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On Hilbert Property of Rings

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Abstract

In this paper, we study a Hilbert property with respect to the skew polynomial extensions. A property $*$ of a ring R is said to be *the Hilbert property* if its polynomial extension possesses the same property $*$.

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1 Introduction

Throughout this paper, R denotes an associative ring with identity denoted 1, unless otherwise stated. Recall that for a ring R with a ring endomorphism, $\alpha : R \longrightarrow R$ and an α-derivation δ of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$, *the Ore extension* $R[x; \alpha, \delta]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x + \delta(r)$ for any $r \in R$. If $\delta = 0$, we write $R[x; \alpha]$. It is called a skew polynomial ring (also Ore extension of endomorphism type.) Some properties of the skew polynomial rings have been studied in [6] and [4]. According to Krempa [8], an endomorphism α of a ring R is called rigid, if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. In Hong et al [5], a ring R is called α-rigid if there exist a rigid endomorphism α of R. Any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced [5]. Hong et al [4] introduced α-skew Armendariz ring, which is a generalization of α-rigid ring and Armendariz ring. A ring R is said to be α-skew Armendariz ring, if for $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ the condition $pq = 0$ implies $a_i\alpha^i(b_j) = 0$ for all i and j.

The Armendariz property of rings has been extended to skew polynomial rings in [6]. Following Hong et al [6], a ring R is called α-Armendariz if for $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ the condition $pq = 0$ implies $a_i b_j = 0$ for all i and j. α- Armendariz ring is a generalization of Armnedariz ring and α- rigid ring. The authors of [6] proved that an α- Armendariz ring is α-skew Armendariz.

In this paper, α is an endomorphism of R unless specially noted. We keep the standard notation $\mathbb{Z}$ for the set of all integers numbers.

2 The Hilbert property

If R is a ring with identity then $R[x;\alpha]$ need not be with identity, unless $\alpha(1) = 1$. This means possessing identity is not the Hilbert property, in general. For example, for any α-Armendariz ring R its extension $R[x;\alpha]$ is a ring with identity, since in this case $\alpha(1) = 1$, according to [6].

The following examples show that the reducibility is not the Hilbert property.

Example 1. *Consider $R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z}\}$ with usual componentwise addition and multiplication. It is easy to see that R is a commutative reduced ring. Let $\alpha : R \longrightarrow R$ be the endomorphism defined by $\alpha((a, b)) = (a, 0)$. For nonzero element $p = (0, b)x$, where $b \neq 0$ of $R[x;\alpha]$, its square p^2 is zero. Hence, $R[x;\alpha]$ is not reduced.*

Remind that α-rigid rings are reduced but the converse doesn't hold in general. Here is an example of reduced ring which is not α-rigid.

Example 2. *$R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z}; a \equiv b(mod2)\}$ is subring of $\mathbb{Z}\oplus\mathbb{Z}$ with usual addition and multiplication. Let $\alpha : R \longrightarrow R$ be defined by $\alpha((a, b)) = (b, a)$. Obviously, $\mathbb{Z}$ is reduced ring so R is reduced but $R[x;\alpha]$ isn't reduced. Because*

$p = (0, b)x \in R[x; \alpha]$ $(b \neq 0)$ is a nonzero nilpotent element. The ring R is not α-rigid, because of $(b, 0)\alpha((b, 0)) = \bar{0}$ with $b \neq 0$ but $(b, 0) \neq \bar{0}$.

2.1 Compatibility and rigidity

By Hashemi and Moussavi [3], a ring R is said to be α-compatible ring (the compatibility condition) if for each $a, b \in R$, the equality $a\alpha(b) = 0$ does hold if and only if $ab = 0$ does. Let now α be an inner automorphism, i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for $r \in R$. This section contains a few facts on relations of the compatibility and the rigidity, which we use in the next section.

