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Counting One-Codimensional Subspaces Generated by Subsets of a Root System

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Abstract

Let Φ be an irreducible (and reduced) root system of a Euclidean space. In this paper, we give a complete classification of one-codimensional subspaces generated by a subset of Φ, and we also give explicitely a basis of such each subspace for classical-type. This is motivated by a classification theory of semisimple finite prehomogeneous vector spaces and we explain an example for an application to it.

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1 Introduction

Let Φ be an irreducible (and reduced) root system of a finite-dimensional Euclidean space V. In this paper, we give a complete classification of the refined lattices of corank one (or, one-codimensional subspaces of V) generated by a subset of Φ. We refer to Bourbaki [1] for terminology on root system. As mentioned in §7, our classification of such lattices (or, subspaces) is motivated by a classification theory [3] of semisimple "finite-prehomogeneous" vector spaces. We have counted such lattices as distinct sets:

Theorem 1.1. *The number of refined lattices of corank one* (*or, one-codimensional subspaces of* V) *generated by a subset of an irreducible* (*and reduced*) *root system* Φ *of a Euclidean space* V *is given by the following table:*

Φ	*# of subspaces*
$\mathbb{A}_n$ $(n \geq 2)$	$2^n - 1$
$\mathbb{B}_n$ $(n \geq 2)$	$(3^n - 1)/2$
$\mathbb{C}_n$ $(n \geq 2)$	$(3^n - 1)/2$
$\mathbb{D}_n$ $(n \geq 4)$	$(3^n - n\,2^{n-1} - 1)/2$

$\mathbb{E}_6$	639
$\mathbb{E}_7$	8,821
$\mathbb{E}_8$	440,880
$\mathbb{F}_4$	120
$\mathbb{G}_2$	6

This theorem can be immediately obtained as a corollary of our theorems for classical-type (see Theorems 3.5, 4.6, and 5.4). In this case, we also give a basis of each corank-one lattice explicitely, as numerical vectors of a Euclidean space. Our approach is based on an observation of standard form of matrices obtained by arranging some roots. The assertion for $\mathbb{G}_2$-type is trivial. The numbers for other exceptional-type has been calculated by computer. That is not hard work; so we will explain only a strategy of computation in §6. We note that it is easier to count the refined lattices of corank one rather than the one-codimensional subspaces.

In particular, when the Dynkin diagram Γ of Φ is simply laced (i.e., it does not have an arrow), our classification is nothing but a classification of dimension vectors which give a necessary and sufficient condition whether a scalar-removed representation associated with Γ is "finite-prehomogeneous" or not; we will give an example in §7 for an application.

On the other hand, Oshima [4] has completely classified the subsystems of a root system. Our result corresponds to his classification of the so-called L-closed subsystems of a root system (see [4, Definition 6.6]). Note that the decomposition of a matrix appearing our theorems corresponds to the decomposition of a reducible root system of a one-codimensional subspace. In fact, the numbers that are presented in Theorem 1.1 can be derived from his result.

2 Preliminaries

Let Φ be an irreducible (and reduced) root system of $V = \mathbb{R}^l$. For vectors $w_1, w_2, \ldots, w_m \in V$, we let L, $(L)_\mathbb{R}$ be the $\mathbb{Z}$-module, $\mathbb{R}$-vector subspace generated by them, respectively;

$$L = \langle w_1, w_2, \ldots, w_m \rangle_\mathbb{Z} = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \cdots + \mathbb{Z}w_m,$$
$$(L)_\mathbb{R} = \langle w_1, w_2, \ldots, w_m \rangle_\mathbb{R} = \mathbb{R}w_1 + \mathbb{R}w_2 + \cdots + \mathbb{R}w_m.$$

We simply call such a $\mathbb{Z}$-module L a *lattice*. The rank (resp. corank) of L is defined as the dimension (resp. codimension) of its corresponding subspace $(L)_\mathbb{R}$. Let $(L)_\mathbb{R}^\perp$ be the ortho-complement of $(L)_\mathbb{R}$ with respect to the standard inner product (or, dot product) on $V = \mathbb{R}^l$. In the case where $(L)_\mathbb{R}$ is of codimension one, we call a basis of $(L)_\mathbb{R}^\perp$ a *normal vector* of L or $(L)_\mathbb{R}$.

Now we define the lexicographical order on $V = \mathbb{R}^l$ with respect to the standard basis; i.e., we have $\boldsymbol{x} > \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in V$ if the first non-zero entry of $\boldsymbol{x} - \boldsymbol{y}$ is positive. We denote by Φ^+ the set of all positive roots of Φ. Let L be a lattice generated by a subset of Φ, and we put $p = \dim(L)_\mathbb{R}$ and $\Psi^+ = (L)_\mathbb{R} \cap \Phi^+$. We will choose p positive roots $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_p$ as follows:

$$\boldsymbol{\alpha}_1 := \min \Psi^+, \text{ and } \boldsymbol{\alpha}_k := \min\bigl(\Psi^+ \smallsetminus \langle \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{k-1} \rangle_\mathbb{R}\bigr) \text{ for } k = 2, 3, \ldots, p.$$

Note that $\Psi = (L)_\mathbb{R} \cap \Phi$ is a root system of $(L)_\mathbb{R}$ and that $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_p$ is nothing but the simple roots of Ψ with respect to the lexicographical order. When we apply our result to classification theory of semisimple finite prehomogeneous vector spaces, the next lemma is fundamental:

Lemma 2.1. *For any element $\boldsymbol{\alpha} \in \Psi^+$, there exist non-negative integers $k_1, k_2, \ldots, k_p$ such that $\boldsymbol{\alpha} = k_1\boldsymbol{\alpha}_1 + k_2\boldsymbol{\alpha}_2 + \cdots + k_p\boldsymbol{\alpha}_p$.*

These $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_p$ are uniquely determined by L; so we will say that the lattice spanned by them is the *refinement* of L, and denote it $\mathrm{Ref}(L)$. We have $L \subseteq \mathrm{Ref}(L)$ in general; and we say that L is *refined* if $L = \mathrm{Ref}(L)$. For two lattices L_1, L_2 generated by positive roots respectively, we have $\mathrm{Ref}(L_1) = \mathrm{Ref}(L_2)$ if and only if $(L_1)_\mathbb{R} = (L_2)_\mathbb{R}$; in the case where L_1, L_2 are of corank one, this condition is equivalent to that the normal vectors of L_1 and L_2 are equal, up to non-zero constant. That is to say, the subspaces of codimension one (or the refined lattices of corank one) are distinguished by their normal vectors.

