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The Number of Fuzzy Subgroups of Group Defined by A Presentation

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Abstract

In this paper, we count the number of fuzzy subgroups of group $G(\wp)$ defined by presentation $\wp = \langle a, b : a^2, b^q, ab = b^r a\rangle$ with q is a prime number and $r < q$. We define an equivalence relation on the fuzzy subgroups of any groups G. Then we identify the form and the order of elements of $G(\wp)$. We determine all subgroups and draw the diagram of subgroups lattice of $G(\wp)$. The number of fuzzy subgroups of $G(\wp)$ can be determined by counting the number of chain on lattice.

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1 Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965 [1]. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971 (in [2]). Without any equivalence relation on fuzzy subgroups of group G, the number of fuzzy subgroups is infinite, even for the trivial group $\{e\}$. Some authors have used the equivalence relation of fuzzy sets to study the equivalence of fuzzy subgroups ([3], [4], [5], [6], [7], [8]). All of them have treated the particular case of finite abelian group.

It is interesting to investigate the number of fuzzy subgroups of nonabelian groups. In the first paper of this topic [9], we have counted the number of fuzzy subgroups of nonabelian symmetric groups S_2, S_3 and alternating group A_4.

In this paper, we will count the number of fuzzy subgroups of group $G(\wp)$ in which group presentation $\wp$ is given by $\wp = \langle a, b : a^2, b^q, ab = b^r a\rangle$ with q is a prime number and $r < q$.

2 Preliminaries

Some basic notions and results that will be used later are given in this section. In this section, a group G is assumed to be a finite group.

Definition 2.1 *(Rosenfeld, [2]). Let X be a nonempty set. A fuzzy set of X is a function μ from X into [0, 1].*

Definition 2.2 *(Rosenfeld, [2]).A fuzzy subset of G is called a fuzzy subgroup of G if $\mu(xy) \geq min\{\mu(x), \mu(y)\}$, $\forall x, y \in G$ and $\mu(x^{-1}) \geq \mu(x)$, $\forall x \in G$.*

Theorem 2.3 *(Rosenfeld, [2]). Let e denote the identity element of G. If μ is a fuzzy subgroup of G, then $\mu(e) \geq \mu(x), \forall x, y \in G$ and $\mu(x^{-1}) = \mu(x), \forall x \in G$.*

Theorem 2.4 *.A fuzzy subset μ of G is a fuzzy subgroup of G if and only if there is a chain of subgroups of G, $P_1(\mu) \leq P_2(\mu) \leq ... \leq P_n(\mu) = G$ such that μ can be written as*

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1(\mu) \\ \theta_2, & x \in P_2(\mu) \\ \vdots & \\ \theta_n, & x \in P_n(\mu) \end{cases} \tag{1}$$

Proof. *Let*

$$\mu(x) = \begin{cases} \theta_1, & x \in B_1 \\ \theta_2, & x \in B_2 \\ \vdots & \\ \theta_n, & x \in B_n \end{cases} \tag{2}$$

where $\bigcup_{1\leq i\leq p} B_i = G$.
Let $B_0 = \emptyset, P_m = \bigcup_{1\leq i\leq m} B_i$ for $1 \leq m \leq p$. Using these symbols we have, $B_m = \{x \in G \mid \mu(x) = \theta_m\}, P_m = \{x \in G \mid \mu(x) \geq \theta_m\}, B_m = P_m \backslash P_{m-1}$ and $P_p = \bigcup_{1\leq i\leq p} B_i = G$.
($\Longrightarrow$) Let m be any arbitrary element in $\{1, 2, ..., p\}$. Note that $e \in P_1$ and hence $e \in P_m$. If $x, y \in P_m$, then $\mu(x) \geq \theta_m$ and $\mu(y) \geq \theta_m$. Hence $\mu(xy) \geq min\{\mu(x), \mu(y)\}$. Thus $\mu(xy) \geq \theta_m$. It means $xy \in P_m$.

