Counting One-Codimensional Subspaces
Generated by Subsets of a Root System

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Abstract

Let $\Phi$ be an irreducible (and reduced) root system of a Euclidean space. In this paper, we give a complete classification of one-codimensional subspaces generated by a subset of $\Phi$, and we also give explicitly a basis of such each subspace for classical-type. This is motivated by a classification theory of semisimple finite prehomogeneous vector spaces and we explain an example for an application to it.

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1 Introduction

Let $\Phi$ be an irreducible (and reduced) root system of a finite-dimensional Euclidean space $V$. In this paper, we give a complete classification of the refined lattices of corank one (or, one-codimensional subspaces of $V$) generated by a subset of $\Phi$. We refer to Bourbaki [1] for terminology on root system. As mentioned in §7, our classification of such lattices (or, subspaces) is motivated by a classification theory [3] of semisimple “finite-prehomogeneous” vector spaces. We have counted such lattices as distinct sets:

Theorem 1.1. The number of refined lattices of corank one (or, one-codimensional subspaces of $V$) generated by a subset of an irreducible (and reduced) root system $\Phi$ of a Euclidean space $V$ is given by the following table:

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th># of subspaces</th>
<th>$\Phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n \ (n \geq 2)$</td>
<td>$2^n - 1$</td>
<td>$E_6$</td>
</tr>
<tr>
<td>$B_n \ (n \geq 2)$</td>
<td>$(3^n - 1)/2$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$C_n \ (n \geq 2)$</td>
<td>$(3^n - 1)/2$</td>
<td>$E_8$</td>
</tr>
<tr>
<td>$D_n \ (n \geq 4)$</td>
<td>$(3^n - n 2^{n-1} - 1)/2$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$G_2$</td>
<td></td>
<td>$G_2$</td>
</tr>
</tbody>
</table>

This theorem can be immediately obtained as a corollary of our theorems for classical-type (see Theorems 3.5, 4.6, and 5.4). In this case, we also give a basis of each corank-one lattice explicitly, as numerical vectors of a Euclidean space. Our approach is based on an observation of standard form of matrices obtained by arranging some roots. The assertion for $G_2$-type is trivial. The numbers for other exceptional-type has been calculated by computer. That is not hard work; so we will explain only a strategy of computation in §6. We note that it is easier to count the refined lattices of corank one rather than the one-codimensional subspaces.

In particular, when the Dynkin diagram $\Gamma$ of $\Phi$ is simply laced (i.e., it does not have an arrow), our classification is nothing but a classification of dimension vectors which give a necessary and sufficient condition whether a scalar-removed representation associated with $\Gamma$ is “finite-prehomogeneous” or not; we will give an example in §7 for an application.

On the other hand, Oshima [4] has completely classified the subsystems of a root system. Our result corresponds to his classification of the so-called $L$-closed subsystems of a root system (see [4, Definition 6.6]). Note that the decomposition of a matrix appearing our theorems corresponds to the decomposition of a reducible root system of a one-codimensional subspace. In fact, the numbers that are presented in Theorem 1.1 can be derived from his result.
2 Preliminaries

Let \( \Phi \) be an irreducible (and reduced) root system of \( V = \mathbb{R}^l \). For vectors \( w_1, w_2, \ldots, w_m \in V \), we let \( L, (L)_{\mathbb{R}} \) be the \( \mathbb{Z} \)-module, \( \mathbb{R} \)-vector subspace generated by them, respectively:

\[
L = \langle w_1, w_2, \ldots, w_m \rangle_{\mathbb{Z}} = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \cdots + \mathbb{Z}w_m,
\]
\[
(L)_{\mathbb{R}} = \langle w_1, w_2, \ldots, w_m \rangle_{\mathbb{R}} = \mathbb{R}w_1 + \mathbb{R}w_2 + \cdots + \mathbb{R}w_m.
\]

We simply call such a \( \mathbb{Z} \)-module \( L \) a lattice. The rank (resp. corank) of \( L \) is defined as the dimension (resp. codimension) of its corresponding subspace \( (L)_{\mathbb{R}} \). Let \( (L)_{\mathbb{R}}^\perp \) be the ortho-complement of \( (L)_{\mathbb{R}} \) with respect to the standard inner product (or, dot product) on \( V = \mathbb{R}^l \). In the case where \( (L)_{\mathbb{R}} \) is of codimension one, we call a basis of \( (L)_{\mathbb{R}}^\perp \) a normal vector of \( L \) or \( (L)_{\mathbb{R}} \).

Now we define the lexicographical order on \( V = \mathbb{R}^l \) with respect to the standard basis; i.e., we have \( x > y \) for \( x, y \in V \) if the first non-zero entry of \( x - y \) is positive. We denote by \( \Phi^+ \) the set of all positive roots of \( \Phi \). Let \( L \) be a lattice generated by a subset of \( \Phi \), and we put \( p = \dim(L)_{\mathbb{R}} \) and \( \Psi^+ = (L)_{\mathbb{R}} \cap \Phi^+ \). We will choose \( p \) positive roots \( \alpha_1, \alpha_2, \ldots, \alpha_p \) as follows:

\[
\alpha_1 := \min \Psi^+, \quad \text{and} \quad \alpha_k := \min(\Psi^+ \setminus \langle \alpha_1, \ldots, \alpha_{k-1} \rangle_{\mathbb{R}}) \quad \text{for} \quad k = 2, 3, \ldots, p.
\]

Note that \( \Psi = (L)_{\mathbb{R}} \cap \Phi \) is a root system of \( (L)_{\mathbb{R}} \) and that \( \alpha_1, \alpha_2, \ldots, \alpha_p \) is nothing but the simple roots of \( \Psi \) with respect to the lexicographical order. When we apply our result to classification theory of semisimple finite prehomogeneous vector spaces, the next lemma is fundamental:

**Lemma 2.1.** For any element \( \alpha \in \Psi^+ \), there exist non-negative integers \( k_1, k_2, \ldots, k_p \) such that \( \alpha = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_p\alpha_p \).

