International Journal of Algebra, Vol. 5, 2011, no. 7, 325 - 334

f-Derivations of Weak BCC-Algebras

Janus Thomys

Hinueberstr. 13 A, 30175 Hannover, Germany
janus.thomys@htp-tel.de

Abstract. We describe f-derivations of weak BCC-algebras in which the condition $(xy)z = (xz)y$ is satisfied in the case when elements x, y belong to the same branch.

Mathematics Subject Classification: 03G25, 06F35

Keywords: weak BCC-algebra; solid weak BCC-algebra; BZ-algebra; derivation

1 Introduction

A weak BCC-algebras (called also BZ-algebras) are a generalization of BCI-algebras. In BCI-algebras an important role plays the identity $(xy)z = (xz)y$ which in weak BCC-algebras is not satisfied. So, results proved for BCI-algebras cannot be transferred to weak BCC-algebras, but as observed W.A. Dudek (see [5]) the part of these results can be transferred to weak BCC-algebras in which the equation $(xy)z = (xz)y$ is satisfied in the case when elements x, y belong to the same branch and z is an arbitrary element of G.

In this short note we prove that the part of results on f-derivations of BCI-algebras can be transferred to some weak BCC-algebras.

2 Preliminaries

The BCC-operation will be denoted by juxtaposition. Dots will be only used to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written in the abbreviated form as $(xy \cdot zy) \cdot xz = 0$.

Definition 2.1. A *weak BCC-algebra* is a system $(G; \cdot, 0)$ of type $(2, 0)$ satisfying the following axioms:

(i) $(xy \cdot zy) \cdot xz = 0$,

(ii) $xx = 0$,

(iii) $x0 = x$,

(iv) $xy = yx = 0 \Longrightarrow x = y$.

By many mathematicians, especially from China and Korea, weak BCC-algebras are called *BZ-algebras* (cf. [8], [14], [15] or [16]), but we save the first name because it coincides with the general concept of names presented in the book [9] for algebras of logic.

A weak BCC-algebra satisfying the identity

(v) $0x = 0$

is called a *BCC-algebra*. A BCC-algebra with the condition

(vi) $(x \cdot xy)y = 0$

is called a *BCK-algebra*.

One can prove (see [2] or [3]) that *a BCC-algebra is a BCK-algebra if and only if it satisfies the identity*

(vii) $xy \cdot z = xz \cdot y$.

An algebra $(G; \cdot, 0)$ of type $(2, 0)$ satisfying the axioms (i), (ii), (iii), (iv) and (vi) is called a *BCI-algebra*. A weak BCC-algebra is a BCI-algebra if and only if it satisfies (vii).

In any weak BCC-algebra we can define a natural partial order $\leqslant$ by putting

$$x \leqslant y \Longleftrightarrow xy = 0.$$

Directly from the axioms of weak BCC-algebras we can see that the following two implications

$(viii)$ $x \leqslant y \Longrightarrow xz \leqslant yz$,

(ix) $x \leqslant y \Longrightarrow zy \leqslant zx$

are valid for all $x, y, z \in G$.

The set of all minimal (with respect to $\leqslant$) elements of G is denoted by $I(G)$. Elements belonging to $I(G)$ are called *initial*.

In the investigation of algebras connected with various types of logics an important role plays the so-called *Dudek's map* φ defined as $\varphi(x) = 0x$. The main properties of this map in the case of weak BCC-algebras are collected in the following theorem proved in [8].

Theorem 2.2. *Let G be a weak BCC-algebra. Then*

(1) $\varphi^2(x) \leqslant x$,

(2) $x \leqslant y \Longrightarrow \varphi(x) = \varphi(y)$,

(3) $\varphi^3(x) = \varphi(x)$,

(4) $\varphi^2(xy) = \varphi^2(x)\varphi^2(y)$,

for all $x, y \in G$. □

Theorem 2.3. $I(G) = \{a \in G : \varphi^2(a) = a\}$. □

The proof of this theorem is given in [6]. Comparing this result with Theorem 2.2 (4) we obtain

Corollary 2.4. $I(G)$ *is a subalgebra of* G. □

Corollary 2.5. $I(G) = \varphi(G)$ *for any weak BCC-algebra* G. □

The set

$$B(a) = \{x \in G : a \leqslant x\},$$

where $a \in I(G)$ is called a *branch* of G initiated by a. The branch initiated by 0 is the greatest BCC-algebra contained in G.

