Abstract

In this paper we introduce the concepts of $B_1$ near-rings and strong $B_1$ near-rings. We say that a right near-ring $N$ is a $B_1$ near-ring if for every $a \in N$, there exists $x \in N^*$ where $N^* = N - \{0\}$, such that $Nax = Nxa$. By the way of generalization, we define $N$ as a strong $B_1$ near-ring if $Nab = Nba$ for all $a, b \in N$. We discuss some of their properties, obtain a characterisation and also a structure theorem.

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1 Introduction

Throughout this paper $N$ stands for a right near-ring $(N, +, .)$, with at least two elements and ‘0’ denotes the identity element of the group $(N, +)$. Obviously $0n = 0$ for all $n \in N$. If, in addition, $n0 = 0$ for all $n \in N$ then we say that $N$ is zero symmetric. A subgroup $(M, +)$ of $(N, +)$ is called an $N$-subgroup of $N$ if $NM \subset M$ and an invariant $N$-subgroup of $N$ if $MN \subset M$ as well. $N$ is called weak commutative if $abc = acb$ for all $a, b, c \in N$ (Definition 9.4, p.289 of Pilz[4]). $N$ is said to be regular if for every $a \in N$ there exists $b \in N$ such that $a = aba$. An element $a$ is said to be nilpotent if $a^k = 0$ for some positive integer $k$. $N$ is called nil if every element of $N$ is nilpotent. $N$ is
called integral if $N$ has no non-zero zero divisors. $N$ is called a $P_k$ near-ring if there exists a positive integer $k$ such that $x^k N = xN x$ for all $x \in N$. For any subset $A$ of $N$, we denote by $A^*$ the set of all non-zero elements of $A$. In particular $N^* = N - \{0\}$. $N$ is called a strong $S_1$ near-ring if $N^* = N_{S_1}(a)$ for all $a \in N$ where $N_{S_1}(a) = \{x \in N^*/axa = xa\}$. For basic concepts and terms used but left undefined in this paper we refer to Pilz.

2 Preliminary Results

We freely make use of the following results from [3], [4] and [5] and designate them as $R(1), R(2), \ldots$,

$R(1)$: $N$ has no non-zero nilpotent elements if and only if $a^2 = 0 \Rightarrow a = 0$ for all $a \in N$ (Problem 14, p.9 of [3]).

$R(2)$: $N$ is zero symmetric if and only if every left ideal of $N$ is an $N$-subgroup of $N$ (Proposition 1.34(b), p.19 of [4]).

$R(3)$: Let $N$ be zero symmetric. Then the following are equivalent:
(i) $N$ has no non-zero nilpotent elements. (ii) $N$ is a subdirect product of integral near-rings (Theorem 9.36, p.302 of [4]).

$R(4)$: If $N$ is a strong $S_1$ near-ring then $N$ is zero symmetric (Proposition 5.6 of [5]).

$R(5)$: $N$ is a strong $S_1$ near-ring if and only if $axa = xa$ for all $a, x \in N$ (Theorem 5.8 of [5]).

3 $B_1$ near-rings

As in [2] a right near-ring $N$ is said to be left bipotent if $Na = Na^2$ for all $a \in N$. Motivated by this we have the following definition.

Definition 3.1 We say that $N$ is a $B_1$ near-ring if for every $a \in N$, there exists $x \in N^*$ such that $Nax = Nxa$.

Examples 3.2 (a) Every constant near-ring is a $B_1$ near-ring.
(b) We consider the near-ring $(\mathbb{Z}_4, +, \cdot)$ where $(\mathbb{Z}_4, +)$ is the group of integers modulo ‘4’ and ‘.’ is defined as follows (scheme(4), p.407 of Pilz [4]).

$$
\begin{array}{c|ccc}
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 3 & 0 \\
3 & 0 & 0 & 2 & 0 \\
\end{array}
$$

This is a $B_1$ near-ring.
**Theorem 3.3** Let $N$ be a near-ring. Each of the following statements implies that $N$ is a $B_1$ near-ring.

(i) $N$ is a zero symmetric nil near-ring.

(ii) $N$ is weak commutative.

(iii) $N$ has identity ‘1’.

(iv) $N$ is a near-field.

**Proof** (i) Let $a \in N$. If $a = 0$, then for any $x \in N^*$, $Nax = Nxa = N0 = \{0\}$. If $a \in N^*$, since $N$ is nil, there exists a positive integer $k$ such that $a^k = 0$. Put $x = a^{k-1} \neq 0$. Now $Nax = Naa^{k-1} = Naa^k = Na^{k-1}a = Nxa = N0 = \{0\}$. Thus $N$ is a $B_1$ near-ring.