These \( \alpha_1, \alpha_2, \ldots, \alpha_p \) are uniquely determined by \( L \); so we will say that the lattice spanned by them is the refinement of \( L \), and denote it \( \text{Ref}(L) \). We have \( L \subseteq \text{Ref}(L) \) in general; and we say that \( L \) is refined if \( L = \text{Ref}(L) \). For two lattices \( L_1, L_2 \) generated by positive roots respectively, we have \( \text{Ref}(L_1) = \text{Ref}(L_2) \) if and only if \( (L_1)_{\mathbb{R}} = (L_2)_{\mathbb{R}} \); in the case where \( L_1, L_2 \) are of corank one, this condition is equivalent to that the normal vectors of \( L_1 \) and \( L_2 \) are equal, up to non-zero constant. That is to say, the subspaces of codimension one (or the refined lattices of corank one) are distinguished by their normal vectors.

Here we will prepare a few terminology and notation needed later. For a column vector \( \mathbf{x} = (x_i)_{i=1}^n \in \mathbb{R}^n \), we put \( \text{supp} \mathbf{x} = \{ i; \ x_i \neq 0 \} \). Then we will call \( s(\mathbf{x}) = \min \text{supp} \mathbf{x} \) the starting of \( \mathbf{x} \) and \( e(\mathbf{x}) = \max \text{supp} \mathbf{x} \) its ending.

For vectors \( w_1, w_2, \ldots, w_m \in \mathbb{R}^n \), we denote by \( L(w_1, w_2, \ldots, w_m) \) the vector...
space generated by them, and we put $M(w_1, w_2, \ldots, w_m)$ the matrix of size $n \times m$ arranged $w_1, w_2, \ldots, w_m$ as columns; that is,

$$L(w_1, w_2, \ldots, w_m) = \langle w_1, w_2, \ldots, w_m \rangle \mathbb{R}, \quad M(w_1, w_2, \ldots, w_m) = [w_1 | w_2 | \cdots | w_m].$$

We write $A \simeq A'$ if a matrix $A'$ can be obtained from a matrix $A$ by right-elementary transformation (i.e., elementary transformation for columns). In addition, we write $B \sim B'$ if a matrix $B'$ can be obtained from a matrix $B$ by right-elementary transformation and replacing rows.

Let $M$ be a matrix of size $(n + 1) \times (n - 1)$. If it can be expressed, by appropriate right-elementary transformation and replacing rows, as follows:

$$M \sim \left[ \begin{array}{c|c} X & O \\ \hline O & Y \end{array} \right],$$

where $X$ is of size $(r + 1) \times r$ and $Y$ of size $(n - r) \times (n - r - 1)$ with $1 \leq r \leq n - 2$, then we will say that $M$ is reducible of type $A$. Similarly we say a matrix $M$ of size $n \times (n - 1)$ is reducible of type $D$ if $M$ can be expressed in the form (2.1) with a square matrix $X$ of degree $r$ and $Y$ of size $(n - r) \times (n - r - 1)$ with $2 \leq r \leq n - 2$. For simplicity we will sometimes omit the words “of type $A$ (or $D$)” when there is no confusion.

We say that a matrix $A$ is sincere if $A$ does not contain a row consisting of zeros. A matrix $A$ of size $m \times n$ is called full-rank if rank $A = \min\{m, n\}$.

### 3 One-codimensional subspaces of type $\tilde{A}_n$

In this section, we denote by $\Phi^{(n+1)} = \Phi(\tilde{A}_n)$ the root system of type $\tilde{A}_n$. We denote by $E^{(n+1)} = \{ \mathbf{x} = (x_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1}; \ x_1 + x_2 + \cdots + x_{n+1} = 0 \}$ the $n$-dimensional subspace perpendicular to the vector $(1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$. We may regard $\Phi^{(n+1)}$ as a finite subset of $E^{(n+1)}$:

$$\Phi^{(n+1)} = \Phi(\tilde{A}_n) = \{ \pm(e_i^{(n+1)} - e_j^{(n+1)}) \in E^{(n+1)}; \ 1 \leq i < j \leq n + 1 \};$$

where we denote by $e_1^{(n+1)}, e_2^{(n+1)}, \ldots, e_{n+1}^{(n+1)}$ the standard basis of $\mathbb{R}^{n+1}$; that is,

$$e_i^{(n+1)} = t(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \quad \text{(only the } i \text{-th entry is not zero}).$$

In particular, we can choose vectors $\alpha_i^{(n+1)} = e_i^{(n+1)} - e_{i+1}^{(n+1)} \ (i = 1, 2, \ldots, n)$ as a basis of $E^{(n+1)}$. We call them the simple roots of $\Phi^{(n+1)}$. 
Lemma 3.1. For \( n \) vectors \( w_1, w_2, \ldots, w_n \in E^{(n+1)} \), put \( M = M(w_1, w_2, \ldots, w_n) \) the matrix of size \( (n+1) \times n \). If \( M \) is full-rank, then we can choose the simple roots \( \alpha_1^{(n+1)}, \alpha_2^{(n+1)}, \ldots, \alpha_n^{(n+1)} \) as a basis of the subspace \( L(w_1, w_2, \ldots, w_n) \). In particular, we have \( M \simeq M(\alpha_1^{(n+1)}, \alpha_2^{(n+1)}, \ldots, \alpha_n^{(n+1)}) \).