Here we will prepare a few terminology and notation needed later. For a column vector $\boldsymbol{x} = (x_i)_{i=1}^n \in \mathbb{R}^n$, we put $\operatorname{supp} \boldsymbol{x} = \{i;\ x_i \neq 0\}$. Then we will call $s(\boldsymbol{x}) = \min \operatorname{supp} \boldsymbol{x}$ the starting of $\boldsymbol{x}$ and $e(\boldsymbol{x}) = \max \operatorname{supp} \boldsymbol{x}$ its ending. For vectors $w_1, w_2, \ldots, w_m \in \mathbb{R}^n$, we denote by $L(w_1, w_2, \ldots, w_m)$ the vector space generated by them, and we put $M(w_1, w_2, \ldots, w_m)$ the matrix of size $n \times m$ arranged $w_1, w_2, \ldots, w_m$ as columns; that is,

$$L(w_1, w_2, \ldots, w_m) = \langle w_1, w_2, \ldots, w_m \rangle_{\mathbb{R}}, \; M(w_1, w_2, \ldots, w_m) = \big[w_1 \big| w_2 \big| \cdots \big| w_m \big].$$

We write $A \simeq A'$ if a matrix A' can be obtained from a matrix A by right-elementary transformation (i.e., elementary transformation for columns). In addition, we write $B \sim B'$ if a matrix B' can be obtained from a matrix B by right-elementary transformation and replacing rows.

Let M be a matrix of size $(n+1) \times (n-1)$. If it can be expressed, by appropriate right-elementary transformation and replacing rows, as follows:

$$M \sim \left[\begin{array}{c|c} X & O \\ \hline O & Y \end{array} \right], \tag{2.1}$$

where X is of size $(r+1) \times r$ and Y of size $(n-r) \times (n-r-1)$ with $1 \le r \le n-2$, then we will say that M is *reducible of type* A. Similarly we say a matrix M of size $n \times (n-1)$ is *reducible of type* D if M can be expressed in the form (2.1) with a square matrix X of degree r and Y of size $(n-r) \times (n-r-1)$ with $2 \le r \le n-2$. For simplicity we will sometimes omit the words "of type $\mathbb{A}$ (or $\mathbb{D}$)" when there is no confusion.

We say that a matrix A is sincere if A does not contain a row consisting of zeros. A matrix A of size $m \times n$ is called full-rank if $\operatorname{rank} A = \min\{m, n\}$.

3 One-codimensional subspaces of type $\mathbb{A}_n$

In this section, we denote by $\Phi^{(n+1)} = \Phi(\mathbb{A}_n)$ the root system of type $\mathbb{A}_n$. We denote by $E^{(n+1)} = \{\boldsymbol{x} = (x_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1};\; x_1 + x_2 + \cdots + x_{n+1} = 0\}$ the n-dimensional subspace perpendicular to the vector ${}^t(1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$. We may regard $\Phi^{(n+1)}$ as a finite subset of $E^{(n+1)}$:

$$\Phi^{(n+1)} = \Phi(\mathbb{A}_n) = \big\{ \pm(\boldsymbol{e}_i^{(n+1)} - \boldsymbol{e}_j^{(n+1)}) \in E^{(n+1)};\; 1 \le i < j \le n+1 \big\},$$

where we denote by $\boldsymbol{e}_1^{(n+1)}, \boldsymbol{e}_2^{(n+1)}, \ldots, \boldsymbol{e}_{n+1}^{(n+1)}$ the standard basis of $\mathbb{R}^{n+1}$; that is,

$$\boldsymbol{e}_i^{(n+1)} = {}^t(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \quad \text{(only the } i\text{-th entry is not zero).}$$

In particular, we can choose vectors $\boldsymbol{\alpha}_i^{(n+1)} = \boldsymbol{e}_i^{(n+1)} - \boldsymbol{e}_{i+1}^{(n+1)}$ $(i = 1, 2, \ldots, n)$ as a basis of $E^{(n+1)}$. We call them the simple roots of $\Phi^{(n+1)}$.

Lemma 3.1. *For n vectors $w_1, w_2, \ldots, w_n \in E^{(n+1)}$, put $M = M(w_1, w_2, \ldots, w_n)$ the matrix of size $(n+1)\times n$. If M is full-rank, then we can choose the simple roots $\boldsymbol{\alpha}_1^{(n+1)}, \boldsymbol{\alpha}_2^{(n+1)}, \ldots, \boldsymbol{\alpha}_n^{(n+1)}$ as a basis of the subspace $L(w_1, w_2, \ldots, w_n)$. In particular, we have $M \simeq M(\boldsymbol{\alpha}_1^{(n+1)}, \boldsymbol{\alpha}_2^{(n+1)}, \ldots, \boldsymbol{\alpha}_n^{(n+1)})$.*

For a subset $I \subseteq \{1, 2, \ldots, n+1\}$, we define the projection $\pi_I : \mathbb{R}^{n+1} \to \mathbb{R}^{\sharp I}$ by $\boldsymbol{x} = (x_i)_{i=1}^{n+1} \mapsto (x_i)_{i\in I}$, and define the subset $\Phi_I^{(n+1)} \subset \Phi^{(n+1)}$ as follows:

$$\Phi_I^{(n+1)} = \bigl\{ w \in \Phi^{(n+1)};\ s(w) \in I \text{ and } e(w) \in I \bigr\}. \tag{3.1}$$

Then we note that the projection π_I induces a bijection $\Phi_I^{(n+1)} \to \Phi^{(\sharp I)}$.

Proposition 3.2. *For $n-1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)}$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$ the corresponding matrix of size $(n+1)\times(n-1)$. If M is not sincere but full-rank, then we have*

$$M \sim \left[\begin{array}{c} M(\boldsymbol{\alpha}_1^{(n)}, \boldsymbol{\alpha}_2^{(n)}, \ldots, \boldsymbol{\alpha}_{n-1}^{(n)}) \\ \hline O \end{array}\right],$$

where $\boldsymbol{\alpha}_1^{(n)}, \boldsymbol{\alpha}_2^{(n)}, \ldots, \boldsymbol{\alpha}_{n-1}^{(n)}$ are the simple roots of $\Phi^{(n)} = \Phi(\mathbb{A}_{n-1})$ and O denotes the zero matrix of appropriate size.