From Theorem 2.3, $\mu(x^{-1}) = \mu(x) \geq \theta_m$. Hence $x^{-1} \in P_m$. Thus, P_m is a subgroup of G. We obtain the chain $P_1(\mu) \leq P_2(\mu) \leq ... \leq P_m(\mu) = G$ of subgroups G and μ as in (2)can be written as in (1).
($\Leftarrow$) Now, let $\forall m \in \{1, 2, ..., p\}$ P_m be a subgroup of G, $x, y \in G$ and $\mu(x) = \theta_k, \mu(y) = \theta_m$ for some natural numbers k and m with $k < m$. Then we have $x \in P_k, y \in P_m$. We conclude that $x, y \in P_m$, since $k < m$. Since P_m is a subgroup of G, then we have $xy \in P_m$. Therefore, $\mu(xy) \geq \theta_m = \mu(y)$. Also $\theta_m < \theta_k = \mu(x)$. Thus, $\mu(xy) \geq min\{\mu(x), \mu(y)\}$.
On the other hand, if $x \in G$ and $x \in B_i$ for some $i \in \{1, 2, ..., p\}$, then $\mu(x) = \theta_i$. Therefore $x \in P_i$. Since P_i is a subgroup of G, then $x^{-1} \in P_i$. Thus, $\mu(x^{-1} \geq \theta_i = \mu(x)$. We conclude that μ is a fuzzy subgroup of G.

Example 2.5 *Consider the group $G = Z_{12}$. Define functions $\mu, \gamma, \alpha, \beta$ as follows:*

$$\mu(x) = \begin{cases} 1, & x \in \{0, 2, 4, 6, 8, 10\} \\ \frac{1}{2}, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases},$$
$$\gamma(x) = \begin{cases} 1, & x \in \{0, 1, 2, 3, 4, 5\} \\ \frac{1}{2}, & x \in \{6, 7, 8, 9, 10, 11\} \end{cases},$$
$$\alpha(x) = \begin{cases} 0, & x \in \{0, 4, 8\} \\ \frac{1}{2}, & x \in \{2, 6, 10\} \\ \frac{1}{3}, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases},$$
$$\beta(x) = \begin{cases} 1, & x \in \{0, 4, 8\} \\ \frac{1}{2}, & x \in \{2, 6, 10\} \\ \frac{1}{4}, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases}.$$

Note that $P_1(\mu) = \{0, 2, 4, 6, 8, 10\}$ and $P_2(\mu) = Z_{12}$ both are subgroup of Z_{12}. According to Theorem 2.3, μ is a fuzzy subgroup of Z_{12}. Similarly, we can show that α and β are fuzzy subgroups of Z_{12}. Note that $P_1(\gamma) = \{1, 2, 3, 4, 5\}$ is not subgroup of Z_{12}. By Theorem 2.3, γ is not fuzzy subgroup of Z_{12} .

Definition 2.6 *(Sulaiman and Abdul Ghafur [9]). Let μ, γ be fuzzy subgroups of G of the form*

$$\mu(x) = \begin{cases} \theta_1, & x \in A_1 \\ \theta_2, & x \in A_2 \\ \vdots & \\ \theta_m, & x \in A_m \end{cases} \quad \text{and } \gamma(x) = \begin{cases} \delta_1, & x \in B_1 \\ \delta_2, & x \in B_2 \\ \vdots & \\ \delta_n, & x \in B_n \end{cases}$$

with $\theta_i, \delta_j \in [0, 1]$, $\theta_k > \theta_l$,$\delta_k > \delta_l$ for $k < l$ and $\bigcup_{1 \leq i \leq m} A_i = G, \bigcup_{1 \leq j \leq n} B_j = G$. Then we define that μ and γ are equivalent if $m = n$ and $A_i = B_i, \forall i \in \{1, 2, ..., n\}$.

It is easy to check that this relation is indeed an equivalence relation. Two fuzzy subgroups of G are said to be different if they are not equivalent. In this paper, the number of fuzzy subgroups of group G is defined to be the number distinct equivalence classes of fuzzy subgroups of group G. Using Theorem 2.4 and Definition 2.6, we conclude that

Lemma 2.7 *The number of fuzzy subgroups of G is equal to the number of chain on the lattice subgroups of G.*

Proof. Use Theorem 2.4 and Definition 2.6.