Definition 2.6. A weak BCC-algebra G is called *solid* if (vii) is valid for all x, y belonging to the same branch and arbitrary $z \in G$.

Such weak BCC-algebras were introduced in [5].

Theorem 2.7. *[11] In solid weak BCC-algebras the map* φ *is a homomorphism.* □

Definition 2.8. A non-empty subset A of a weak BCC-algebra G is called a *BCC-ideal* if

(1) $0 \in A$,
(2) $y \in A$ and $xy \cdot z \in A$ imply $xz \in A$.

Using $(viii)$ and (ix) it is not difficult to see that $B(0)$ is a BCC-ideal of each weak BCC-algebra. The relation $\sim$ defined by

$$x \sim y \iff xy,\, yx \in B(0)$$

is a congruence on G. Its equivalence classes coincide with branches of G, i.e., $B(a) = C_a$ for any $a \in I(G)$ (cf. [6]). So, $B(a)B(b) = B(ab)$ and $xy \in B(ab)$ for $x \in B(a)$, $y \in B(b)$.

In the following part of this paper, we will need those two propositions proved in [6].

Proposition 2.9. *Elements $x, y \in G$ are in the same branch if and only if $xy \in B(0)$.* □

Proposition 2.10. *If $x, y \in B(a)$, then also $x \cdot xy$ and $y \cdot yx$ are in $B(a)$.* □

One of important classes of weak BCC-algebras is the class of *group-like weak BCC-algebras* called also *anti-grouped BZ-algebras* [15], i.e., weak BCC-algebras containing only one-element branches. A special case of such algebras are group-like BCI-algebras described in [1].

The conditions under which a weak BCC-algebra is group-like are found in [6] and [15]. Below we present some of these conditions.

Theorem 2.11. *A weak BCC-algebra G is group-like if and only if at least one of the following conditions is satisfied:*

(1) $\varphi^2(x) = x$ *for all* $x \in G$,

(2) $\varphi(xy) = yx$ *for all* $x, y \in G$,

(3) $\operatorname{Ker}\varphi = \{0\}$. □

3 f-derivations of weak BCC-algebras

Now we present a generalization of the concept of derivations.

Definition 3.1. Let f be an endomorphism of a weak BCC-algebra G. A map $d_f : G \longrightarrow G$ satisfying the identity

$$d_f(xy) = d_f(x)f(y) \wedge f(x)d_f(y)$$

is called a *left-right f-derivation* (briefly, *(l, r)-f-derivation*) of G.

Similarly is defined a *right-left f-derivation* (briefly, *(r, l)-f-derivation*) of G. A self-map d_f is called a *f-derivation* of G if it is both a (l, r)- and a (r, l)-f-derivation. A f-derivation d_f with the property $d_f(0) = 0$ is called *regular*.

For $f(x) = x$ we obtain a derivation considered in [13].

Example 3.2. Consider a weak BCC-algebra G defined in the Table 3.1. Since it is solid, $\varphi(x) = 0x$ is its endomorphism (see Theorem 2.7).

$\cdot$	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Table 3.1

The map d_φ defined by

$$d_\varphi(x) = \begin{cases} 2 & if \quad x \in \{0,1\}, \\ 0 & if \quad x \in \{2,3,4,5\} \end{cases}$$

– as it can easily be checked – is a non-regular derivation. Direct computation shows that it is also a φ-derivation of G. □

Example 3.3. Let G and d_φ be as in the previous example. Then obviously d_φ is a derivation of G but it is not a φ-derivation for an endomorphism $\varphi(x) = 0$ since $d_\varphi(2 \cdot 3) = d_\varphi(0) = 2 \neq d_\varphi(2)\varphi(3) \wedge \varphi(2)d_\varphi(3) = 0$. □

Example 3.4. The set $G = \{0, 1, 2, 3, 4, 5\}$ with the operation $\cdot$ defined by the table:

$*$	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Table 3.2

is a solid weak BCC-algebra.