Proof. By assumption, the matrix M has a unique row (say the i-th row) consisting of zero entries. Let $\pi_I : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the corresponding projection, where we put $I = \{1, 2, \ldots, n+1\} \smallsetminus \{i\}$. Then the matrix $M\bigl(\pi_I(w_1), \pi_I(w_2), \ldots, \pi_I(w_{n-1})\bigr)$ of size $n \times (n-1)$ is full-rank, and moreover each $\pi_I(w_k)$ is contained in $\Phi^{(n)} \subset E^{(n)}$. Hence, applying Lemma 3.1, we obtain our assertion. □

Lemma 3.3. *For $n-1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)}$ with $n \geq 3$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$. If M is full-rank and sincere, then M is reducible.*

Proof. Our assertion for $n = 3$ can be immediately proven; hence we let $n \geq 4$. First we can, by appropriate right-elementary transformation and replacing rows, express M as follows:

$$M \sim M(\boldsymbol{e}_1^{(n+1)} - \boldsymbol{e}_2^{(n+1)},\ w_2',\ w_3',\ \ldots,\ w_{n-1}'), \tag{3.2}$$

where the w_k''s are elements in $\Phi^{(n+1)}$ satisfying $s(w_k') \geq 2$ $(k = 2, 3, \ldots, n-1)$; that is, the entries except for the $(1,1)$-entry of the first row of the right-hand side of (3.2) are zeros. Then, for a subset $R = \{2, 3, \ldots, n+1\}$, we consider the matrix $M' = M\bigl(\pi_R(w_2'), \pi_R(w_3'), \ldots, \pi_R(w_{n-1}')\bigr)$ of size $n \times (n-2)$. If M' is not sincere, we must have $s(w_k') \geq 3$, since M is sincere; then M is reducible. In the case where M' is sincere, we see that M' is reducible by the assumption of induction; thus M is reducible. □

Proposition 3.4. *For $n-1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)}$ with $n \geq 3$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$. If M is full-rank and sincere, then we have*

$$M \sim M(\boldsymbol{\alpha}_1^{(n+1)}, \boldsymbol{\alpha}_2^{(n+1)}, \ldots, \boldsymbol{\alpha}_r^{(n+1)}, \boldsymbol{\alpha}_{r+2}^{(n+1)}, \ldots, \boldsymbol{\alpha}_n^{(n+1)}).$$

Proof. It follows from Lemma 3.3 that M is reducible. Hence we can, by appropriate right-elementary transformation and replacing rows, express M as (2.1). In the right-hand side of (2.1), X is a matrix of size $(r+1) \times r$ and Y of size $(n-r) \times (n-r-1)$ with $1 \leq r \leq n-2$, and moreover, both X and Y are full-rank. Then each column of the right-hand side of (2.1) is an element of $E^{(n+1)}$; hence each column of X (resp. Y) is an element of $E^{(r+1)}$ (resp. $E^{(n-r)}$). Applying Lemma 3.1 to X and Y respectively, we obtain our assertion. □

Theorem 3.5. *Let $L = L(w_1, w_2, \ldots, w_{n-1})$ be a vector space of codimension one spanned by roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)} = \Phi(\mathbb{A}_n)$ and $M = M(w_1, w_2, \ldots, w_{n-1})$ the corresponding matrix.*

(1) *If M is not sincere, then there exists a number $i \in \{1, 2, \ldots, n+1\}$ such that L can be expressed in the form*

$$L = L\big(\pi_I^{-1}(\boldsymbol{\alpha}_1^{(n)}), \pi_I^{-1}(\boldsymbol{\alpha}_2^{(n)}), \ldots, \pi_I^{-1}(\boldsymbol{\alpha}_n^{(n)})\big),$$

where we put $I = \{1, 2, \ldots, n+1\} \setminus \{i\}$. A normal vector $\boldsymbol{n}_L \in E^{(n+1)}$ of L is determined by two conditions

$$\pi_I(\boldsymbol{n}_L) = {}^t(1, 1, \ldots, 1) \in \mathbb{R}^n \text{ and } \pi_{\{i\}}(\boldsymbol{n}_L) = (-n) \in \mathbb{R}^1.$$

(2) *If M is sincere, then there exists a subset $R \subset \{1, 2, \ldots, n+1\}$ having $\sharp R = r+1$ elements $(r = 1, 2, \ldots, n-2)$ such that L can be expressed in the form*

$$L = L\big(\pi_R^{-1}(\boldsymbol{\alpha}_1^{(r+1)}), \ldots, \pi_R^{-1}(\boldsymbol{\alpha}_r^{(r+1)}), \pi_S^{-1}(\boldsymbol{\alpha}_1^{(s+1)}), \ldots, \pi_S^{-1}(\boldsymbol{\alpha}_s^{(s+1)})\big),$$

where we put $S = \{1, 2, \ldots, n+1\} \setminus R$ and $s = n-r-1$. A normal vector $\boldsymbol{n}_L \in E^{(n+1)}$ of L is determined by two conditions

$$\pi_R(\boldsymbol{n}_L) = {}^t(s+1, \ldots, s+1) \in \mathbb{R}^{r+1} \text{ and } \pi_S(\boldsymbol{n}_L) = -{}^t(r+1, \ldots, r+1) \in \mathbb{R}^{s+1}.$$

Proof. Our assertion follows immediately from Propositions 3.2 and 3.4. □

Therefore the number of distinct subspaces of codimension one is given by

$$(n+1) + \frac{1}{2}\sum_{r=1}^{n-2} \binom{n+1}{r+1} = 2^n - 1.$$

4 One-codimensional subspaces of type $\mathbb{D}_n$

In this section, we denote by $\Phi^{(n)} = \Phi(\mathbb{D}_n)$ the root system of type $\mathbb{D}_n$, which can be regarded as a finite subset of $\mathbb{R}^n$:

$$\Phi^{(n)} = \Phi(\mathbb{D}_n) = \left\{\pm(\boldsymbol{e}_i^{(n)} \pm \boldsymbol{e}_j^{(n)});\ 1 \leq i < j \leq n\right\},$$

where we denote by $\boldsymbol{e}_1^{(n)}, \boldsymbol{e}_2^{(n)}, \dots, \boldsymbol{e}_n^{(n)}$ the standard basis of $\mathbb{R}^n$. In particular, we can choose vectors

$$\boldsymbol{\alpha}_1^{(n)} = \boldsymbol{e}_1^{(n)} - \boldsymbol{e}_2^{(n)}, \quad \dots, \quad \boldsymbol{\alpha}_{n-1}^{(n)} = \boldsymbol{e}_{n-1}^{(n)} - \boldsymbol{e}_n^{(n)}, \quad \boldsymbol{\alpha}_n^{(n)} = \boldsymbol{e}_{n-1}^{(n)} + \boldsymbol{e}_n^{(n)}$$

as a basis of $\mathbb{R}^n$. We call them the simple roots of $\Phi^{(n)}$.

