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B_1 Near-Rings

R.Balakrishnan

Department of Mathematics
V.O. Chidambaram College, Thoothukudi, India

S. Silviya

ssilviyafdo@yahoo.co.in

T. Tamizh Chelvam

Department of Mathematics
Manonmaniam Sundaranar University, Tirunelveli, India

Abstract

In this paper we introduce the concepts of B_1 near-rings and strong B_1 near-rings. We say that a right near-ring N is a B_1 near-ring if for every $a \in N$, there exists $x \in N^*$ where $N^* = N - \{0\}$, such that $Nax = Nxa$. By the way of generalization, we define N as a strong B_1 near-ring if $Nab = Nba$ for all $a, b \in N$. We discuss some of their properties, obtain a characterisation and also a structure theorem.

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1 Introduction

Throughout this paper N stands for a right near-ring $(N, +, .)$, with at least two elements and '0' denotes the identity element of the group $(N, +)$. Obviously $0n = 0$ for all $n \in N$. If, in addition, $n0 = 0$ for all $n \in N$ then we say that N is zero symmetric. A subgroup $(M, +)$ of $(N, +)$ is called an N-subgroup of N if $NM \subset M$ and an invariant N-subgroup of N if $MN \subset M$ as well. N is called weak commutative if $abc = acb$ for all $a, b, c \in N$ (Definition 9.4, p.289 of Pilz[4]). N is said to be regular if for every $a \in N$ there exists $b \in N$ such that $a = aba$. An element a is said to be nilpotent if $a^k = 0$ for some positive integer k. N is called nil if every element of N is nilpotent. N is called integral if N has no non-zero zero divisors. N is called a P_k near-ring[1] if there exists a positive integer k such that $x^k N = xNx$ for all $x \in N$. For any subset A of N, we denote by A^* the set of all non-zero elements of A. In particular $N^* = N - \{0\}$. N is called a strong S_1 near-ring[5] if $N^* = N_{S_1}(a)$ for all $a \in N$ where $N_{S_1}(a) = \{x \in N^*/axa = xa\}$. For basic concepts and terms used but left undefined in this paper we refer to Pilz[4].

2 Preliminary Results

We freely make use of the following results from [3], [4] and [5] and designate them as **R(1)**,**R(2)**..etc.,
R(1): N has no non-zero nilpotent elements if and only if $a^2 = 0 \Rightarrow a = 0$ for all $a \in N$ (Problem 14, p.9 of [3]).
R(2): N is zero symmetric if and only if every left ideal of N is an N-subgroup of N (Proposition 1.34(b), p.19 of [4]).
R(3): Let N be zero symmetric. Then the following are equivalent:
(i) N has no non-zero nilpotent elements. **(ii)** N is a subdirect product of integral near-rings (Theorem 9.36, p.302 of [4]).
R(4): If N is a strong S_1 near-ring then N is zero symmetric (Proposition 5.6 of [5]).
R(5): N is a strong S_1 near-ring if and only if $axa = xa$ for all $a, x \in N$ (Theorem 5.8 of [5]).

3 B_1 near-rings

As in [2] a right near-ring N is said to be left bipotent if $Na = Na^2$ for all $a \in N$. Motivated by this we have the following definition.

Definition 3.1 We say that N is a B_1 near-ring if for every $a \in N$, there exists $x \in N^*$ such that $Nax = Nxa$.

Examples 3.2 (a) Every constant near-ring is a B_1 near-ring.
(b) We consider the near-ring $(Z_4, +, .)$ where $(Z_4, +)$ is the group of integers modulo '4' and '.' is defined as follows (scheme(4), p.407 of Pilz [4]).

.	0	1	2	3
0	0	0	0	0
1	0	0	1	0
2	0	0	3	0
3	0	0	2	0

This is a B_1 near-ring.

Theorem 3.3 Let N be a near-ring. Each of the following statements implies that N is a B_1 near-ring.
(i) N is a zero symmetric nil near-ring.
(ii) N is weak commutative.
(iii) N has identity '1'.
(iv) N is a near-field.