For a subset \( I \subset \{1, 2, \ldots, n+1\} \), we define the projection \( \pi_I : \mathbb{R}^{n+1} \to \mathbb{R}^n \) by \( x = (x_i)_{i=1}^{n+1} \mapsto (x_i)_{i \in I} \), and define the subset \( \Phi_{I}^{(n+1)} \subset \Phi^{(n+1)} \) as follows:

\[
\Phi_{I}^{(n+1)} = \{ w \in \Phi^{(n+1)} ; s(w) \in I \text{ and } e(w) \in I \}. \tag{3.1}
\]

Then we note that the projection \( \pi_I \) induces a bijection \( \Phi_{I}^{(n+1)} \to \Phi^{(n)} \).

Proposition 3.2. For \( n - 1 \) roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)} \), we put \( M = M(w_1, w_2, \ldots, w_{n-1}) \) the corresponding matrix of size \( (n+1) \times (n-1) \). If \( M \) is not sincere but full-rank, then we have

\[
M \sim \begin{bmatrix} M(\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_{n-1}^{(n)}) \\ O \end{bmatrix},
\]

where \( \alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_{n-1}^{(n)} \) are the simple roots of \( \Phi^{(n)} = \Phi(\mathfrak{A}_{n-1}) \) and \( O \) denotes the zero matrix of appropriate size.

Proof. By assumption, the matrix \( M \) has a unique row (say the \( i \)-th row) consisting of zero entries. Let \( \pi_I : \mathbb{R}^{n+1} \to \mathbb{R}^n \) be the corresponding projection, where we put \( I = \{1, 2, \ldots, n+1\} \setminus \{i\} \). Then the matrix \( M(\pi_I(w_1), \pi_I(w_2), \ldots, \pi_I(w_{n-1})) \) of size \( n \times (n-1) \) is full-rank, and moreover each \( \pi_I(w_k) \) is contained in \( \Phi^{(n)} \subset E^{(n)} \). Hence, applying Lemma 3.1, we obtain our assertion. \( \square \)

Lemma 3.3. For \( n - 1 \) roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)} \) with \( n \geq 3 \), we put \( M = M(w_1, w_2, \ldots, w_{n-1}) \). If \( M \) is full-rank and sincere, then \( M \) is reducible.

Proof. Our assertion for \( n = 3 \) can be immediately proven; hence we let \( n \geq 4 \). First we can, by appropriate right-elementary transformation and replacing rows, express \( M \) as follows:

\[
M \sim M(e_1^{(n+1)} - e_2^{(n+1)}, w_2', w_3', \ldots, w_{n-1}'), \tag{3.2}
\]

where the \( w_k' \)'s are elements in \( \Phi^{(n+1)} \) satisfying \( s(w_k') \geq 2 \) \((k = 2, 3, \ldots, n-1)\); that is, the entries except for the \((1, 1)\)-entry of the first row of the right-hand side of (3.2) are zeros. Then, for a subset \( R = \{2, 3, \ldots, n+1\} \), we consider the matrix \( M' = M(\pi_R(w_2'), \pi_R(w_3'), \ldots, \pi_R(w_{n-1}')) \) of size \( n \times (n-2) \). If \( M' \) is not sincere, we must have \( s(w_k') \geq 3 \), since \( M \) is sincere; then \( M \) is reducible. In the case where \( M' \) is sincere, we see that \( M' \) is reducible by the assumption of induction; thus \( M \) is reducible. \( \square \)
Proposition 3.4. For \( n - 1 \) roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)} \) with \( n \geq 3 \), we put \( M = M(w_1, w_2, \ldots, w_{n-1}) \). If \( M \) is full-rank and sincere, then we have

\[
M \sim M(\alpha_1^{(n+1)}, \alpha_2^{(n+1)}, \ldots, \alpha_r^{(n+1)}, \alpha_{r+2}^{(n+1)}, \ldots, \alpha_n^{(n+1)}).
\]

Proof. It follows from Lemma 3.3 that \( M \) is reducible. Hence we can, by appropriate right-elementary transformation and replacing rows, express \( M \) as (2.1). In the right-hand side of (2.1), \( X \) is a matrix of size \((r+1) \times r\) and \( Y \) of size \((n-r) \times (n-r-1)\) with \( 1 \leq r \leq n-2 \), and moreover, both \( X \) and \( Y \) are full-rank. Then each column of the right-hand side of (2.1) is an element of \( E^{(n+1)} \); hence each column of \( X \) (resp. \( Y \)) is an element of \( E^{(r+1)} \) (resp. \( E^{(n-r)} \)). Applying Lemma 3.1 to \( X \) and \( Y \) respectively, we obtain our assertion. \( \Box \)

Theorem 3.5. Let \( L = L(w_1, w_2, \ldots, w_{n-1}) \) be a vector space of codimension one spanned by roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n+1)} = \Phi(\mathbb{A}_n) \) and \( M = M(w_1, w_2, \ldots, w_{n-1}) \) the corresponding matrix.

1. If \( M \) is not sincere, then there exists a number \( i \in \{1, 2, \ldots, n+1\} \) such that \( L \) can be expressed in the form

\[
L = L(\pi_I^{-1}(\alpha_1^{(n)}), \pi_I^{-1}(\alpha_2^{(n)}), \ldots, \pi_I^{-1}(\alpha_n^{(n)})),
\]

where we put \( I = \{1, 2, \ldots, n+1\} \setminus \{i\} \). A normal vector \( n_L \in E^{(n+1)} \) of \( L \) is determined by two conditions

\[
\pi_I(n_L) = \begin{bmatrix} 1, 1, \ldots, 1 \end{bmatrix} \in \mathbb{R}^n \text{ and } \pi_{\{i\}}(n_L) = (-n) \in \mathbb{R}^{1}.
\]