Lemma 4.1. *For n vectors $w_1, w_2, \dots, w_n \in \mathbb{R}^n$, put $M = M(w_1, w_2, \dots, w_n)$ the matrix of size $n \times n$. If M is non-singular, then we can choose the simple roots $\boldsymbol{\alpha}_1^{(n)}, \boldsymbol{\alpha}_2^{(n)}, \dots, \boldsymbol{\alpha}_n^{(n)}$ as a basis of the subspace $L(w_1, w_2, \dots, w_n)$. In particular, we have $M \simeq M(\boldsymbol{\alpha}_1^{(n)}, \boldsymbol{\alpha}_2^{(n)}, \dots, \boldsymbol{\alpha}_n^{(n)})$.*

For a subset $I \subseteq \{1, 2, \dots, n\}$, we define the projection $\pi_I : \mathbb{R}^n \to \mathbb{R}^{\sharp I}$ by $\boldsymbol{x} = (x_i)_{i=1}^n \mapsto (x_i)_{i \in I}$, and define the subset $\Phi_I^{(n)} \subset \Phi^{(n)}$ by a similar manner to (3.1). Then the projection π_I induces a bijection $\Phi_I^{(n)} \to \Phi^{(\sharp I)}$. The following proposition can be proved in a similar way to Proposition 3.2.

Proposition 4.2. *For $n-1$ roots $w_1, w_2, \dots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \dots, w_{n-1})$ the corresponding matrix of size $n \times (n-1)$. If M is not sincere but full-rank, then we have*

$$M \sim \left[\frac{M(\boldsymbol{\alpha}_1^{(n-1)}, \boldsymbol{\alpha}_2^{(n-1)}, \dots, \boldsymbol{\alpha}_{n-1}^{(n-1)})}{O}\right],$$

where $\boldsymbol{\alpha}_1^{(n-1)}, \boldsymbol{\alpha}_2^{(n-1)}, \dots, \boldsymbol{\alpha}_{n-1}^{(n-1)}$ are the simple roots of $\Phi^{(n-1)} = \Phi(\mathbb{D}_{n-1})$.

Lemma 4.3. *For $n-1$ roots $w_1, w_2, \dots, w_{n-1} \in \Phi^{(n)}$ with $n \geq 4$, we put $M = M(w_1, w_2, \dots, w_{n-1})$. Assume that M is full-rank and sincere. If $s(w_1) = s(w_2)$ and $e(w_1) = e(w_2)$, then M is reducible.*

Proof. It follows from the assumption that we can, by replacing rows, express M as follows:

$$M \sim \left[\begin{array}{c|c} X & Z \\ \hline O & Y \end{array}\right] \quad \text{with a square matrix } X \text{ of degree two.}$$

Here we note that X is non-singular; hence we may have $Z = O$. □

Proposition 4.4. *For $n-1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$. Assume that M is full-rank, sincere, and irreducible. Then we can choose positive roots $v_k = \boldsymbol{e}_k^{(n)} - \boldsymbol{e}_{k+1}^{(n)}$ or $\boldsymbol{e}_k^{(n)} + \boldsymbol{e}_{k+1}^{(n)}$ for each $k = 1, 2, \ldots, n-1$ as a basis of the subspace $L = L(w_1, w_2, \ldots, w_{n-1})$. In particular, we have $M \simeq M(v_1, v_2, \ldots, v_{n-1})$.*

Proof. First we may assume that $1 \leq s(w_1) \leq s(w_2) \leq \cdots \leq s(w_{n-1}) \leq n-1$. If there exists a number i satisfying $s(w_i) = s(w_{i+1})$, we have $e(w_i) \neq e(w_{i+1})$ by Lemma 4.3. We put $w'_{i+1} = w_i \pm w_{i+1} \in \Phi^{(n)}$. Then we have $s(w_i) < s(w'_{i+1})$, and we see that the matrix $M(w_1, \ldots, w_i, w'_{i+1}, w_{i+2}, \ldots, w_{n-1})$ which is obtained by replacing the $(i+1)$-th column of M with w'_{i+1} is also irreducible. Therefore, for a prescribed basis $w_1, w_2, \ldots, w_{n-1}$ of L, we may assume that $1 \leq s(w_1) < s(w_2) < \cdots < s(w_{n-1}) \leq n-1$. Then we have $s(w_k) = k$ for $k = 1, 2, \ldots, n-1$. In fact, we have $w_{n-1} = \pm\boldsymbol{e}_{n-1}^{(n)} \pm \boldsymbol{e}_n^{(n)}$, since $e(w_{n-1}) = n$ and $w_{n-1} \in \Phi^{(n)}$. By choosing $n-2$ vectors $w'_k \in \Phi^{(n)}$ satisfying $s(w'_k) = k$ and $e(w'_k) \leq n-1$ $(k = 1, 2, \ldots, n-2)$, we can consider $w'_1, w'_2, \ldots, w'_{n-2}, w_{n-1}$ to be a basis of L. Then we should have $w'_{n-2} = \pm\boldsymbol{e}_{n-2}^{(n)} \pm \boldsymbol{e}_{n-1}^{(n)}$ because $e(w'_{n-2}) = n-1$. By repeating this and by multiplying (-1) to appropriate columns if necessary, we obtain our assertion. □

Proposition 4.5. *For $n-1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$. If M is full-rank, sincere, and reducible, then we can, by appropriate right-elementary transformation and replacing rows, express M as follows:*

$$M \sim \left[\begin{array}{c|c} M(\boldsymbol{\alpha}_1^{(r)}, \boldsymbol{\alpha}_2^{(r)}, \ldots, \boldsymbol{\alpha}_r^{(r)}) & O \\ \hline O & M(v_1, v_2, \ldots, v_{n-r-1}) \end{array}\right],$$

where the $\boldsymbol{\alpha}_k^{(r)}$'s are the simple roots of $\Phi^{(r)}$, and each v_k is either $\boldsymbol{e}_k^{(n-r)} + \boldsymbol{e}_{k+1}^{(n-r)}$ or $\boldsymbol{e}_k^{(n-r)} - \boldsymbol{e}_{k+1}^{(n-r)} \in \Phi^{(n-r)}$.