Proof **(i)** Let $a \in N$. If $a = 0$, then for any $x \in N^*$, $Nax = Nxa = N0 = \{0\}$. If $a \in N^*$, since N is nil, there exists a positive integer k such that $a^k = 0$. Put $x = a^{k-1} \neq 0$. Now $Nax = Naa^{k-1} = Na^k = Na^{k-1}a = Nxa = N0 = \{0\}$. Thus N is a B_1 near-ring.
(ii) Let $a \in N$. For any $x \in N^*$, $y \in Nax \Rightarrow y = nax$ where $n \in N$. Since N is weak commutative, $y = nxa \in Nxa$. Therefore $Nax \subset Nxa$. Similarly $Nxa \subset Nax$ and **(ii)** follows.
(iii) Follows by taking $x = 1$ in the Definition 3.1.
(iv) Follows from **(iii)**

Theorem 3.4 Let N be a B_1-near-ring. If N is a strong S_1 near-ring without non-zero zero divisors then the following are true.
(i) Every non-zero N-subgroup of N is an B_1 near-ring.
(ii) Every non-zero ideal of N is an B_1 near-ring.

Proof Since N is a strong S_1 near-ring , by R(4) N is zero symmetric and by R(5) $aba = ba$ for all $a, b \in N$ ....(1).
(i) Let M be an N-subgroup of N and let $m \in M$. If $m = 0$ then for any $x \in N^*$, $Nmx = N0 = \{0\}$ [since N is zero symmetric] $= Nxm$.
For $m \neq 0$, since N is a B_1 near-ring, there exists $y \in N^*$ such that $Nmy = Nym$ ....(2). Let $n = ym$. It follows that $n \in M^*$. Now $Mmn = Mm(ym) \subset Nm(ym) = (Nmy)m = (Nym)m$ [by(2)] $= N(mym)m$ [by (1)] $= Nm(ym)m \subset M(ym)m = Mnm$. That is $Mmn \subset Mnm$ ....(3). In a similar fashion we get $Mnm \subset Mmn$...............(4). From (3) and (4) we get $Mmn = Mnm$. Consequently M is a B_1 near-ring.

(ii) Since N is zero symmetric, R(2) demands that every ideal of N is an N-subgroup of N and now (ii) follows from (i).

Proposition 3.5 Let N be a B_1 near-ring. Then for every $a \in N$, there exists $x \in N^*$ such that the following are true.
(i) There exists $n \in N$ such that $axa = nax$.
(ii) $Nax \subset Na \cap Nx$.
(iii) If N is Boolean then $Naxa = Nxa$.
(iv) If N is a strong S_1 near-ring then there exists $n \in N$ such that $xa = nax$.

Proof Let $a \in N$. Since N is a B_1 near-ring, there exists $x \in N^*$ such that $Nax = Nxa$ ....(1).
(i) Since $axa \in Nxa$, by using (1) we get $axa = nax$ for some $n \in N$ and (i) follows.
(ii) From (1) we get, $Nax = Nxa \subset Na$. Obviously $Nax \subset Nx$. Therefore $Nax \subset Na \cap Nx$.
(iii) When N is Boolean, $Nxa = Nxa^2 = (Nxa)a = (Nax)a$ [by (1)] and (iii) follows.
(iv) Since N is a strong S_1 near-ring, the result follows from R(5) and (i).

4 Strong B_1 near-rings

By generalizing the concept of B_1 near-rings, we introduce strong B_1 near-rings. We also study some of its important properties, obtain a simple characterisation under a condition and also a structure theorem.

Definition 4.1 We say that N is a strong B_1 near-ring if $Nab = Nba$ for all $a, b \in N$.

Examples 4.2 (a) Every commutative near-ring is a strong B_1 near-ring.
(b) We consider the near-ring $(N, +, .)$,where $(N, +)$ is the Klein's four group { 0, a, b, c } and '.' is defined as follows (scheme (20), p.408 of Pilz [4])

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

This is a strong B_1 near-ring.

Proposition 4.3 Every strong B_1 near-ring is a B_1 near-ring.

Proof Straight forward.

Remark 4.4 Converse of Proposition 4.3 is not valid. For example we consider the near-ring $(N, +, .)$ where $(N, +)$ is the Klein's four group { 0, a, b, c } and '.' is defined as follows (scheme(14), p.408 of Pilz [4]).