Proposition 1. *Let α be an inner automorphism of a commutative ring R, then R is α- compatible.*

Proof. Let α be an inner automorphism and $a\alpha(b) = 0$ for $a, b \in R$, so there exist an invertible element $u \in R$ such that $\alpha(b) = u^{-1}bu$. Hence $a\alpha(b) = au^{-1}bu = au^{-1}ub = ab = 0$, therefore R is α- compatible. □

Proposition 2. *Let α be an inner automorphism of a reduced ring R, then R is α-rigid.*

Proof. Suppose that $r\alpha(r) = 0$, $r \in R$. Since α is an inner automorphism there exists an invertible element $u \in R$, such that $r\alpha(r) = ru^{-1}ru = 0$, so $ru^{-1}r = 0$. This implies that $(ru^{-1})^2 = 0$ and since R is reduced we get $ru^{-1} = 0$, which means $r = u$. Then $\alpha(r) = r^{-1}r^2$, so $r\alpha(r) = r^2 = 0$, hence $r = 0$. Therefore R is α-rigid. □

Proposition 3. *Let R be a reduced α- compatible ring, then R is α-rigid.*

Proof. Let $a\alpha(a) = 0$. By the hypothesis R satisfies the compatibility condition so $a^2 = 0$. Since R is reduced one gets $a = 0$. Therefore R is α-rigid. □

Theorem 1. *Reduced α-compatible rings are α-Armendariz, so α-skew Armendariz.*

Proof. Let R be a reduced α- compatible ring. Suppose that $p = \sum_{i=0}^{m} a_i x^i$, $q = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha]$ with $pq = 0$. This follows that $a_0 b_0 = 0$ and since R is reduced we get $b_0 a_0 = 0$. If we multiply the equation $a_0 b_1 + a_1 \alpha(b_0) = 0$ on the left hand side by b_0, then $b_0 a_1 \alpha(b_0) = 0$. By using the α-compatibility and the reducibility of R we have $b_0 a_1 b_0 = 0$ and $b_0 a_1 = 0$, consequently. Multiplying the relation $a_0 b_2 + a_1 \alpha(b_1) + a_2 \alpha^2(b_0) = 0$ from the left hand side by b_0 we have $b_0 a_2 \alpha^2(b_0) = 0$, $b_0 a_2 b_0 = 0$ and $b_0 a_2 = 0$, by the same reason as above.

Continuing this process, we have $b_0a_i = a_ib_0 = 0$ for each $0 \le i \le m$. The compatibility condition gives

$$a_ib_0 = b_0a_i = 0 \Leftrightarrow a_i\alpha(b_0) = b_0\alpha(a_i) = 0 \ \ \forall i.$$

So we have $a_0b_1 = 0$ and $a_0b_2 + a_1\alpha(b_1) = 0$. Multiplying the last from the left hand side by b_1 we obtain $b_1a_1\alpha(b_1) = 0$ then $b_1a_1 = 0$. Continuing this process, we have $b_1a_i = a_ib_1 = 0$ for each i. By the same argument as above we get $a_ib_j = 0$ for each $0 \le i \le m$, and $0 \le j \le n$. Therefore R is α-Armendariz and hence it is α-skew Armendariz. □

2.2 The Hilbert property

Jokanović [7], proved that, if R is α-rigid then $R[x]$ is α'-rigid, where

$$\alpha'(\sum_{i=0}^{n} a_ix^i) = \sum_{i=0}^{n} \alpha(a_i)x^i.$$

Theorem 2. *If R is reduced α-compatible ring then $R[x,\alpha]$ is α'-rigid.*

Proof. Note that, if R is α-compatible ring, then for each $a, b \in R$ and any non-negative integer m, $a\alpha^m(b) = 0$ if and only if $ab = 0$. Let $p = a_0 + a_1x + ... + a_nx^n \in R[x;\alpha]$ and $p\alpha'(p) = 0$. We claim that $p = 0$. From the relation $(a_0+a_1x+...+a_nx^n)(\alpha(a_0)+\alpha(a_1)x+...+\alpha(a_n)x^n) = 0$, we have $a_0\alpha(a_0) = 0$, so $a_0 = 0$. Since the coefficient of x^2 has to be zero, which means $a_1\alpha^2(a_1) = 0$, then $a_1^2 = 0$ (by compatibility condition), so $a_1 = 0$ (by reducibility). We obtain $a_2 = 0$ by using the same argument as above. Continuing this process, we have $a_0 = a_1 = ... = a_n = 0$, hence $p = 0$. Therefore, $R[x,\alpha]$ is α'-rigid. □