Proof. Since M is reducible, we can express it as (2.1), where X in (2.1) is a non-singular matrix of degree r and Y of size $(n-r) \times (n-r-1)$ with $2 \leq r \leq n-2$. In addition, we may assume that Y is full-rank, sincere, and irreducible. Then we have $X \simeq M(\boldsymbol{\alpha}_1^{(r)}, \boldsymbol{\alpha}_2^{(r)}, \ldots, \boldsymbol{\alpha}_r^{(r)})$ by Lemma 4.1. On the other hand, the subspace generated by the column vectors consisting of Y is nothing but the subspace generated by elements of $\Phi^{(n-r)}$; hence our assertion follows from Proposition 4.4. □

Theorem 4.6. *Let $L = L(w_1, w_2, \ldots, w_{n-1})$ be a vector space of codimension one spanned by roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)} = \Phi(\mathbb{D}_n)$ and $M = M(w_1, w_2, \ldots, w_{n-1})$ the corresponding matrix.*

(1) *If M is not sincere, then there exists a number $i \in \{1, 2, \ldots, n\}$ such that L can be expressed in the form*

$$L = L\big(\pi_I^{-1}(\boldsymbol{\alpha}_1^{(n-1)}),\, \pi_I^{-1}(\boldsymbol{\alpha}_2^{(n-1)}),\, \ldots,\, \pi_I^{-1}(\boldsymbol{\alpha}_{n-1}^{(n-1)})\big),$$

where we put $I = \{1, 2, \ldots, n\} \setminus \{i\}$. A normal vector of L is given by $\boldsymbol{e}_i^{(n)}$.

(2) *If M is sincere and irreducible, then L can be expressed in the form*

$$L = L\big(v_1,\, v_2,\, \ldots,\, v_{n-1}\big),$$

where each v_k is either $\boldsymbol{e}_k^{(n)} + \boldsymbol{e}_{k+1}^{(n)}$ or $\boldsymbol{e}_k^{(n)} - \boldsymbol{e}_{k+1}^{(n)} \in \Phi^{(n)}$. A normal vector $\boldsymbol{n}_L$ is of the form $\boldsymbol{n}_L = {}^t(1, \pm 1, \pm 1, \ldots, \pm 1) \in \mathbb{R}^n$ and signatures $\pm$ are determined by the conditions ${}^t\boldsymbol{n}_L v_k = 0$ for all $k = 1, 2, \ldots, n-1$.

(3) *If M is sincere and reducible, then there exists a subset $R \subset \{1, 2, \ldots, n\}$ having $\sharp R = r$ elements ($r = 2, 3, \ldots, n-2$) such that L can be expressed in the form*

$$L = L\big(\pi_R^{-1}(\boldsymbol{\alpha}_1^{(r)}), \ldots, \pi_R^{-1}(\boldsymbol{\alpha}_r^{(r)}),\, \pi_S^{-1}(v_1), \ldots, \pi_S^{-1}(v_{s-1})\big),$$

where we put $S = \{1, 2, \ldots, n\} \setminus R$, $s = n - r$, and v_k is either $\boldsymbol{e}_k^{(s)} + \boldsymbol{e}_{k+1}^{(s)}$ or $\boldsymbol{e}_k^{(s)} - \boldsymbol{e}_{k+1}^{(s)} \in \Phi^{(s)}$. A normal vector $\boldsymbol{n}_L \in \mathbb{R}^n$ is determined by the conditions

$$\pi_R(\boldsymbol{n}_L) = {}^t(0,\, 0,\, \ldots,\, 0) \in \mathbb{R}^r, \quad \pi_S(\boldsymbol{n}_L) = {}^t(1,\, \pm 1,\, \ldots,\, \pm 1) \in \mathbb{R}^s,$$

and ${}^t\pi_S(\boldsymbol{n}_L) v_k = 0$ for all $k = 1, 2, \ldots, s-1$.

Proof. Our assertion follows from Propositions 4.2, 4.4, and 4.5. □

Therefore the number of distinct subspaces of codimension one is given by

$$n + 2^{n-1} + \sum_{r=2}^{n-2} \binom{n}{r} \times 2^{n-r-1} = \frac{1}{2}(3^n - n \cdot 2^{n-1} - 1).$$

5 One-codimensional subspaces of type $\mathbb{B}_n$ or $\mathbb{C}_n$

In this section, we classify one-codimensional subspaces generated by a subset of the root system of type $\mathbb{B}_n$ or $\mathbb{C}_n$. Our classification of both types are parallel. So we denote by $\Phi^{(n)}$ the root system either $\Phi(\mathbb{B}_n)$ or $\Phi(\mathbb{C}_n)$, which can be respectively regarded as a finite subset of $\mathbb{R}^n$:

$$\Phi(\mathbb{B}_n) = \big\{\pm(\boldsymbol{e}_i^{(n)} \pm \boldsymbol{e}_j^{(n)});\ 1 \le i < j \le n\big\} \cup \big\{\pm\boldsymbol{e}_1^{(n)},\, \pm\boldsymbol{e}_2^{(n)},\, \ldots,\, \pm\boldsymbol{e}_n^{(n)}\big\},$$
$$\Phi(\mathbb{C}_n) = \big\{\pm(\boldsymbol{e}_i^{(n)} \pm \boldsymbol{e}_j^{(n)});\ 1 \le i < j \le n\big\} \cup \big\{\pm 2\boldsymbol{e}_1^{(n)},\, \pm 2\boldsymbol{e}_2^{(n)},\, \ldots,\, \pm 2\boldsymbol{e}_n^{(n)}\big\}.$$

For a subset $I \subseteq \{1, 2, \ldots, n\}$, we define the projection $\pi_I : \mathbb{R}^n \to \mathbb{R}^{\sharp I}$ and the subset $\Phi_I^{(n)} \subset \Phi^{(n)}$ as in the previous section.

Proposition 5.1. *For $n-1$ roots $w_1, w_2, \dots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \dots, w_{n-1})$ the corresponding matrix of size $n \times (n-1)$. If M is not sincere but full-rank, then we have*

$$M \sim \left[\frac{M(\boldsymbol{e}_1^{(n-1)}, \boldsymbol{e}_2^{(n-1)}, \dots, \boldsymbol{e}_{n-1}^{(n-1)})}{O}\right],$$

where $\boldsymbol{e}_1^{(n-1)}, \boldsymbol{e}_2^{(n-1)}, \dots, \boldsymbol{e}_{n-1}^{(n-1)}$ are the standard basis of $\mathbb{R}^{n-1}$.