Example 2.8 *Let μ, α, β be fuzzy subgroups as in Example 2.5. Since $|(\mu)| \neq |(\alpha)|$, by Definition 2.6, μ is not equivalent to α. Fuzzy subgroups α and β are not equivalent too. Note that $|\alpha| = |\beta|$, $P_1(\alpha) = P_1(\beta)$ and $P_2(\alpha) = P_2(\beta)$. Thus α equivalent to β.*

3 Group Presentations

Lex X be a non empty set (alphabet). A word W on X is of the form

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots\, x_n^{\varepsilon_n} \tag{3}$$

with $n \geq 0, x_i \in X, \varepsilon_i = \pm 1$.
If $n = 0$, then this word is called *empty word* and denoted by 1. The inverse of the word W as in (3) is defined to be

$$W^{-1} = x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \ldots\, x_1^{-\varepsilon_1}.$$

A word W is *cyclically reduce* if the firs symbol of W is not the inverse of the last symbol of W.

Let $\Re$ be a set of cyclically reduce word in X. A *group presentation* $\wp$ is a pair $< X : \Re >$ where set $\Re$ is called *relation*. There are four operations on words, namely: elementary deletion (ED) for word is the operation to delete the inverse pairs $x^{\varepsilon} x^{-\varepsilon}$, the elementary insertion (EI) for word is the operation to insert the inverse pairs, deletion of $\Re$ word (DR) is the operation to delete some sub word $R \in \Re$ on word W and insertion of $\Re$ word (IR) is the operation to insert some sub word $R \in \Re$ on word W.

Definition 3.1 *Two words U and V on X is said to equivalent and denoted by $U \approx V$ if V can be obtained from U by a finite number of words operations ED, EI, DR or IR.*

Lemma 3.2 *The equivalent of words using ED, EI, DR or IR in $\wp$ is an equivalent relation.*

The equivalent class containing U is denoted by $[U]$.

Theorem 3.3 *The set of all equivalent of words in* $\wp$ *with binary operation*

$$[U] \bullet [V] = [UV]$$

form a group. This group is named group defined (presented) by $\wp$ *and denoted by* $G(\wp)$*.*

Alternative definition of group defined by $\wp$ is a factor group $G = F(X)/N$ with N is the normal closure of R in free group $F(X)$ (for more detail, see [10]) and we can show that $G(\wp) \cong F(X)/N$.

Example 3.4 *Consider group presentation* $\wp =< a, b|a^2, b^3, aba^{-1}b^{-1} >$*. We can check that* $ab \approx ba$*. Hence, if* W *is a word in* $\wp$*, then* W *can be expressed as* $W \approx a^p b^q$*. Since* $a^2 \approx 1$ *and* $b^3 \approx 1$ *then we conclude that* $G(\wp) = \{[1], [a], [b], [b^2], [ab], [ab^2]\}$*. We can show that* $G(\wp) \simeq Z_6$*.*

For simplification, we will write $[U] \in G(\wp)$ by U. Hence, group $G(\wp)$ as in Example 3.4 will be written by $G(\wp) = \{1, a, b, b^2, ab, ab^2\}$ instead of $\{[1], [a], [b], [b^2], [ab], [ab^2]\}$.

4 Main Results

In this section, the group presentation $\wp = \langle a, b : a^2, b^q, ab = b^r a\rangle$ with q is a prime number and $r < q$ will be considered. We determine the number of fuzzy subgroups of group $G(\wp)$. Firstly, we derive theorem on order of elements $G(\wp)$. Then we draw diagram of subgroups lattice of $G(\wp)$. We observe two cases, namely for $q - r > 1$ and another for $q - r = 1$.

Theorem 4.1 *The order of* $G(\wp)$ *is* $2q$*.*
Proof. *By using the relation* $ab = b^r a$*, every elements of* $G(\wp)$ *is equivalent to a word of the form* $b^m a^n$*. The relations* $a^2 = e, b^q = e$ *give us the conclusion that* $m \in \{0, 1, 2, ..., q-1\}$ *and* $n \in \{0, 1\}$*. It is easy to check that:*

1. $b^i a \neq b^j a$ *for* $i \neq j$ *where* $0 < i, j \leq q - 1$.
2. $b^i a \neq b^j, b^i a \neq a, \forall i, j$ *where* $0 \leq i, j \leq q - 1$.