Consider the map:

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 2 & 5 & 4 \end{pmatrix}.$$

Routine calculations show that f is an endomorphism of G which is not a derivation, but the map $d_f = f$ is a f-derivation of G. □

Example 3.5. The endomorphism φ^2, where φ is the Dudek's map, is a regular derivation of each solid weak BCC-algebra (see [13]). It is also a φ^2-derivation but it is not a φ-derivation. Indeed, φ is an endomorphism of each solid weak BCC-algebra but for a solid waek BCC-algebra defined in Example 3.4 we have

$$\varphi^2(1)\varphi(5)\wedge\varphi(1)\varphi^2(5) = \varphi(1)\varphi^2(5)\cdot(\varphi(1)\varphi^2(5)\cdot\varphi^2(1)\varphi(5)) = (1\cdot3)\cdot((1\cdot3)\cdot(0\cdot4)) = 3$$

and $\varphi^2(1 \cdot 5) = 2$. Hence, φ^2 is not a (l, r)-φ-derivation. It is also not a (r, l)-φ-derivation because

$$\varphi(2)\varphi^2(5)\wedge\varphi^2(2)\varphi(5) = \varphi^2(2)\varphi(5)\cdot(\varphi^2(2)\varphi(5)\cdot\varphi(2)\varphi^2(5)) = (2\cdot2)\cdot((2\cdot2)\cdot(3\cdot3)) = 0$$

and $\varphi^2(2 \cdot 5) = 3$. □

Proposition 3.6. *Each endomorphism f of a weak BCC-algebra G is its f-derivation.*

Proof. Straightforward. □

Theorem 3.7. *Let d_f be a self-map of weak BCC-algebra G. Then:*

(1) *if d_f is a regular (l,r)-f-derivation of G, then $d_f(x) = d_f(x) \wedge f(x)$,*

(2) *if d_f is a (r,l)-f-derivation of G, then $d_f(x) = f(x) \wedge d_f(x)$ if and only if d_f is regular.*

Proof. (1) Let d_f be a regular (l,r)-f-derivation of G. Then

$$d_f(x) = d_f(x0) = d_f(x)f(0) \wedge f(x)d_f(0) = d_f(x) \wedge f(x).$$

(2) If d_f is a (r,l)-f-derivation of G and $d_f(x) = f(x) \wedge d_f(x)$, then

$$d_f(0) = f(0) \wedge d_f(0) = 0 \wedge d_f(0) = 0.$$

Conversely, if $d_f(0) = 0$, then

$$d_f(x) = d_f(x0) = f(x)d_f(0) \wedge d_f(x)f(0) = f(x) \wedge d_f(x).$$

□

Theorem 3.8. *If d_f is a regular (r,l)-f-derivation of a solid weak BCC-algebra G, then $d_f(B(a)) \subset B(f(a))$.*

Proof. Let $x \in B(a)$. Then $a \leqslant x$, and consequently

$$0 = d_f(0) = d_f(ax) = f(a)d_f(x) \wedge d_f(a)f(x) = d_f(a)f(x) \cdot (d_f(a)f(x) \cdot f(a)d_f(x)),$$

i.e., $d_f(a)f(x) \leqslant d_f(a)f(x) \cdot f(a)d_f(x)$.

From this, applying $(viii)$, we obtain

$$0 = d_f(a)f(x) \cdot d_f(a)f(x) \leqslant (d_f(a)f(x) \cdot f(a)d_f(x)) \cdot d_f(a)f(x) = 0 \cdot f(a)d_f(x)$$

because G is solid. Hence, by Theorem 2.2, we have $0 \leqslant f(a)d_f(x)$. Thus $f(a)d_f(x) \in B(0)$, which means (Proposition 2.9) that $f(a)$ and $d_f(x)$ are in the same branch.