(ii) Let $a \in N$. For any $x \in N^*$, $y \in Nax \Rightarrow y = nax$ where $n \in N$. Since $N$ is weak commutative, $y = nxa \in Nxa$. Therefore $Nax \subset Nxa$. Similarly $Nxa \subset Nax$ and (ii) follows.

(iii) Follows by taking $x = 1$ in the Definition 3.1.

(iv) Follows from (iii).

**Theorem 3.4** Let $N$ be a $B_1$-near-ring. If $N$ is a strong $S_1$ near-ring without non-zero zero divisors then the following are true.

(i) Every non-zero $N$-subgroup of $N$ is an $B_1$ near-ring.

(ii) Every non-zero ideal of $N$ is an $B_1$ near-ring.

**Proof** Since $N$ is a strong $S_1$ near-ring, by R(4) $N$ is zero symmetric and by R(5) $aba = ba$ for all $a, b \in N$ .......(1).

(i) Let $M$ be an $N$-subgroup of $N$ and let $m \in M$. If $m = 0$ then for any $x \in N^*$, $Nmx = N0 = \{0\}$ [since $N$ is zero symmetric] = $Nxm$.

For $m \neq 0$, since $N$ is a $B_1$ near-ring, there exists $y \in N^*$ such that $Nmy = Nym$ ..............(2). Let $n = ym$. It follows that $n \in M^*$. Now $Mmn = Mm(ym) \subset Nm(ym) = (Nmy)m = (Nym)m$ [by (2)] = $N(mym)m$ [ by (1)] = $Nm(ym)m \subset M(ym)m = Mnm$. That is $Mmn \subset Mnm$ ..............(3). In a similar fashion we get $Mnm \subset Mmn$..............(4). From (3) and (4) we get $Mmn = Mnm$. Consequently $M$ is a $B_1$ near-ring.

(ii) Since $N$ is zero symmetric, R(2) demands that every ideal of $N$ is an $N$-subgroup of $N$ and now (ii) follows from (i).

**Proposition 3.5** Let $N$ be a $B_1$ near-ring. Then for every $a \in N$, there exists $x \in N^*$ such that the following are true.

(i) There exists $n \in N$ such that $axa = nax$.

(ii) $Nax \subset Na \cap Nx$.

(iii) If $N$ is Boolean then $Nxa = Nxa$.

(iv) If $N$ is a strong $S_1$ near-ring then there exists $n \in N$ such that $xa = nax$. 
Proof Let \( a \in N \). Since \( N \) is a \( B_1 \) near-ring, there exists \( x \in N^* \) such that \( Nax = Nxa \) .....................(1).

(i) Since \( axa \in Nxa \), by using (1) we get \( axa = nax \) for some \( n \in N \) and (i) follows.

(ii) From (1) we get, \( Nax = Nxa \subset Na \). Obviously \( Nax \subset Nx \). Therefore \( Nax \subset Na \cap Nx \).

(iii) When \( N \) is Boolean, \( Nxa = Nxa^2 = (Nxa)a = (Nax)a \) [by (1)] and (iii) follows.

(iv) Since \( N \) is a strong \( S_1 \) near-ring, the result follows from R(5) and (i).

4 Strong \( B_1 \) near-rings

By generalizing the concept of \( B_1 \) near-rings, we introduce strong \( B_1 \) near-rings. We also study some of its important properties, obtain a simple characterisation under a condition and also a structure theorem.

Definition 4.1 We say that \( N \) is a strong \( B_1 \) near-ring if \( Nab = Nba \) for all \( a, b \in N \).

Examples 4.2 (a) Every commutative near-ring is a strong \( B_1 \) near-ring.

(b) We consider the near-ring \( (N, +, .) \), where \( (N, +) \) is the Klein’s four group \{ 0, a, b, c \} and ‘.’ is defined as follows (scheme (20), p.408 of Pilz [4])

\[
\begin{array}{c|cccc}
. & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & 0 & a & b & c \\
c & a & 0 & c & b \\
\end{array}
\]

This is a strong \( B_1 \) near-ring.

Proposition 4.3 Every strong \( B_1 \) near-ring is a \( B_1 \) near-ring.

Proof Straight forward.

Remark 4.4 Converse of Proposition 4.3 is not valid. For example we consider the near-ring \( (N, +, .) \) where \( (N, +) \) is the Klein’s four group \{ 0, a, b, c \} and ‘.’ is defined as follows (scheme(14), p.408 of Pilz [4]).

\[
\begin{array}{c|cccc}
. & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & c \\
b & 0 & 0 & 0 & 0 \\
c & 0 & a & 0 & c \\
\end{array}
\]
This is a $B_1$ near-ring. But it is not a strong $B_1$ near-ring [since $Nac \neq Nca$].