2. If \( M \) is sincere, then there exists a subset \( R \subset \{1, 2, \ldots, n+1\} \) having \( \#R = r + 1 \) elements \((r = 1, 2, \ldots, n-2)\) such that \( L \) can be expressed in the form

\[
L = L(\pi_R^{-1}(\alpha_1^{(r+1)}), \ldots, \pi_R^{-1}(\alpha_r^{(r+1)}), \pi_S^{-1}(\alpha_1^{(s+1)}), \ldots, \pi_S^{-1}(\alpha_s^{(s+1)})),
\]

where we put \( S = \{1, 2, \ldots, n+1\} \setminus R \) and \( s = n-r-1 \). A normal vector \( n_L \in E^{(n+1)} \) of \( L \) is determined by two conditions

\[
\pi_R(n_L) = \begin{bmatrix} s+1, \ldots, s+1 \end{bmatrix} \in \mathbb{R}^{r+1} \text{ and } \pi_S(n_L) = -\begin{bmatrix} r+1, \ldots, r+1 \end{bmatrix} \in \mathbb{R}^{s+1}.
\]

Proof. Our assertion follows immediately from Propositions 3.2 and 3.4. \( \Box \)

Therefore the number of distinct subspaces of codimension one is given by

\[
(n + 1) + \frac{1}{2} \sum_{r=1}^{n-2} \binom{n+1}{r+1} = 2^n - 1.
\]
4 One-codimensional subspaces of type $\mathbb{D}_n$

In this section, we denote by $\Phi^{(n)} = \Phi(\mathbb{D}_n)$ the root system of type $\mathbb{D}_n$, which can be regarded as a finite subset of $\mathbb{R}^n$:

$$\Phi^{(n)} = \Phi(\mathbb{D}_n) = \{ \pm (e_i^{(n)} \pm e_j^{(n)}) ; 1 \leq i < j \leq n \},$$

where we denote by $e_1^{(n)}, e_2^{(n)}, \ldots, e_n^{(n)}$ the standard basis of $\mathbb{R}^n$. In particular, we can choose vectors

$$\alpha_1^{(n)} = e_1^{(n)} - e_2^{(n)}, \ldots, \alpha_{n-1}^{(n)} = e_{n-1}^{(n)} - e_n^{(n)}, \alpha_n^{(n)} = e_n^{(n)} + e_{n-1}^{(n)}$$

as a basis of $\mathbb{R}^n$. We call them the simple roots of $\Phi^{(n)}$.

**Lemma 4.1.** For $n$ vectors $w_1, w_2, \ldots, w_n \in \mathbb{R}^n$, put $M = M(w_1, w_2, \ldots, w_n)$ the matrix of size $n \times n$. If $M$ is non-singular, then we can choose the simple roots $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_n^{(n)}$ as a basis of the subspace $L(w_1, w_2, \ldots, w_n)$. In particular, we have $M \simeq M(\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_n^{(n)})$.

For a subset $I \subseteq \{1, 2, \ldots, n\}$, we define the projection $\pi_I : \mathbb{R}^n \to \mathbb{R}^{|I|}$ by

$$x = (x_i)_{i=1}^n \mapsto (x_i)_{i \in I},$$

and define the subset $\Phi_I^{(n)} \subset \Phi^{(n)}$ by a similar manner to (3.1). Then the projection $\pi_I$ induces a bijection $\Phi_I^{(n)} \to \Phi^{(|I|)}$. The following proposition can be proved in a similar way to Proposition 3.2.

**Proposition 4.2.** For $n - 1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$ the corresponding matrix of size $n \times (n - 1)$. If $M$ is not sincere but full-rank, then we have

$$M \simeq \begin{bmatrix} M(\alpha_1^{(n-1)}, \alpha_2^{(n-1)}, \ldots, \alpha_{n-1}^{(n-1)}) \\ O \end{bmatrix},$$

where $\alpha_1^{(n-1)}, \alpha_2^{(n-1)}, \ldots, \alpha_{n-1}^{(n-1)}$ are the simple roots of $\Phi^{(n-1)} = \Phi(\mathbb{D}_{n-1})$.

**Lemma 4.3.** For $n - 1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)}$ with $n \geq 4$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$. Assume that $M$ is full-rank and sincere. If $s(w_1) = s(w_2)$ and $e(w_1) = e(w_2)$, then $M$ is reducible.

**Proof.** It follows from the assumption that we can, by replacing rows, express $M$ as follows:

$$M \sim \begin{bmatrix} X & Z \\ O & Y \end{bmatrix}$$

with a square matrix $X$ of degree two.

Here we note that $X$ is non-singular; hence we may have $Z = O$. $\square$
Proposition 4.4. For $n - 1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$. Assume that $M$ is full-rank, sincere, and irreducible. Then we can choose positive roots $v_k = e_k^{(n)} - e_{k+1}^{(n)}$ or $e_k^{(n)} + e_{k+1}^{(n)}$ for each $k = 1, 2, \ldots, n - 1$ as a basis of the subspace $L = L(w_1, w_2, \ldots, w_{n-1})$. In particular, we have $M \simeq M(v_1, v_2, \ldots, v_{n-1})$.

Proof. First we may assume that $1 \leq s(w_1) \leq s(w_2) \leq \cdots \leq s(w_{n-1}) \leq n - 1$. If there exists a number $i$ satisfying $s(w_i) = s(w_{i+1})$, we have $e(w_i) = e(w_{i+1})$ by Lemma 4.3. We put $w'_{i+1} = w_i \pm w_{i+1} \in \Phi^{(n)}$. Then we have $s(w_i) < s(w'_{i+1})$, and we see that the matrix $M(w_1, w_i, w'_{i+1}, w_{i+2}, \ldots, w_{n-1})$ which is obtained by replacing the $(i+1)$-th column of $M$ with $w'_{i+1}$ is also irreducible. Therefore, for a prescribed basis $w_1, w_2, \ldots, w_{n-1}$ of $L$, we may assume that $1 \leq s(w_1) < s(w_2) < \cdots < s(w_{n-1}) \leq n - 1$. Then we have $s(w_k) = k$ for $k = 1, 2, \ldots, n - 1$. In fact, we have $w_{n-1} = \pm e_{n-1}^{(n)} \pm e_n^{(n)}$, since $e(w_{n-1}) = n$ and $w_{n-1} \in \Phi^{(n)}$. By choosing $n - 2$ vectors $w'_k \in \Phi^{(n)}$ satisfying $s(w'_k) = k$ and $e(w'_k) \leq n - 1$ ($k = 1, 2, \ldots, n - 2$), we can consider $w'_1, w'_2, \ldots, w'_{n-2}, w_{n-1}$ to be a basis of $L$. Then we should have $w'_{n-2} = \pm e_{n-2}^{(n)} \pm e_{n-1}^{(n)}$ because $e(w'_{n-2}) = n - 1$. By repeating this and by multiplying $(-1)$ to appropriate columns if necessary, we obtain our assertion. \hfill \Box