Hence, we have $G(\wp) = \{e, a, b, b^2, ..., b^{q-1}, ba, b^2 a, ..., b^{q-1}a\}$*. Thus the order of* $G(\wp)$ *is* $2q$*.*

Theorem 4.2 *For* $m < q$*,* $a(b^m a) = b^{mr}$*.*
Proof. *Note that* $a(b^m a) = ab(b^{m-1}a) = b^r a(b^{m-1}a) = b^r(ab)b^{m-2}a = b^{2r}ab^{m-2}a = b^{2r}(ab)b^{m-3}a = b^{2r}(b^r a)b^{m-3}a = b^{3r}ab^{m-3}a$*.*
By doing substitution $ab = b^r a$ *for* $m-$*times, we obtain* $a(b^m a) = b^{mr}a(b^{m-m}a) = b^{mr}a^2 = b^{mr}$*.*

Using Theorem 4.2, we will derive $(b^m a)^n$ into the form $b^i a^j$ for $n \in N$. In general, we have

Theorem 4.3 *For $k \in N$, we have:*
$(i)(b^m a)^{2k} = b^{km+k(mr)}$
$(ii)(b^m a)^{2k-1} = b^{(k-1)mr+km} a.$
Proof. *We prove by induction on k.*
(i) *Clearly $(b^m a)^2 = b^m a b^m a = b^m(a(b^m a)) = b^{m+mr}$. Assumed that the statement is true for $k = n$. It means $(b^m a)^{2n} = b^{nm+n(mr)}$. For $k = n+1$, $(b^m a)^{2(n+1)} = (b^m a)^{2n+2} = (b^m a)^{2n}(b^m a)^2 = b^{nm+n(mr)} b^{m+mr} = b^{(n+1)m+(n+1)r}$. This complete the induction.*
(ii) *Clearly the statement is true for $k = 1$. Assumed that the statement is true for $k = n$, that is $b^m a)^{2n-1} = b^{(n-1)mr+nm} a$. For $k = n+1$, $(b^m a^{2(n+1)-1} = (b^m a)^2)(b^m a)^{2n+1} = b^{(n+1)m+nmr} a$. Thus the induction is complete.*

Theorem 4.3 shows that the order of $(b^m a)$ must be even.

Theorem 4.4 *If $q - r > 1$ and $0 < m < q$ then $o(b^m a) = 2q$.*
Proof. *Note that $(b^m a)^{2q} = e$. Assumed that there is some $k \in N$, $k < 2q$ such that $o(b^m a) = k$. Thus $(b^m a)^k = e$. According to the theorem 5.3, k must be an even number. Let $k = 2p$ for some $p \in N$, then $p < q$. Using theorem 5.3 (i), we get $(b^m a)^k = (b^m a)^{2p} = b^{pm+(pm)r} = e$. So, $pm + (pm)r = pm(1 + r)$ must be multiple of q. Since $p < q, m < q, (1 + r) < q$ and q is a prime number, then q is not a factor of $pm(1 + r)$ and hence a contradiction. Thus, $o(b^m a) = 2q$.*

Theorem 4.5 *If $q - r > 1$ then the number of fuzzy subgroup of $G(\wp)$ is 6.*
Proof. *From Theorem 4.1, the order of $G(\wp)$ is $2q$. Hence every subgroup of $G(\wp)$ must be of order $1, 2, q$ or $2q$. According to the Theorem 4.4, we have $G(\wp) = \langle ba\rangle = \langle b^2 a\rangle = ... = \langle b^{q-1} a\rangle$. Therefore there are only four subgroups of $G(\wp)$. Those are $\{e\}, < a \mid a^2 >, < b \mid b^q >$ and $G(\wp)$ of order $1, 2, q$ and $2q$, respectively. Diagram of lattice subgroups of $G(\wp)$ as in Fig. 1. Now, we will construct all of different fuzzy subgroups of $G(\wp)$. Let μ be a fuzzy subgroup of $G(\wp)$ and $\theta_i \in [0, 1]$ for $i \in \{1, 2, 3\}$. Since the length of maximal chain on the lattice subgroups of $G(\wp)$ is 3, then μ must be of length 1, 2 or 3. We have six chains on the lattice subgroups of $G(\wp)$, namely $G(\wp), \langle a\rangle \leq G(\wp)$, $\langle b\rangle \leq G(\wp), \{e\} \leq G(\wp), \{e\} \leq \langle a\rangle \leq G(\wp)$ and $\{e\} \leq \langle a\rangle \leq G(\wp)$. Therefore, we get six fuzzy subgroups of $G(\wp)$, namely*