Proof. This can be proved by a similar manner to Proposition 3.2. □

Proposition 5.2. *For $n-1$ roots $w_1, w_2, \dots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \dots, w_{n-1})$. Assume that M is full-rank, sincere, and irreducible. Then we can choose positive roots $v_k = \boldsymbol{e}_k^{(n)} - \boldsymbol{e}_{k+1}^{(n)}$ or $\boldsymbol{e}_k^{(n)} + \boldsymbol{e}_{k+1}^{(n)}$ for each $k = 1, 2, \dots, n-1$ as a basis of the subspace $L = L(w_1, w_2, \dots, w_{n-1})$. In particular, we have $M \simeq M(v_1, v_2, \dots, v_{n-1})$.*

Proof. We note that Lemma 4.3 holds even for $\mathbb{B}_n$ or $\mathbb{C}_n$-type with $n \geq 2$. Since the starting of each column of M is different from its own ending, our assertion can be proved by a similar manner to Proposition 4.4. □

Proposition 5.3. *For $n-1$ roots $w_1, w_2, \dots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \dots, w_{n-1})$. If M is full-rank, sincere, and reducible, then we can, by appropriate right-elementary transformation and replacing rows, express M as follows:*

$$M \sim \left[\begin{array}{c|c} M(\boldsymbol{e}_1^{(r)}, \boldsymbol{e}_2^{(r)}, \dots, \boldsymbol{e}_r^{(r)}) & O \\ \hline O & M(v_1, v_2, \dots, v_{n-r-1}) \end{array}\right],$$

where the $\boldsymbol{e}_k^{(r)}$'s are the standard basis of $\mathbb{R}^r$, and each v_k is either $\boldsymbol{e}_k^{(n-r)} + \boldsymbol{e}_{k+1}^{(n-r)}$ or $\boldsymbol{e}_k^{(n-r)} - \boldsymbol{e}_{k+1}^{(n-r)} \in \Phi^{(n-r)}$.

Proof. Since M is reducible, we can express it as (2.1), where X in (2.1) is a square matrix of degree r and Y of size $(n-r) \times (n-r-1)$ with $1 \leq r \leq n-2$. In addition, we may assume that Y is full-rank, sincere, and irreducible. Then we have $X \simeq M(\boldsymbol{e}_1^{(r)}, \boldsymbol{e}_2^{(r)}, \dots, \boldsymbol{e}_r^{(r)})$, since X is non-singular. Our assertion follows from Proposition 5.2 for irreducible Y. □

Theorem 5.4. *Let $\Phi^{(n)}$ be the root system of type $\mathbb{B}_n$ or $\mathbb{C}_n$, $L = L(w_1, w_2, \dots, w_{n-1})$ a vector space of codimension one spanned by roots $w_1, w_2, \dots, w_{n-1} \in \Phi^{(n)}$, and $M = M(w_1, w_2, \dots, w_{n-1})$ the corresponding matrix.*

(1) *If M is not sincere, then there exists a number $i \in \{1, 2, \dots, n\}$ such that L can be expressed in the form*

$$L = L\big(\pi_I^{-1}(\boldsymbol{e}_1^{(n-1)}),\ \pi_I^{-1}(\boldsymbol{e}_2^{(n-1)}),\ \dots,\ \pi_I^{-1}(\boldsymbol{e}_{n-1}^{(n-1)})\big),$$

where we put $I = \{1, 2, \ldots, n\} \smallsetminus \{i\}$*. A normal vector of* L *is given by* $\boldsymbol{e}_i^{(n)}$.

(2) *If* M *is sincere and irreducible, then* L *can be expressed in the form*

$$L = L\big(v_1,\, v_2,\, \ldots,\, v_{n-1}\big),$$

where each v_k *is either* $\boldsymbol{e}_k^{(n)} + \boldsymbol{e}_{k+1}^{(n)}$ *or* $\boldsymbol{e}_k^{(n)} - \boldsymbol{e}_{k+1}^{(n)} \in \Phi^{(n)}$*. A normal vector* $\boldsymbol{n}_L$ *is of the form* $\boldsymbol{n}_L = {}^t(1,\, \pm 1,\, \pm 1,\, \ldots,\, \pm 1) \in \mathbb{R}^n$ *and signatures* $\pm$ *are determined by the conditions* ${}^t\boldsymbol{n}_L v_k = 0$ *for all* $k = 1, 2, \ldots, n-1$.

(3) *If* M *is sincere and reducible, then there exists a subset* $R \subset \{1, 2, \ldots, n\}$ *having* $\sharp R = r$ *elements* $(r = 1, 2, \ldots, n-2)$ *such that* L *can be expressed in the form*

$$L = L\big(\pi_R^{-1}(\boldsymbol{e}_1^{(r)}), \ldots, \pi_R^{-1}(\boldsymbol{e}_r^{(r)}),\, \pi_S^{-1}(v_1), \ldots, \pi_S^{-1}(v_{s-1})\big),$$

where we put $S = \{1, 2, \ldots, n\} \smallsetminus R$*,* $s = n - r$*, and* v_k *is either* $\boldsymbol{e}_k^{(s)} + \boldsymbol{e}_{k+1}^{(s)}$ *or* $\boldsymbol{e}_k^{(s)} - \boldsymbol{e}_{k+1}^{(s)} \in \Phi^{(s)}$*. A normal vector* $\boldsymbol{n}_L \in \mathbb{R}^n$ *is determined by the conditions*

$$\pi_R(\boldsymbol{n}_L) = {}^t(0,\, 0,\, \ldots,\, 0) \in \mathbb{R}^r, \quad \pi_S(\boldsymbol{n}_L) = {}^t(1,\, \pm 1,\, \ldots,\, \pm 1) \in \mathbb{R}^s,$$

and ${}^t\pi_S(\boldsymbol{n}_L) v_k = 0$ *for all* $k = 1, 2, \ldots, s-1$.

Proof. Our assertion follows from Propositions 5.1, 5.2, and 5.3. □

Therefore the number of distinct subspaces of codimension one is given by

$$n + 2^{n-1} + \sum_{r=1}^{n-2} \binom{n}{r} \times 2^{n-r-1} = \frac{1}{2}(3^n - 1).$$

6 A strategy for computing the numbers for exceptional-type

In this section, we explain a strategy for computing the number of one-codimensional subspaces for exceptional-type. Let Φ be the root system of an arbitrary exceptional-type (other than $\mathbb{G}_2$-type) and Φ^+ the subset consisting of all positive roots with respect to a fixed lexicographical order. We denote by n (resp. r) the number of simple roots (resp. positive roots) of Φ. We number the positive roots as $\Phi^+ = \{\boldsymbol{a}_1,\, \boldsymbol{a}_2,\, \ldots,\, \boldsymbol{a}_r\}$ with $\boldsymbol{a}_1 < \boldsymbol{a}_2 < \cdots < \boldsymbol{a}_r$. For $k = 1, 2, \ldots, r$, we put $B(k) = \{j;\ \boldsymbol{a}_j - \boldsymbol{a}_k \notin \Phi^+ \text{ and } k < j \le r\}$. If a lattice that is generated by $n-1$ roots $\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots, \boldsymbol{a}_{i_{n-1}}$ is refined, then we must necessarily have

$$i_k \in B(i_1) \cap \cdots \cap B(i_{k-1}) \quad \text{for} \quad k = 2, 3, \ldots, n-1. \tag{6.1}$$

As mentioned in §2, to classify one-codimensional subspaces generated by a subset of Φ, it is sufficient to classify the refined lattices of corank one.