Proposition 4.5. For $n - 1$ roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)}$, we put $M = M(w_1, w_2, \ldots, w_{n-1})$. If $M$ is full-rank, sincere, and reducible, then we can, by appropriate right-elementary transformation and replacing rows, express $M$ as follows:

$$M \sim \begin{bmatrix} M(\alpha_1^{(r)}, \alpha_2^{(r)}, \ldots, \alpha_r^{(r)}) & O \\ O & M(v_1, v_2, \ldots, v_{n-1}) \end{bmatrix},$$

where the $\alpha_k^{(r)}$’s are the simple roots of $\Phi^{(r)}$, and each $v_k$ is either $e_k^{(n-r)} + e_{k+1}^{(n-r)}$ or $e_k^{(n-r)} - e_{k+1}^{(n-r)} \in \Phi^{(n-r)}$.

Proof. Since $M$ is reducible, we can express it as (2.1), where $X$ in (2.1) is a non-singular matrix of degree $r$ and $Y$ of size $(n - r) \times (n - r - 1)$ with $2 \leq r \leq n - 2$. In addition, we may assume that $Y$ is full-rank, sincere, and irreducible. Then we have $X \simeq M(\alpha_1^{(r)}, \alpha_2^{(r)}, \ldots, \alpha_r^{(r)})$ by Lemma 4.1. On the other hand, the subspace generated by the column vectors consisting of $Y$ is nothing but the subspace generated by elements of $\Phi^{(n-r)}$; hence our assertion follows from Proposition 4.4. \hfill \Box

Theorem 4.6. Let $L = L(w_1, w_2, \ldots, w_{n-1})$ be a vector space of codimension one spanned by roots $w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)} = \Phi(\mathbb{D}_n)$ and $M = M(w_1, w_2, \ldots, w_{n-1})$ the corresponding matrix.
(1) If $M$ is not sincere, then there exists a number $i \in \{1, 2, \ldots, n\}$ such that $L$ can be expressed in the form

$$L = L(\pi_I^{-1}(\alpha_1^{(n-1)}), \ldots, \pi_I^{-1}(\alpha_r^{(n-1)})), \quad \pi_I^{-1}(\alpha_r^{(n-1)}), \ldots, \pi_I^{-1}(\alpha_{n-1}^{(n-1)})),$$

where we put $I = \{1, 2, \ldots, n\} \setminus \{i\}$. A normal vector of $L$ is given by $e_i^{(n)}$.

(2) If $M$ is sincere and irreducible, then $L$ can be expressed in the form

$$L = L(v_1, v_2, \ldots, v_{n-1}),$$

where each $v_k$ is either $e_k^{(n)} + e_{k+1}^{(n)}$ or $e_k^{(n)} - e_{k+1}^{(n)} \in \Phi^{(n)}$. A normal vector $n_L$ is defined by

$$\pi_R(n_L) = t(0, 0, \ldots, 0) \in \mathbb{R}^r, \quad \pi_S(n_L) = t(1, \pm 1, \ldots, \pm 1) \in \mathbb{R}^s,$$

and

$$\pi_R(n_L)v_k = 0$$

for all $k = 1, 2, \ldots, s-1$.

Proof. Our assertion follows from Propositions 4.2, 4.4, and 4.5. \hfill \Box

Therefore the number of distinct subspaces of codimension one is given by

$$n + 2^{n-1} + \sum_{r=2}^{n-2} \binom{n}{r} \times 2^{n-r-1} = \frac{1}{2} (3^n - n \cdot 2^{n-1} - 1).$$

5 One-codimensional subspaces of type $\mathbb{B}_n$ or $\mathbb{C}_n$

In this section, we classify one-codimensional subspaces generated by a subset of the root system of type $\mathbb{B}_n$ or $\mathbb{C}_n$. Our classification of both types are parallel. So we denote by $\Phi^{(n)}$ the root system either $\Phi(\mathbb{B}_n)$ or $\Phi(\mathbb{C}_n)$, which can be respectively regarded as a finite subset of $\mathbb{R}^n$:

$$\Phi(\mathbb{B}_n) = \{\pm (e_i^{(n)} \pm e_j^{(n)}); 1 \leq i < j \leq n\} \cup \{\pm e_1^{(n)}, \pm e_2^{(n)}, \ldots, \pm e_n^{(n)}\},$$

$$\Phi(\mathbb{C}_n) = \{\pm (e_i^{(n)} \pm e_j^{(n)}); 1 \leq i < j \leq n\} \cup \{\pm 2e_1^{(n)}, \pm 2e_2^{(n)}, \ldots, \pm 2e_n^{(n)}\}.$$
 Proposition 5.1. For \( n - 1 \) roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)} \), we put \( M = M(w_1, w_2, \ldots, w_{n-1}) \) the corresponding matrix of size \( n \times (n - 1) \). If \( M \) is not sincere but full-rank, then we have

\[
M \sim \begin{bmatrix}
    M(e_1^{(n-1)}, e_2^{(n-1)}, \ldots, e_{n-1}^{(n-1)}) \\
    O
\end{bmatrix},
\]

where \( e_1^{(n-1)}, e_2^{(n-1)}, \ldots, e_{n-1}^{(n-1)} \) are the standard basis of \( \mathbb{R}^{n-1} \).