$$\mu_1(x) = \theta_1, \forall x \in G,\ \mu_2(x) = \begin{cases} \theta_1, & x \in \langle a\rangle \\ \theta_2, & x \in G\backslash\langle a\rangle \end{cases},$$

$$\mu_3(x) = \begin{cases} \theta_1, & x \in \langle b\rangle \\ \theta_2, & x \in G\backslash\langle b\rangle \end{cases},\ \mu_4(x) = \begin{cases} \theta_1, & x \in \{e\} \\ \theta_2, & x \in G\backslash\{e\} \end{cases},$$

$$\mu_5(x) = \begin{cases} \theta_1, & x \in \{e\} \\ \theta_2, & x \in \langle a \rangle \backslash \{e\} \\ \theta_3, & x \in G \backslash \langle a \rangle \end{cases}, \ \mu_6(x) = \begin{cases} \theta_1, & x \in \{e\} \\ \theta_2, & x \in \langle b \rangle \backslash \{e\} \\ \theta_3, & x \in G \backslash \langle b \rangle \end{cases}.$$

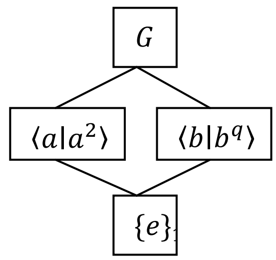

Figure 1: Subgroup lattice of $G(\wp)$ for $q - r > 1$.

Theorem 4.6 *If* $q - r = 1$ *then:*

$(i) o(ba) = 2$

$(ii) o(b^m a) = 2, \forall m < q$

Proof.

$(i)(ba)(ba) = b(b^r a)a = e.$

$(ii)(b^m a)(b^m a) = b^m(a(b^m a))$. *According to Theorem 4.2, we obtain* $(b^m a)(b^m a) = b^m b^{mr} = e$.

Theorem 4.7 *If* $q - r = 1$ *then the number of fuzzy subgroups of* $G(\wp)$ *is* $2(q+2)$.

Proof. *The order of* $G(\wp)$ *is* $2q$. *Therefore, nontrivial subgroups of* $G(\wp)$ *must be of order 2 or* q. *According to Theorem 4.6, subgroups of* $G(\wp)$ *of order 2 are cyclic groups* $\langle a \rangle, \langle ba \rangle, \langle b^2 a \rangle, ..., \langle b^{q-1} a \rangle$. *We have only one subgroup of* $G(\wp)$ *of order* q, *namely* $\langle b \rangle$. *We obtain diagram of lattice subgroups of* $G(\wp)$ *as in Fig. 2. We have only one chain of length 1 namely* $G(\wp)$. *There are* $(q+2)$ *chain of length 2, namely* $\langle a \rangle \leq G(\wp), \langle b \rangle \leq G(\wp), \langle ba \rangle \leq G(\wp), \langle b^2 a \rangle \leq G(\wp), ..., \langle b^{q-1} a \rangle \leq G(\wp)$ *and* $\{e\} \leq G(\wp)$ *whereas the number of chain of length 3 is* $(q+1)$, *namely* $\{e\} \leq \langle a \rangle \leq G(\wp), \{e\} \leq \langle b \rangle \leq G(\wp), \{e\} \leq \langle ba \rangle \leq G(\wp), \{e\} \leq \langle b^2 a \rangle \leq G(\wp), ..., \{e\} \leq \langle b^{q-1} a \rangle \leq G(\wp)$. *The total number of chains on the lattice subgroups of* $G(\wp)$ *is* $2(q+1)$. *From Lemma 2.7, we have proved this theorem.*

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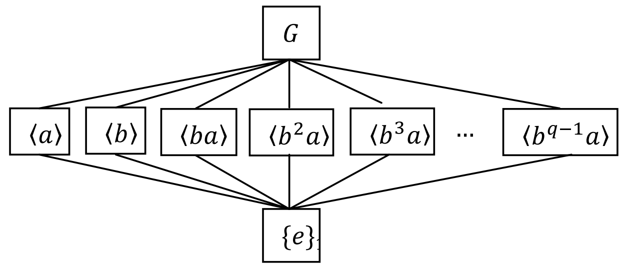

Figure 2: Subgroup lattice of $G(\wp)$ for $q - r = 1$.

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