Therefore, to do this, first we find out all pairs $I = (i_1, i_2, \ldots, i_{n-1})$ with $i_1 < i_2 < \cdots < i_{n-1}$ satisfying the condition (6.1). The number of such pairs is not so large as compared with the binomial coefficient $\binom{r}{n-1}$. Next we want to check whether each lattice $L(I) = \langle \boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots, \boldsymbol{a}_{i_{n-1}} \rangle_{\mathbb{Z}}$ generated by roots corresponding to such a pair I is refined or not. If $L = L(I)$ is of corank one, we can express $(L)_{\mathbb{R}} \cap \Phi^+$ as follows:

$$(L)_{\mathbb{R}} \cap \Phi^+ = \{\boldsymbol{a} \in \Phi^+;\ \boldsymbol{n}_L \cdot \boldsymbol{a} = 0\}, \tag{6.2}$$

where $\boldsymbol{n}_L$ is a normal vector of $L = L(I)$. Putting $R(I;k) = \{\boldsymbol{a} \in (L)_{\mathbb{R}} \cap \Phi^+;\ \boldsymbol{a} < \boldsymbol{a}_{i_k}\}$ for $k = 2, 3, \ldots, n-1$, we have an easy criterion whether $L = L(I)$ is a refined lattice or not:

Lemma 6.1. *Such $L = L(I)$ is a refined lattice if and only if*

$$\dim\langle\, R(I;k)\,\rangle_{\mathbb{R}} = k - 1 \quad \textit{for} \quad k = 2, 3, \ldots, n-1. \tag{6.3}$$

Now we can easily check the conditions (6.1)—(6.3) by using computer. Thus we obtain the numbers for exceptional-type that are presented in Theorem 1.1.

7 An application to classification of semisimple finite prehomogeneous vector spaces

In this section, first we review on representations associated with a quiver. The last half-part of this section, we will give an example of our result to classification theory of semisimple prehomogeneous vector spaces.

Let $Q = (Q_0, Q_1)$ be a connected finite acyclic quiver; that is, Q_0 and Q_1 are finite sets of vertices and arrows, respectively. Let n be the number of vertices of Q. For an n-tuple $\boldsymbol{d} = (d_i)_{i \in Q_0} \in \mathbb{Z}^n_{\geq 0}$ of non-negative integers, the direct product of general linear algebraic group $G_{\boldsymbol{d}} = \prod_{i \in Q_0} GL(d_i)$ acts naturally on $R_{\boldsymbol{d}}(Q) = \bigoplus_{s \to t \text{ in } Q} M(d_t, d_s)$, where $M(d_t, d_s)$ denotes the vector space consisting of $d_t \times d_s$ matrices (in the case of $d_i = 0$, the corresponding things should be trivial); we consider them over an algebraically closed field of characteristic zero. We call $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}}(Q))$ a *representation associated with* Q and $\boldsymbol{d}$ its dimension vector (we sometimes write as $\mathbf{dim}\, v = \boldsymbol{d}$). We say that a dimension vector $\boldsymbol{d}$ is sincere if each entry of $\boldsymbol{d}$ is positive. Note that each point $v \in R_{\boldsymbol{d}}(Q)$ is considered to be a *representation of* Q; hence its $G_{\boldsymbol{d}}$-orbit in $R_{\boldsymbol{d}}(Q)$ is nothing but the isomorphic class of v.

For such a quiver $Q = (Q_0, Q_1)$, its quadratic form $q_Q(\boldsymbol{x})$ on $\mathbb{Z}^n$ is defined by $q_Q(\boldsymbol{x}) = \sum_{i \in Q_0} {x_i}^2 - \sum_{s \to t \text{ in } Q} x_s x_t$ for $\boldsymbol{x} = (x_i)_{i \in Q_0} \in \mathbb{Z}^n$, which does not depend on the choice of an orientation of arrows of Q. A vector $\boldsymbol{x} = (x_i)_{i \in Q_0} \in \mathbb{Z}^n$ is called a root of Q if $q_Q(\boldsymbol{x}) = 1$. We say that a root $\boldsymbol{x}$ of

Q is positive if each entry of $\boldsymbol{x}$ is non-negative. We denote by Φ_Q the set consisting of all positive roots of Q. It is known that, for each indecomposable representation v of Q, its dimension vector $\mathbf{dim}\, v$ is a positive root. Moreover, in the case where Q is a Dynkin quiver (i.e., its underlying graph is one of Dynkin diagrams of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$), the map $\mathbf{dim}$ induces a bijection between the isomorphism classes of indecomposable representations and the positive roots of Q. Then, since Φ_Q is a finite set, each representation $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}}(Q))$ having dimension vector $\boldsymbol{d}$ has a finitely many $G_{\boldsymbol{d}}$-orbits; hence it is a *finite prehomogeneous vector space* (abbreviated FP).

However, it is known by [3] that a *scalar-removed* representation; i.e., the action of the direct product of special linear algebraic group $S_{\boldsymbol{d}} = \prod_{i \in Q_0} SL(d_i)$ may not have a dense $S_{\boldsymbol{d}}$-orbit. For a Dynkin quiver Q, the condition whether $(S_{\boldsymbol{d}}, R_{\boldsymbol{d}}(Q))$ is an FP or not can be characterized by using refined lattices of corank one generated by positive roots of Q. More precisely, for a dimension vector $\boldsymbol{d}$, the corresponding scalar-removed representation $(S_{\boldsymbol{d}}, R_{\boldsymbol{d}}(Q))$ is not an FP if and only if there exist linearly independent $n-1$ positive roots of Q such that $\boldsymbol{d}$ can be expressed as a linear combination of them with coefficients in non-negative integers; see [3, Theorem 3.4]. By Lemma 2.1, such a dimension vector $\boldsymbol{d}$ is contained in a refined lattice of corank one. Thus, to know such a characterization for $\boldsymbol{d}$, it is sufficient to classify the refined lattices of corank one spanned by positive roots of Q; as mentioned in §2, each of them can be distinguished by a normal vector of it.