**Remark 4.5** It is obvious that the property of $N$ being strong $B_1$ is preserved under near-ring homomorphisms.

Consequently we have the following Theorem:

**Theorem 4.6** Every strong $B_1$ near-ring is isomorphic to a subdirect product of subdirectly irreducible strong $B_1$ near-rings.

**Proof** By Theorem 1.62, p.26 of Pilz [4] we get, $N$ is isomorphic to a subdirect product of subdirectly irreducible near-rings $N_i$'s, say, and each $N_i$ is a homomorphic image of $N$ under the usual projection map $\pi_i$. The desired result now follows from Remark 4.5.

**Lemma 4.7** If $N$ is a strong $B_1$ near-ring if and only if for all $a, b, c \in N$, there exists $n \in N$ such that $abc = ncb$.

**Proof** For the ‘only if’ part, let $a, b, c \in N$. Now $abc \in Nbc$. Since $N$ is a strong $B_1$ near-ring, $Nbc = Ncb$. Therefore $abc \in Ncb$ and this implies that $abc = ncb$ for some $n \in N$.

For the ‘if’ part, let $a, b, c \in N$. Now $abc \in Nbc$. From our assumption there exists $n \in N$ such that $abc = ncb \in Ncb$. Therefore $Nbc \subset Ncb$. In a similar fashion we get $Ncb \subset Nbc$. Thus $N$ is a strong $B_1$ near-ring.

**Theorem 4.8** Let $N$ be a strong $B_1$ near-ring. If $N$ is regular then we have the following:

(i) For every $a \in N$, there exists $x \in N$ such that $a = a^2x$.

(ii) $N$ has no non-zero nilpotent elements.

(iii) Any two principal $N$-subgroups of $N$ commute with each other.

(iv) $N$ is a $P_1$ near-ring.

(v) $N$ is left bipotent.

**Proof** Since $N$ is regular, for every $a \in N$, there exists $x \in N$ such that $a = axa$ .................(1).

(i) Since $N$ is a strong $B_1$ near-ring, Lemma 4.7 guarantees that there exists $n \in N$ such that $axa = nax$ .................(2). From (1) and (2) we get $a = nax$ .................(3). Now $na = n(axa)$ [by (1)] = $(nax)a = aa$ [by (3)] = $a^2$. That is $a^2 = na$ .................(4). Using (4) in (3) we get $a = a^2x$.

(ii) Let $a \in N$. Suppose $a^2 = 0$. Now (i) demands that there exists $x \in N$ such that $a = a^2x$ and therefore $a = 0$. Now R(1) guarantees that $N$ has no non-zero nilpotent elements.

(iii) First we show that $NaN = Na$ for all $a \in N$. Let $y \in NaN$. Then
\( y = nan' \) for some \( n, n' \in N \) ............\( (5) \). Now Lemma 4.7 demands that \( nan' = zn'a \) for some \( z \in N \)............\( (6) \). Combining \( (5) \) and \( (6) \) we get, 
\( y = zn'a = (zn')a \in Na \). Therefore \( NaN \subset Na \) ............\( (7) \). Also from \( (1) \) we get \( Na = Naxa = Na(xa) \subset NaN \). That is \( Na \subset NaN \) ............\( (8) \). From \( (7) \) and \( (8) \) we get \( NaN = Na \) ............\( (9) \). Let \( b, c \in N \). Now \( NbNc = (NbN)c = (Nb)c \) [by \( (9) \)] = \( Nbc = Ncb \) [since \( N \) is a strong \( B_1 \) near-ring] = \( (Nc)b = (NcN)b \) [by\( (9) \)] = \( NcNb \). That is \( NbNc = NcNb \) and \( (iii) \) follows.

\( (iv) \) For any \( a \in N \), let \( y \in aN \). Then there exists \( z \in N \) such that \( y = az = (axa)z \) [by \( (1) \)] = \( a(xaz) \). That is \( y = a(xaz) \) ............\( (10) \). Now Lemma 4.7 demands that there exists \( n \in N \) such that \( xaz = nza \) ............\( (11) \). From \( (10) \) and \( (11) \) we get \( y = a(nz)a \in aNa \). Therefore \( aNa \subset aNa \) ............\( (12) \). 

Obviously \( aNa \subset aN \) ............\( (13) \). From \( (12) \) and \( (13) \) we get \( aNa = aN \).

Thus \( N \) is a \( P_1 \)-near-ring.