But, by Theorem 2.3, for every $a \in I(G)$ we have $f(a) = f(0 \cdot 0a) = 0 \cdot 0f(a)$. So, $f(a) \in I(G)$. Consequently, $d_f(x) \in B(f(a))$. Therefore $d_f(B(a)) \subset B(f(a))$. □

Theorem 3.9. *If a (r,l)-f-derivation of a solid weak BCC-algebra G is regular, then $d_f(a) = f(a)$ for every $a \in I(G)$.*

Proof. Since $d_f(B(a)) \subset B(f(a))$ for every $a \in I(G)$, we have $f(a) \leqslant d_f(a)$ i.e., the elements $f(a)$ and $d_f(a)$ are in the same branch.

But $d_f(a) = f(a) \wedge d_f(a) = d_f(a) \cdot d_f(a)f(a)$. Hence

$$d_f(a)f(a) = (d_f(a) \cdot d_f(a)f(a))f(a) = d_f(a)f(a) \cdot d_f(a)f(a) = 0.$$

Thus $d_f(a) \leqslant f(a)$. □

Theorem 3.10. *For any f-derivation d_f of a solid weak BCC-algebra G elements $f(x)$ and $d_f(d_f(x))$ are in the same branch.*

Proof. Let us assume $f(y) = d_f(x)$. Then from Definition 3.1 we obtain

$$d_f(xy) = d_f(x)f(y) \wedge f(x)d_f(y) = f(y)f(y) \wedge f(x)d_f(y) =$$

$$0 \wedge f(x)d_f(y) = f(x)d_f(y) \cdot (f(x)d_f(y) \cdot 0) = 0.$$

Thus $d_f(xy) = 0$ for $f(y) = d_f(x)$. This together with Theorem 2.2 gives

$$0 = d_f(xy) = f(x)d_f(y) \wedge d_f(x)f(y) = f(x)d_f(y) \wedge 0 = 0(0 \cdot f(x)d_f(y)) \leqslant f(x)d_f(y).$$

Therefore, $f(x)d_f(y) \in B(0)$, which shows (by Proposition 2.9) that the elements $f(x)$ and $d_f(y) = d_f(d_f(x))$ are in the same branch. □

Theorem 3.11. *A (r,l)-f-derivation (resp. (l,r)-f-derivation) d_f of a solid weak BCC-algebra G is regular if and only if for every $x \in G$ elements $f(x)$ and $d_f(x)$ belong to the same branch.*

Proof. Let d_f be a regular (r,l)-f-derivation of a solid weak BCC-algebra G. Then for any $x \in G$ we have

$$0 = d_f(xx) = f(x)d_f(x) \wedge d_f(x)f(x) = d_f(x)f(x) \cdot (d_f(x)f(x) \cdot f(x)d_f(x)),$$

which implies $d_f(x)f(x) \leqslant d_f(x)f(x) \cdot f(x)d_f(x)$. From this, by $(viii)$, we obtain

$$0 = d_f(x)f(x) \cdot d_f(x)f(x) \leqslant (d_f(x)f(x) \cdot f(x)d_f(x)) \cdot d_f(x)f(x)$$

$$= (d_f(x)f(x) \cdot d_f(x)f(x)) \cdot f(x)d_f(x) = 0 \cdot f(x)d_f(x).$$

So, $f(x)d_f(x) \in B(0)$. This, according to Proposition 2.9, means that $f(x)$ and $d_f(x)$ are in the same branch.