.	0	a	b	c
0	0	0	0	0
a	0	a	0	c
b	0	0	0	0
c	0	a	0	c

This is a B_1 near-ring. But it is not a strong B_1 near-ring [since Nac $\neq N$ca].

Remark 4.5 It is obvious that the property of N being strong B_1 is preserved under near-ring homomorphisms.

Consequently we have the following Theorem:

Theorem 4.6 Every strong B_1 near-ring is isomorphic to a subdirect product of subdirectly irreducible strong B_1 near-rings.

Proof By Theorem 1.62, p.26 of Pilz [4] we get, N is isomorphic to a subdirect product of subdirectly irreducible near-rings N_i's, say, and each N_i is a homomorphic image of N under the usual projection map π_i. The desired result now follows from Remark 4.5.

Lemma 4.7 If N is a strong B_1 near-ring if and only if for all $a, b, c \in N$, there exists $n \in N$ such that $abc = ncb$.

Proof For the 'only if' part, let $a, b, c \in N$. Now $abc \in Nbc$. Since N is a strong B_1 near-ring, $Nbc = Ncb$. Therefore $abc \in Ncb$ and this implies that $abc = ncb$ for some $n \in N$.
For the 'if' part, let $a, b, c \in N$. Now $abc \in Nbc$. From our assumption there exists $n \in N$ such that $abc = ncb \in Ncb$. Therefore $Nbc \subset Ncb$. In a similar fashion we get $Ncb \subset Nbc$. Thus N is a strong B_1 near-ring.

Theorem 4.8 Let N be a strong B_1 near-ring. If N is regular then we have the following:
(i) For every $a \in N$, there exists $x \in N$ such that $a = a^2x$.
(ii) N has no non-zero nilpotent elements.
(iii) Any two principal N-subgroups of N commute with each other.
(iv) N is a P_1 near-ring.
(v) N is left bipotent.

Proof Since N is regular, for every $a \in N$, there exists $x \in N$ such that $a = axa$ ....(1).
(i) Since N is a strong B_1 near-ring, Lemma 4.7 guarantees that there exists $n \in N$ such that $axa = nax$ ....(2). From (1) and (2) we get $a = nax$ ....(3). Now $na = n(axa)$ [by (1)]$= (nax)a = aa$ [by (3)] $= a^2$. That is $a^2 = na$ ....(4). Using (4) in (3) we get $a = a^2x$.
(ii) Let $a \in N$. Suppose $a^2 = 0$. Now (i) demands that there exists $x \in N$ such that $a = a^2x$ and therefore $a = 0$. Now R(1) guarantees that N has no non-zero nilpotent elements.
(iii) First we show that $NaN = Na$ for all $a \in N$. Let $y \in NaN$. Then

$y = nan'$ for some $n, n' \in N$ ....(5). Now Lemma 4.7 demands that $nan' = zn'a$ for some $z \in N$................ (6). Combining (5) and (6) we get, $y = zn'a = (zn')a \in Na$. Therefore $NaN \subset Na$ ....(7). Also from (1) we get $Na = Naxa = Na(xa) \subset NaN$. That is $Na \subset NaN$ ....(8). From (7) and (8) we get $NaN = Na$ ....(9). Let $b, c \in N$. Now $NbNc = (NbN)c = (Nb)c$ [by (9)] $= Nbc = Ncb$ [since N is a strong B_1 near-ring] $= (Nc)b = (NcN)b$ [by(9)] $= NcNb$. That is $NbNc = NcNb$ and (iii) follows.
(iv) For any $a \in N$, let $y \in aN$. Then there exists $z \in N$ such that $y = az = (axa)z$ [by (1)] $= a(xaz)$. That is $y = a(xaz)$ ....(10). Now Lemma 4.7 demands that there exists $n \in N$ such that $xaz = nza$ ....(11). From (10) and (11) we get $y = a(nz)a \in aNa$. Therefore $aN \subset aNa$ ....(12). Obviously $aNa \subset aN$ ....(13). From (12) and (13) we get $aNa = aN$. Thus N is a P_1-near-ring.
(v) From (1), we get $Na = Naxa = (Nax)a = (Nxa)a$ [since N is a strong B_1 near-ring] $= Nxa^2 \subset Na^2$ [since $Nx \subset N$]. Therefore $Na \subset Na^2$. Consequently $Na = Na^2$. Thus N is left bipotent.