Proof. This can be proved by a similar manner to Proposition 3.2.

 Proposition 5.2. For \( n - 1 \) roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)} \), we put \( M = M(w_1, w_2, \ldots, w_{n-1}) \). Assume that \( M \) is full-rank, sincere, and irreducible. Then we can choose positive roots \( v_k = e_k^{(n)} - e_{k+1}^{(n)} \) or \( e_k^{(n)} + e_{k+1}^{(n)} \) for each \( k = 1, 2, \ldots, n - 1 \) as a basis of the subspace \( L = L(w_1, w_2, \ldots, w_{n-1}) \). In particular, we have \( M \simeq M(v_1, v_2, \ldots, v_{n-1}) \).

Proof. We note that Lemma 4.3 holds even for \( \mathbb{B}_n \) or \( \mathbb{C}_n \)-type with \( n \geq 2 \). Since the starting of each column of \( M \) is different from its own ending, our assertion can be proved by a similar manner to Proposition 4.4.

 Proposition 5.3. For \( n - 1 \) roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)} \), we put \( M = M(w_1, w_2, \ldots, w_{n-1}) \). If \( M \) is full-rank, sincere, and reducible, then we can, by appropriate right-elementary transformation and replacing rows, express \( M \) as follows:

\[
M \sim \begin{bmatrix}
    M(e_1^{(r)}, e_2^{(r)}, \ldots, e_r^{(r)}) \\
    O
\end{bmatrix},
\]

where the \( e_k^{(r)} \)'s are the standard basis of \( \mathbb{R}^r \), and each \( v_k \) is either \( e_k^{(n-r)} + e_{k+1}^{(n-r)} \) or \( e_k^{(n-r)} - e_{k+1}^{(n-r)} \) \( \in \Phi^{(n-r)} \).

Proof. Since \( M \) is reducible, we can express it as (2.1), where \( X \) in (2.1) is a square matrix of degree \( r \) and \( Y \) of size \( (n-r) \times (n-r-1) \) with \( 1 \leq r \leq n-2 \). In addition, we may assume that \( Y \) is full-rank, sincere, and irreducible. Then we have \( X \simeq M(e_1^{(r)}, e_2^{(r)}, \ldots, e_r^{(r)}) \), since \( X \) is non-singular. Our assertion follows from Proposition 5.2 for irreducible \( Y \).

 Theorem 5.4. Let \( \Phi^{(n)} \) be the root system of type \( \mathbb{B}_n \) or \( \mathbb{C}_n \), \( L = L(w_1, w_2, \ldots, w_{n-1}) \) a vector space of codimension one spanned by roots \( w_1, w_2, \ldots, w_{n-1} \in \Phi^{(n)} \), and \( M = M(w_1, w_2, \ldots, w_{n-1}) \) the corresponding matrix.

(1) If \( M \) is not sincere, then there exists a number \( i \in \{1, 2, \ldots, n\} \) such that \( L \) can be expressed in the form

\[
L = L(\pi_i^{-1}(e_1^{(n-1)}), \pi_i^{-1}(e_2^{(n-1)}), \ldots, \pi_i^{-1}(e_{n-1}^{(n-1)})),
\]
where we put $I = \{1, 2, \ldots, n\} \setminus \{i\}$. A normal vector of $L$ is given by $e_i^{(n)}$.

(2) If $M$ is sincere and irreducible, then $L$ can be expressed in the form

$$L = L(v_1, v_2, \ldots, v_{n-1}),$$

where each $v_k$ is either $e_k^{(n)} + e_{k+1}^{(n)}$ or $e_k^{(n)} - e_{k+1}^{(n)} \in \Phi(n)$. A normal vector $n_L$ is of the form $n_L = t(1, \pm 1, \pm 1, \ldots, \pm 1) \in \mathbb{R}^n$ and signatures $\pm$ are determined by the conditions $t, n_{L}, v_k = 0$ for all $k = 1, 2, \ldots, n - 1$.

(3) If $M$ is sincere and reducible, then there exists a subset $R \subset \{1, 2, \ldots, n\}$ having $\Phi = r$ elements ($r = 1, 2, \ldots, n - 2$) such that $L$ can be expressed in the form

$$L = L(\pi_R^{-1}(e_1^{(r)}), \ldots, \pi_R^{-1}(e_r^{(r)}), \pi_s^{-1}(v_1), \ldots, \pi_s^{-1}(v_{s-1})),

where we put $S = \{1, 2, \ldots, n\} \setminus R$, $s = n - r$, and $v_k$ is either $e_k^{(s)} + e_{k+1}^{(s)}$ or $e_k^{(s)} - e_{k+1}^{(s)} \in \Phi(s)$. A normal vector $n_L \in \mathbb{R}^n$ is determined by the conditions

$$\pi_R(n_L) = t(0, 0, \ldots, 0) \in \mathbb{R}^r, \quad \pi_S(n_L) = t(1, \pm 1, \ldots, \pm 1) \in \mathbb{R}^s,$

and $t, \pi_S(n_L)v_k = 0$ for all $k = 1, 2, \ldots, s - 1$.

Proof. Our assertion follows from Propositions 5.1, 5.2, and 5.3.