Now we will give an example. Let Q be a quiver of type $\mathbb{A}_4$ with arbitrarily oriented arrows. Its quadratic form is $q_Q(\boldsymbol{x}) = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 - x_1x_2 - x_2x_3 - x_3x_4$; hence Q has exactly ten positive roots, which are given as follows:

$$\begin{aligned}
&\boldsymbol{a}_1 = {}^t(0,0,0,1), \quad \boldsymbol{a}_2 = {}^t(0,0,1,0), \quad \boldsymbol{a}_3 = {}^t(0,0,1,1), \quad \boldsymbol{a}_4 = {}^t(0,1,0,0),\\
&\boldsymbol{a}_5 = {}^t(0,1,1,0), \quad \boldsymbol{a}_6 = {}^t(0,1,1,1), \quad \boldsymbol{a}_7 = {}^t(1,0,0,0), \quad \boldsymbol{a}_8 = {}^t(1,1,0,0),\\
&\boldsymbol{a}_9 = {}^t(1,1,1,0), \quad \boldsymbol{a}_{10} = {}^t(1,1,1,1).
\end{aligned}$$

We defined the isomorphism $\varphi : \mathbb{R}^4 \to E^{(5)}$ by $\varphi(\boldsymbol{e}_i^{(4)}) = \boldsymbol{\alpha}_i^{(5)}$, where the $\boldsymbol{\alpha}_i^{(5)}$'s are the simple roots of the root system $\Phi(\mathbb{A}_4)$ (see §3). Its representation matrix with respect to bases $\boldsymbol{e}_1^{(4)}, \boldsymbol{e}_2^{(4)}, \boldsymbol{e}_3^{(4)}, \boldsymbol{e}_4^{(4)}$ of $\mathbb{R}^4$ and $\boldsymbol{\alpha}_1^{(5)}, \boldsymbol{\alpha}_2^{(5)}, \boldsymbol{\alpha}_3^{(5)}, \boldsymbol{\alpha}_4^{(5)}$ of $E^{(5)}$ is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix};$$

hence we have $\varphi(\Phi_Q) = \{P\boldsymbol{a}_1, P\boldsymbol{a}_2, \ldots, P\boldsymbol{a}_{10}\}$, which is the positive roots of $\Phi(\mathbb{A}_4)$. According to Theorem 3.5, there exist distinct $15 = 2^4 - 1$ normal vectors (up to non-zero constant) of three-dimensional subspace of $E^{(5)}$ (or, the refined lattices of corank one) generated by elements in $\Phi(\mathbb{A}_4)$; they are given as follows:

$$\begin{aligned}
&\boldsymbol{n}_1 = {}^t(-4,1,1,1,1), && \boldsymbol{n}_2 = {}^t(1,-4,1,1,1), && \boldsymbol{n}_3 = {}^t(1,1,-4,1,1),\\
&\boldsymbol{n}_4 = {}^t(1,1,1,-4,1), && \boldsymbol{n}_5 = {}^t(1,1,1,1,-4), && \boldsymbol{n}_6 = {}^t(-3,-3,2,2,2),\\
&\boldsymbol{n}_7 = {}^t(-3,2,-3,2,2), && \boldsymbol{n}_8 = {}^t(-3,2,2,-3,2), && \boldsymbol{n}_9 = {}^t(-3,2,2,2,-3),\\
&\boldsymbol{n}_{10} = {}^t(2,-3,-3,2,2), && \boldsymbol{n}_{11} = {}^t(2,-3,2,-3,2), && \boldsymbol{n}_{12} = {}^t(2,-3,2,2,-3),\\
&\boldsymbol{n}_{13} = {}^t(2,2,-3,-3,2), && \boldsymbol{n}_{14} = {}^t(2,2,-3,2,-3), && \boldsymbol{n}_{15} = {}^t(2,2,2,-3,-3).
\end{aligned}$$

Thus we see that each normal vector of three-dimensional subspaces of $\mathbb{R}^4$ generated by positive roots of Q is one of ${}^tP \cdot \boldsymbol{n}_k$ for $k = 1, 2, \ldots, 15$. We denote by L_k the refined lattice having a normal vector ${}^tP \cdot \boldsymbol{n}_k$; then all the refined lattices of corank one of type $\mathbb{A}_4$ is given by

$$\begin{aligned}
&L_1 = \langle \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4 \rangle_{\mathbb{Z}}, && L_2 = \langle \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_8 \rangle_{\mathbb{Z}}, && L_3 = \langle \boldsymbol{a}_1, \boldsymbol{a}_5, \boldsymbol{a}_7 \rangle_{\mathbb{Z}},\\
&L_4 = \langle \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_7 \rangle_{\mathbb{Z}}, && L_5 = \langle \boldsymbol{a}_2, \boldsymbol{a}_4, \boldsymbol{a}_7 \rangle_{\mathbb{Z}}, && L_6 = \langle \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_7 \rangle_{\mathbb{Z}},\\
&L_7 = \langle \boldsymbol{a}_1, \boldsymbol{a}_5, \boldsymbol{a}_8 \rangle_{\mathbb{Z}}, && L_8 = \langle \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_9 \rangle_{\mathbb{Z}}, && L_9 = \langle \boldsymbol{a}_2, \boldsymbol{a}_4, \boldsymbol{a}_{10} \rangle_{\mathbb{Z}},\\
&L_{10} = \langle \boldsymbol{a}_1, \boldsymbol{a}_4, \boldsymbol{a}_9 \rangle_{\mathbb{Z}}, && L_{11} = \langle \boldsymbol{a}_3, \boldsymbol{a}_5, \boldsymbol{a}_8 \rangle_{\mathbb{Z}}, && L_{12} = \langle \boldsymbol{a}_2, \boldsymbol{a}_6, \boldsymbol{a}_8 \rangle_{\mathbb{Z}},\\
&L_{13} = \langle \boldsymbol{a}_2, \boldsymbol{a}_6, \boldsymbol{a}_7 \rangle_{\mathbb{Z}}, && L_{14} = \langle \boldsymbol{a}_3, \boldsymbol{a}_5, \boldsymbol{a}_7 \rangle_{\mathbb{Z}}, && L_{15} = \langle \boldsymbol{a}_1, \boldsymbol{a}_4, \boldsymbol{a}_7 \rangle_{\mathbb{Z}}.
\end{aligned}$$

Therefore, a scalar-removed representation $(S_{\boldsymbol{d}}, R_{\boldsymbol{d}}(Q))$ with dimension vector $\boldsymbol{d} = (d_1, d_2, d_3, d_4)$ is not an FP if and only if $\boldsymbol{d}$ is contained in at least one of the 15 lattices listed above. Note that, if $\boldsymbol{d}$ is sincere, it cannot be contained in L_1, L_5, L_6, and L_{15}. In a similar manner to this, we can also obtain sufficiently concrete conditions for such $\boldsymbol{d}$; see and compare with [2, Example 4.3].

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