Conversely, let for every $x \in G$ elements $f(x)$ and $d_f(x)$ be in the same branch. Then $d_f(x)$, $d_f(d_f(x))$ are in the same branch, too. So, $f(x)d_f(x), d_f(x)f(x)$

and $f(x)d_f(d_f(x))$ are in $B(0)$. Thus $0 = 0 \cdot f(x)d_f(x) = 0 \cdot d_f(x)f(x) = 0 \cdot f(x)d_f(d_f(x))$. Hence

$$\begin{aligned} d_f(0) &= d_f(0 \cdot f(x)d_f(x)) = (0 \cdot d_f(f(x)d_f(x))) \wedge (d_f(0) \cdot f(x)d_f(x)) \\ &= (0 \cdot ((f(x)d_f(d_f(x))) \wedge (d_f(f(x))d_f(x)))) \wedge (d_f(0) \cdot f(x)d_f(x)) \\ &= (0 \cdot (d_f(f(x))d_f(x) \cdot (d_f(f(x))d_f(x) \cdot f(x)d_f(d_f(x))))) \wedge (d_f(0) \cdot f(x)d_f(x)) \\ &= ((0 \cdot d_f(f(x))d_f(x))((0 \cdot d_f(f(x))d_f(x))(0 \cdot f(x)d_f(d_f(x))))) \wedge (d_f(0) \cdot f(x)d_f(x)) \\ &= ((0 \cdot d_f(f(x))d_f(x))((0 \cdot d_f(f(x))d_f(x))0)) \wedge (d_f(0) \cdot f(x)d_f(x)) \\ &= ((0 \cdot d_f(f(x))d_f(x))(0 \cdot d_f(f(x))d_f(x))) \wedge (d_f(0) \cdot f(x)d_f(x)) \\ &= 0 \wedge (d_f(0) \cdot f(x)d_f(x)) \\ &= (d_f(0) \cdot f(x)d_f(x))((d_f(0) \cdot f(x)d_f(x))0) \\ &= (d_f(0) \cdot f(x)d_f(x))(d_f(0) \cdot f(x)d_f(x)) = 0 \end{aligned}$$

i.e., d_f is regular.

In a similar way we can prove the theorem for a (l, r)-f-derivation. □

Corollary 3.12. *A f-derivation d_f of a solid weak BCC-algebra G is regular if and only if for every $x \in G$ elements $f(x)$ and $d_f(x)$ are in the same branch.* □

Proposition 3.13. *Let d_f be a self-map of a solid weak BCC-algebra G defined by $d_f(x) = \varphi^2(f(x))$ for all $x \in B(a), a \in I(G)$. Then d_f is a branchwise regular f-derivation of G.*

Proof. Firstly observe, that if $x \in B(a)$, then $f(x) \in f(B(a)) = B(f(a))$. And for $x \in B(a)$ we have $d_f(x) = \varphi^2(f(x)) = \varphi^2(f(a)) = f(a)$. Then for $x, y \in B(a)$

$$\begin{aligned} &d_f(x)f(y) \wedge f(x)d_f(y) = f(a)f(y) \wedge f(x)f(a) = \\ &f(x)f(a) \cdot (f(x)f(a) \cdot f(a)f(y)) = f(x)f(a) \cdot (f(x)f(a) \cdot f(0)) = f(0) = 0. \end{aligned}$$

On the other hand $d_f(xy) = \varphi^2(f(xy)) = 0$, because $f(xy) = f(x)f(y) \in B(0)$. Hence, we have $d_f(xy) = d_f(x)f(y) \wedge f(x)d_f(y)$, which proves that d_f is (l, r)-f-derivation of G.

Now, we have

$$f(x)d_f(y) \wedge d_f(x)f(y) = f(x)f(a) \wedge f(a)f(y) = f(x)f(a) \wedge 0 = 0.$$

Hence $d_f(x)$ is (r, l)-f-derivation of G.

Since $d_f(0) = f(0) = 0$, then $d_f(x)$ is regular. □

Theorem 3.14. *A solid weak BCC-algebra G is group-like if and only if $Ker(d_f) = \{0\}$ for every regular derivation d_f of G.*

Proof. (⇒) If G is group-like then $G = I(G)$ and $B(0) = \{0\}$. From Theorem 3.8 $d_f(0) = 0$. This implies the regularity of every f-derivation of G.