Therefore the number of distinct subspaces of codimension one is given by

$$n + 2^{n-1} + \sum_{r=1}^{n-2} \binom{n}{r} \times 2^{n-r-1} = \frac{1}{2}(3^n - 1).$$

6. A strategy for computing the numbers for exceptional-type

In this section, we explain a strategy for computing the number of one-codimensional subspaces for exceptional-type. Let $\Phi$ be the root system of an arbitrary exceptional-type (other than $G_2$-type) and $\Phi^+$ the subset consisting of all positive roots with respect to a fixed lexicographical order. We denote by $n$ (resp. $r$) the number of simple roots (resp. positive roots) of $\Phi$. We number the positive roots as $\Phi^+ = \{a_1, a_2, \ldots, a_r\}$ with $a_1 < a_2 < \cdots < a_r$. For $k = 1, 2, \ldots, r$, we put $B(k) = \{j; a_j - a_k \not\in \Phi^+ \text{ and } k < j \leq r\}$. If a lattice that is generated by $n - 1$ roots $a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}$ is refined, then we must necessarily have

$$i_k \in B(i_1) \cap \cdots \cap B(i_{k-1}) \quad \text{for} \quad k = 2, 3, \ldots, n - 1. \quad (6.1)$$

As mentioned in §2, to classify one-codimensional subspaces generated by a subset of $\Phi$, it is sufficient to classify the refined lattices of corank one.
Therefore, to do this, first we find out all pairs $I = (i_1, i_2, \ldots, i_{n-1})$ with $i_1 < i_2 < \cdots < i_{n-1}$ satisfying the condition (6.1). The number of such pairs is not so large as compared with the binomial coefficient $\binom{n-1}{n-1}$. Next we want to check whether each lattice $L(I) = (a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}})$ generated by roots corresponding to such a pair $I$ is refined or not. If $L = L(I)$ is of corank one, we can express $(L)_{\mathbb{R}} \cap \Phi^+$ as follows:

$$
(L)_{\mathbb{R}} \cap \Phi^+ = \{ a \in \Phi^+; \, n_L \cdot a = 0 \},
$$

(6.2)

where $n_L$ is a normal vector of $L = L(I)$. Putting $R(I; k) = \{ a \in (L)_{\mathbb{R}} \cap \Phi^+; \, a < a_k \}$ for $k = 2, 3, \ldots, n-1$, we have an easy criterion whether $L = L(I)$ is a refined lattice or not:

**Lemma 6.1.** Such $L = L(I)$ is a refined lattice if and only if

$$
\dim(R(I; k))_{\mathbb{R}} = k - 1 \quad \text{for} \quad k = 2, 3, \ldots, n-1.
$$

(6.3)

Now we can easily check the conditions (6.1)—(6.3) by using computer. Thus we obtain the numbers for exceptional-type that are presented in Theorem 1.1.

## 7 An application to classification of semisimple finite prehomogeneous vector spaces

In this section, first we review on representations associated with a quiver. The last half-part of this section, we will give an example of our result to classification theory of semisimple prehomogeneous vector spaces.

Let $Q = (Q_0, Q_1)$ be a connected finite acyclic quiver; that is, $Q_0$ and $Q_1$ are finite sets of vertices and arrows, respectively. Let $n$ be the number of vertices of $Q$. For an $n$-tuple $d = (d_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^n$ of non-negative integers, the direct product of general linear algebraic group $G_d = \prod_{i \in Q_0} GL(d_i)$ acts naturally on $R_d(Q) = \bigoplus_{s-t \in Q} M(d_t, d_s)$, where $M(d_t, d_s)$ denotes the vector space consisting of $d_t \times d_s$ matrices (in the case of $d_t = 0$, the corresponding things should be trivial); we consider them over an algebraically closed field of characteristic zero. We call $(G_d, R_d(Q))$ a representation associated with $Q$ and $d$ its dimension vector (we sometimes write as $\dim v = d$). We say that a dimension vector $d$ is sincere if each entry of $d$ is positive. Note that each point $v \in R_d(Q)$ is considered to be a representation of $Q$; hence its $G_d$-orbit in $R_d(Q)$ is nothing but the isomorphic class of $v$.

For such a quiver $Q = (Q_0, Q_1)$, its quadratic form $q_Q(x)$ on $\mathbb{Z}^n$ is defined by $q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{s-t \in Q} x_s x_t$ for $x = (x_i)_{i \in Q_0} \in \mathbb{Z}^n$, which does not depend on the choice of an orientation of arrows of $Q$. A vector $x = (x_i)_{i \in Q_0} \in \mathbb{Z}^n$ is called a root of $Q$ if $q_Q(x) = 1$. We say that a root $x$ of
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$Q$ is positive if each entry of $\mathbf{x}$ is non-negative. We denote by $\Phi_Q$ the set consisting of all positive roots of $Q$. It is known that, for each indecomposable representation $v$ of $Q$, its dimension vector $\text{dim } v$ is a positive root. Moreover, in the case where $Q$ is a Dynkin quiver (i.e., its underlying graph is one of Dynkin diagrams of type $A_n, D_n, E_6, E_7$, or $E_8$), the map $\text{dim}$ induces a bijection between the isomorphism classes of indecomposable representations and the positive roots of $Q$. Then, since $\Phi_Q$ is a finite set, each representation $(G_\mathbf{d}, R_\mathbf{d}(Q))$ having dimension vector $\mathbf{d}$ has a finitely many $G_\mathbf{d}$-orbits; hence it is a finite prehomogeneous vector space (abbreviated FP).

However, it is known by [3] that a scalar-removed representation; i.e., the action of the direct product of special linear algebraic group $S\mathbf{d} = \prod_{i \in Q_0} SL(d_i)$ may not have a dense $S\mathbf{d}$-orbit. For a Dynkin quiver $Q$, the condition whether $(S\mathbf{d}, R_\mathbf{d}(Q))$ is an FP or not can be characterized by using refined lattices of corank one generated by positive roots of $Q$. More precisely, for a dimension vector $\mathbf{d}$, the corresponding scalar-removed representation $(S\mathbf{d}, R_\mathbf{d}(Q))$ is not an FP if and only if there exist linearly independent $n - 1$ positive roots of $Q$ such that $\mathbf{d}$ can be expressed as a linear combination of them with coefficients in non-negative integers; see [3, Theorem 3.4]. By Lemma 2.1, such a dimension vector $\mathbf{d}$ is contained in a refined lattice of corank one. Thus, to know such a characterization for $\mathbf{d}$, it is sufficient to classify the refined lattices of corank one spanned by positive roots of $Q$; as mentioned in §2, each of them can be distinguished by a normal vector of it.

