (left, two-sided) ideal of S is an $(\in, \in \vee q)$-fuzzy semiprime.*

Proof. $(i) \Longrightarrow (ii)$ This part follows by Lemma 6.6.

$(ii) \Longrightarrow (i)$: Let every $(\in, \in \vee q)$-fuzzy right (left, two-sided) ideal of the AG-groupoid S with left identity be a $(\in, \in \vee q)$-fuzzy semiprime ideal od S. Since $(a^2S]$ is a right and also a left ideal of S, therefore, by Lemma 6.8, we can easily see that $(C_{(a^2S]})_{0.5}$ is an $(\in, \in \vee q)$-semiprime. Now, it is clear that $a^2 \in (a^2S]$, therefore, $a \in (a^2S]$, which shows that the groupoid S is left regular. ∎

In closing this paper, we state a characterization theorem of an AG-groupoid S.

Theorem 6.10. *The following statements are equivalent for an Abel-Grassmann groupoid S with left identity.*

(i) *The AG-groupoid S is left regular.*
(ii) *Every $(\in, \in \vee q)$-fuzzy right ideal of S is $(\in, \in \vee q)$-fuzzy semiprime.*
(iii) *Every $(\in, \in \vee q)$-fuzzy left ideal of S is $(\in, \in \vee q)$-fuzzy semiprime.*

Proof. $(i) \Longrightarrow (iii)$ and $(ii) \Longrightarrow (i)$ can be followed by Lemma 6.9.

$(iii) \Longrightarrow (ii)$: Let f be an $(\in, \in \vee q)$-fuzzy right ideal of S with left identity. Then , by Lemma 6.1, f is an $(\in, \in \vee q)$-fuzzy left ideal of S and therefore by the given assumption , f is an $(\in, \in \vee q)$-fuzzy semiprime. ∎

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