(⇐) Let $Ker(d_f) = \{0\}$ for every regular f-derivation d_f of G. Let us define a self-map $d_{f,a}$ of G by $d_{f,a}(x) = f(a)$ for $x \in B(a), a \in I(G)$. From Proposition 3.13 $d_{f,a}$ is a regular f-derivation of G and $Ker(d_{f,a}) = \{0\}$. From the definition of the map $d_{f,0}$ for every $x \in B(0)$ $d_{f,0}(x) = 0$, thus $B(0) \subseteq Ker(d_{f,0}) = \{0\}$. From Theorem 2.11 G is group-like. □

Theorem 3.15. *If f is an endomorphism of a weak BCC-algebra G such that $f(G) = G$, then a f-derivation d_f of G is regular if and only if $d_f(A) \subset f(A)$ for all BCC-ideals A of G.*

Proof. For a regular f-derivation of a solid weak BCC-algebra by Theorem 3.7 (2) and Corollary 3.12 we have $d_f(x) = f(x) \wedge d_f(x) \leqslant f(x)$ for all $x \in G$. Let $x \in A$ for some BCC-ideal A of G. Then, $f(A)$ is a BCC-ideal in $f(G) = G$ and $f(x) \in f(A)$. Hence, $d_f(x)f(x) = 0 = f(0) \in f(A)$, and consequently $d_f(x) \in f(A)$. Thus, $d_f(A) \subset f(A)$ for any BCC-ideal A of G.

Conversely, if $d_f(A) \subset f(A)$ for each BCC-ideal A of G, then also $d_f(\{0\}) \subset f(\{0\}) = \{0\}$. Hence $d_f(0) = 0$, i.e., d_f is regular. □

References

[1] W.A. Dudek, *On group-like BCI-algebras*, Demonstratio Math., 21 (1988), 369 – 376.

[2] W.A. Dudek, *On BCC-algebras*, Logique et Analyse, 129-130 (1990), 103 – 111.

[3] W.A. Dudek, *On proper BCC-algebras*, Bull. Inst. Math. Acad. Sinica, 20 (1992), 137 – 150.

[4] W.A. Dudek, *Remarks on the axioms system for BCI-algebras*, Prace Naukowe WSP w Czestochowie, ser. Matematyka, 2 (1996), 46 – 61.

[5] W.A. Dudek, *Solid weak BCC-algebras*, Internat. J. Computer Math., in print.

[6] W.A. Dudek, B. Karamdin, S.A. Bhatti, *Branches and ideals of weak BCC-algebras*, Algebra Coll. (2011), in print.

[7] W.A. Dudek, X.H. Zhang, *On ideals and congruences in BCC-algebras*, Czechoslovak Math. J., 48(123) (2000), 21 – 29.

[8] W.A.Dudek, X.H.Zhang, Y.Q.Wang, *Ideals and atoms of BZ-algebras*, Math. Slovaca, 59 (2009), 387 – 404.

[9] A. Iorgulescu, *Algebras of logic as BCK algebras*, Acad. Econom. Studies, Bucharest, 2008.

[10] Y.B. Jun, X.L. Xin, *On derivators of BCI-algebras*, Information Sciences, 159 (2004), 167 – 176.

[11] B. Karamdin, J. Thomys, *Quasi-commutative weak BCC-algebras*, in preparation.

[12] Ch. Prabpayak, U. Lerrawat, *On derivations of BCC-algebras*, Kasetsart J. (Nat. Sci.), 43 (2009), 398 – 401.

[13] J.Thomys, *Derivations of weak BCC-algebras*, Scientiae Mathematicae Japonicae (SCMJ), in print.

[14] R.F. Ye, *On BZ-algebras*, Selected Papers on BCI, BCK-algebras and Computer Logics, Shanghai Jiaotong Univ. Press, (1991), 21 – 24.

[15] X.H. Zhang, R. Ye, *BZ-algebras and groups*, J. Math. Phys. Sci. 29 (1995), 223 – 233.

[16] X.H. Zhang, Y.Q. Wang, W.A. Dudek, *T-ideals in BZ-algebras and T-type BZ-algebras*, Indian J. Pure Appl. Math. 34 (2003), 1559 – 1570.

Received: October, 2010