Now we will give an example. Let $Q$ be a quiver of type $A_4$ with arbitrarily oriented arrows. Its quadratic form is $q_0(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_2x_3 - x_3x_4$; hence $Q$ has exactly ten positive roots, which are given as follows:

- $\mathbf{a}_1 = t(0, 0, 0, 1)$
- $\mathbf{a}_2 = t(0, 0, 1, 0)$
- $\mathbf{a}_3 = t(0, 0, 1, 1)$
- $\mathbf{a}_4 = t(0, 1, 0, 0)$
- $\mathbf{a}_5 = t(0, 1, 1, 0)$
- $\mathbf{a}_6 = t(0, 1, 1, 1)$
- $\mathbf{a}_7 = t(1, 0, 0, 0)$
- $\mathbf{a}_8 = t(1, 1, 0, 0)$
- $\mathbf{a}_9 = t(1, 1, 1, 0)$
- $\mathbf{a}_{10} = t(1, 1, 1, 1)$

We defined the isomorphism $\varphi : \mathbb{R}^4 \rightarrow E^{(5)}$ by $\varphi(\mathbf{e}_i^{(4)}) = \mathbf{a}_i^{(5)}$, where the $\mathbf{a}_i^{(5)}$'s are the simple roots of the root system $\Phi(A_4)$ (see §3). Its representation matrix with respect to bases $\mathbf{e}_1^{(4)}, \mathbf{e}_2^{(4)}, \mathbf{e}_3^{(4)}, \mathbf{e}_4^{(4)}$ of $\mathbb{R}^4$ and $\mathbf{a}_1^{(5)}, \mathbf{a}_2^{(5)}, \mathbf{a}_3^{(5)}, \mathbf{a}_4^{(5)}$ of $E^{(5)}$ is given by

$$P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix};$$

hence we have $\varphi(\Phi_Q) = \{Pa_1, Pa_2, \ldots, Pa_{10}\}$, which is the positive roots of $\Phi(A_4)$. According to Theorem 3.5, there exist 15 = 2^4 - 1 normal vectors (up to non-zero constant) of three-dimensional subspace of $E^{(5)}$ (or, the refined lattices of corank one) generated by elements in $\Phi(A_4)$; they are
given as follows:

\[ n_1 = \langle -4, 1, 1, 1 \rangle, \quad n_2 = \langle 1, -4, 1, 1 \rangle, \quad n_3 = \langle 1, 1, -4, 1 \rangle, \]
\[ n_4 = \langle 1, 1, 1, -4 \rangle, \quad n_5 = \langle 1, 1, 1, -4 \rangle, \quad n_6 = \langle -3, -3, 2, 2 \rangle, \]
\[ n_7 = \langle -3, 2, -3, 2 \rangle, \quad n_8 = \langle -3, 2, 2, -3, 2 \rangle, \quad n_9 = \langle -3, 2, 2, -3 \rangle, \]
\[ n_{10} = \langle 2, -3, -3, 2 \rangle, \quad n_{11} = \langle 2, -3, 2, -3, 2 \rangle, \quad n_{12} = \langle 2, -3, 2, -3 \rangle, \]
\[ n_{13} = \langle 2, 2, -3, -3, 2 \rangle, \quad n_{14} = \langle 2, 2, -3, -3, 3 \rangle, \quad n_{15} = \langle 2, 2, 2, -3, 3 \rangle. \]

Thus we see that each normal vector of three-dimensional subspaces of \( \mathbb{R}^4 \) generated by positive roots of \( Q \) is one of \( ^tP \cdot n_k \) for \( k = 1, 2, \ldots, 15 \). We denote by \( L_k \) the refined lattice having a normal vector \( ^tP \cdot n_k \); then all the refined lattices of corank one of type \( A_4 \) is given by

\[ L_1 = \langle a_1, a_2, a_4 \rangle, \quad L_2 = \langle a_1, a_2, a_5 \rangle, \quad L_3 = \langle a_1, a_5, a_7 \rangle, \]
\[ L_4 = \langle a_3, a_4, a_6 \rangle, \quad L_5 = \langle a_3, a_4, a_7 \rangle, \quad L_6 = \langle a_1, a_2, a_7 \rangle, \]
\[ L_7 = \langle a_1, a_5, a_8 \rangle, \quad L_8 = \langle a_3, a_4, a_9 \rangle, \quad L_9 = \langle a_2, a_4, a_{10} \rangle, \]
\[ L_{10} = \langle a_1, a_4, a_9 \rangle, \quad L_{11} = \langle a_3, a_5, a_8 \rangle, \quad L_{12} = \langle a_2, a_6, a_8 \rangle, \]
\[ L_{13} = \langle a_2, a_6, a_7 \rangle, \quad L_{14} = \langle a_3, a_5, a_7 \rangle, \quad L_{15} = \langle a_1, a_4, a_7 \rangle. \]

Therefore, a scalar-removed representation \((S_d, R_d(Q))\) with dimension vector \( d = (d_1, d_2, d_3, d_4) \) is not an FP if and only if \( d \) is contained in at least one of the 15 lattices listed above. Note that, if \( d \) is sincere, it cannot be contained in \( L_1, L_5, L_6, \) and \( L_{15} \). In a similar manner to this, we can also obtain sufficiently concrete conditions for such \( d \); see and compare with [2, Example 4.3].

References


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