Corollary 4.9 Let N be a zero symmetric strong B_1 near-ring. If N is regular then N is the subdirect product of integral near-rings.

Proof Let N be a strong B_1 near-ring. Since N is regular, Theorem 4.8(ii) guarantees that, N has no non-zero nilpotent elements. As N is zero symmetric, the desired result now follows from R(3).

Theorem 4.10 Let N be a strong B_1 near-ring. If N is Boolean then the following are true.
(i) NaNb = Nab for all $a, b \in N$.
(ii) All principal N-subgroups of N commute with one another.
(iii) Every ideal of N is a strong B_1 near-ring.
(iv) Every N-subgroup of N is a strong B_1 near-ring.
(v) Every N-subgroup of N is an invariant N-subgroup of N.

Proof Since N is a strong B_1 near-ring, $Nab = Nba$ .... (1).
(i) Let $a, b \in N$. Since N is Boolean, $a = a^2 \in aN$. Thus we have $a \in aN \Rightarrow Na \subset NaN \Rightarrow Nab \subset NaNb$. For the reverse inclusion $y \in NaNb \Rightarrow$ $y = nan'b$ for some $n, n' \in N$ .... (2). Since N is a strong B_1 near ring, by using Lemma 4.7, we get $nan' = zn'a$ where $z \in N$. Therefore from (2) we get $y = zn'ab = (zn')ab \in Nab$. The desired result now follows.
(ii) Let $a, b \in N$. Now $NaNb = Nab$ [by (i)] $= Nba$ [by (1)] $= NbNa$ [by (i)] and (ii) follows.
(iii) Let I be any ideal of N. Let $a, b \in I$. Now $Iab = Ia^2b$ [since N is Boolean] $= (Ia)ab \subset I(Nab) = I(Nba)$ [by (1)] $\subset Iba$. That is $Iab \subset Iba$. Similarly we get $Iba \subset Iab$. Consequently I is a strong B_1 near-ring.

(iv) Let M be an N-subgroup of N. Therefore $NM \subset M$ ....(3). Let $x, y \in M$. Let $z \in Mxy \subset Nxy = Nyx$ [by (1)] $= Ny^2x$ [since N is Boolean] $= (Ny)yx \subset (NM)yx \subset Myx$ [by (3)]. Therefore $Mxy \subset Myx$. Similarly we get $Myx \subset Mxy$. Consequently M is a strong B_1 near-ring.
(v) Let M be an N-subgroup of N. Let $z \in MN \Rightarrow z = mn = m^2n$ for some $m \in M$ and $n \in N$ ....(4). Since N is a strong B_1 near-ring, Lemma 4.7 demands that there exists $n' \in N$ such that $m^2n = n'nm$ ....(5). From (4) and (5) we get $z = n'nm \in NM \subset M$ [since M is an N-subgroup of N]. Therefore $MN \subset M$. Thus M is an invariant N-subgroup of N.

We conclude our discussion with the following characterisation of strong B_1 near-rings.

Theorem 4.11 Let N be a Boolean near-ring. Then N is a strong B_1 near-ring if and only if $Na \cap Nb = Nab$ for all $a, b \in N$.

Proof For the 'only if' part, let $y \in Na \cap Nb$. Therefore $y = na = n'b$ for some $n, n' \in N$. Now by Lemma 4.7, there exists $z \in N$ such that $y^2 = (na)(n'b) = (nan')b = (zn'a)b = (zn')ab \in Nab$. Since N is Boolean, this yields $y \in Nab$. Thus $Na \cap Nb \subset Nab$ .... (1). Since N is a strong B_1 near-ring, $Nab = Nba$. But $Nba \subset Na$ and $Nab \subset Nb$. Hence $Nab \subset Na \cap Nb$ .... (2). From (1) and (2) we get $Na \cap Nb = Nab$.
For the 'if part', let $a, b \in N$. Now $Nab = Na \cap Nb = Nb \cap Na = Nba$. Thus N is a strong B_1 near-ring.

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