\( (v) \) From \( (1) \), we get \( Na = Naxa = (Nax)a = (Nxa)a \) [since \( N \) is a strong \( B_1 \) near-ring] = \( Nxa^2 \subset Na^2 \) [since \( Nx \subset N \)]. Therefore \( Na \subset Na^2 \). Consequently \( Na = Na^2 \). Thus \( N \) is left bipotent.

**Corollary 4.9** Let \( N \) be a zero symmetric strong \( B_1 \) near-ring. If \( N \) is regular then \( N \) is the subdirect product of integral near-rings.

**Proof** Let \( N \) be a strong \( B_1 \) near-ring. Since \( N \) is regular, Theorem 4.8(ii) guarantees that, \( N \) has no non-zero nilpotent elements. As \( N \) is zero symmetric, the desired result now follows from R(3).

**Theorem 4.10** Let \( N \) be a strong \( B_1 \) near-ring. If \( N \) is Boolean then the following are true.

\( (i) \) \( NaNb = Nab \) for all \( a, b \in N \).

\( (ii) \) All principal \( N \)-subgroups of \( N \) commute with one another.

\( (iii) \) Every ideal of \( N \) is a strong \( B_1 \) near-ring.

\( (iv) \) Every \( N \)-subgroup of \( N \) is a strong \( B_1 \) near-ring.

\( (v) \) Every \( N \)-subgroup of \( N \) is an invariant \( N \)-subgroup of \( N \).

**Proof** Since \( N \) is a strong \( B_1 \) near-ring, \( Nab = Nba \) ............\( (1) \).

\( (i) \) Let \( a, b \in N \). Since \( N \) is Boolean, \( a = a^2 \in aN \). Thus we have \( a \in aN \Rightarrow Na \subset NaN \Rightarrow Nab \subset NaNb \). For the reverse inclusion \( y \in NaNb \Rightarrow y = nan'b \) for some \( n, n' \in N \) ............\( (2) \). Since \( N \) is a strong \( B_1 \) near ring, by using Lemma 4.7, we get \( nan' = zn'a \) where \( z \in N \). Therefore from \( (2) \) we get \( y = zn'ab = (zn')ab \in Nab \). The desired result now follows.

\( (ii) \) Let \( a, b \in N \). Now \( NaNb = Nab \) [by \( (i) \)] = \( Nba \) [by \( (1) \)] = \( NbNa \) [by \( (i) \)] and \( (ii) \) follows.

\( (iii) \) Let \( I \) be any ideal of \( N \). Let \( a, b \in I \). Now \( Iab = Ia^2b \) [since \( N \) is Boolean] = \( (Ia)ab \subset I(Nab) = I(Nba) \) [by \( (1) \)] \( \subset Iba \). That is \( Iab \subset Iba \). Similarly, we get \( Iba \subset Iab \). Consequently \( I \) is a strong \( B_1 \) near-ring.
(iv) Let $M$ be an $N$-subgroup of $N$. Therefore $NM \subseteq M$ \hfill (3). Let $x, y \in M$. Let $z \in Mxy \subseteq Nxy = Nyx$ \hfill (iv) \hfill (4) \hfill (v)

Let $z \in Mxy \subseteq Nxy = Nyx$ \hfill (1) \hfill (iii) \hfill (vi)

Therefore $Mxy \subseteq Myx$ \hfill (2) \hfill (vii) \hfill (vii)

We conclude our discussion with the following characterisation of strong $B_1$ near-rings.

**Theorem 4.11** Let $N$ be a Boolean near-ring. Then $N$ is a strong $B_1$ near-ring if and only if $Na \cap Nb = Nab$ for all $a, b \in N$.

**Proof** For the ‘only if’ part, let $y \in Na \cap Nb$. Therefore $y = na = n'b$ for some $n, n' \in N$. Now by Lemma 4.7, there exists $z \in N$ such that $y^2 = (na)(n'b) = (n'an')b = (zn'a)b = (zn')ab \in Nab$. Since $N$ is Boolean, this yields $y \in Nab$. Thus $Na \cap Nb \subseteq Nab$ \hfill (1) \hfill (vi) \hfill (vii)

Since $N$ is a strong $B_1$ near-ring, $Nab = Nba$. But $Nba \subseteq Na$ and $Nab \subseteq Nb$. Hence $Nab \subseteq Na \cap Nb$ \hfill (2) \hfill (vii)

For the ‘if part’, let $a, b \in N$. Now $Nab = Na \cap Nb = Nb \cap Na = Nba$. Thus $N$ is a strong $B_1$ near-ring.

**References**


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