Corollary 1. *If R is an α-rigid ring with compatibility condition then $R[x,\alpha]$ is α'-rigid.*

Proof. It follows from previous Theorem and the fact that, α-rigid rings are reduced [5]. □

Theorem 3. *Let R be an α-compatible reduced ring. Then $R[x;\alpha]$ is reduced.*

Proof. Let $p = a_0+a_1x+...+a_nx^n \in R[x;\alpha]$. Suppose that $p^2 = 0$. Then $a_0^2 = 0$, and we have $a_0 = 0$, due to reducibility of R. Hence $a_0a_2+a_1\alpha(a_1)+a_2\alpha^2(a_0) = 0$ yields $a_1\alpha(a_1) = 0$. Since R is reduced and α-compatible then, $a_1 = 0$. Since the coefficient of x^4 in p^2 has to be zero, we have $a_2\alpha(a_2) = 0$, and then $a_2^2 = 0$ and $a_2 = 0$, consequently. By the same argument as above, continuing the process we obtain $a_0 = a_1 = ... = a_n = 0$. Hence $p = 0$, so $R[x;\alpha]$ is reduced. □

Corollary 2. *Let α be an inner automorphism of a commutative reduced ring R, then $R[x;\alpha]$ is reduced.*

Proof. The proof is an immediate consequence of the previous theorem and Proposition 1. □

The endomorphism α can be extended to endomorphism α' of $R[x]$ by

$$\alpha'(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \alpha(a_i) x^i.$$

Now we show that how the notion of α-compatibility can be extended from R to $R[x;\alpha]$.

Theorem 4. *If R is a reduced α-compatible ring then $R[x]$ is α'-compatible.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j$ be polynomials from $R[x]$. It is sufficient to show that

$$f(x)g(x) = 0 \Leftrightarrow f(x)\alpha'(g(x)) = 0.$$

Indeed, let $f(x)\alpha'(g(x)) = (a_0+a_1x+...+a_nx^n)(\alpha(b_0)+\alpha(b_1)x+...+\alpha(b_m)x^m) = 0$. Then we have $a_0\alpha(b_0) = 0$. Since R is reduced α-compatible we get $a_0b_0 = 0$, and hence $b_0a_0 = 0$. The coefficient of x has to be zero, therefore $a_0\alpha(b_1) + a_1\alpha(b_0) = 0$, so $b_0a_1\alpha(b_0) = 0$, and then $b_0a_1 = a_0b_1 = 0$. From the relation $a_0\alpha(b_2) + a_1\alpha(b_1) + a_2\alpha(b_0) = 0$ we have $b_0a_2 = 0$ and $a_2\alpha(b_0) = 0$, which implies that $a_1b_1 = 0$ and $a_0b_2 = 0$. Continuing this way it can be shown that $f(x)g(x) = 0$. The converse is similar. □

Theorem 5. *If R is a reduced α-compatible ring then $R[x;\alpha]$ is α'-compatible.*

Proof. The proof is similar that of Theorem 4, if one takes account the fact that, for α-compatible rings for each $a, b \in R$ and any non-negative integer m, the equality $a\alpha^m(b) = 0$ does hold if and only if $ab = 0$ does. □

Recall that, a ring is *semi-commutative* if it is satisfies the following condition: Whenever elements a, b in R satisfy $ab = 0$, then $acb = 0$ for each element c of R. A ring R is called *symmetric*, if $abc = 0 \Rightarrow bac = 0$ for all $a, b, c \in R$, *reversible*, if $ab = 0 \Rightarrow ba = 0$ for all $a, b \in R$.

The following implications hold by a simple computation [9]:

$$Reduced \Rightarrow symmetric \Rightarrow reversible \Rightarrow semi-commutative \Rightarrow abelian.$$

Baser [1], called a ring R right (left) α- reversible if whenever $ab = 0$ for $a, b \in R$ then $b\alpha(a) = 0$ $(\alpha(b)a = 0)$. R is called α- reversible if it is both right and left α- reversible. Ben Yakoub and Louzari [2], called a ring R satisfies the condition (C_α) if whenever $a\alpha(b) = 0$ with $a, b \in R$, then $ab = 0$. Clearly, α-compatible ring satisfies the condition (C_α). In [2] the authors proved that, for a ring R with (C_α) condition, R is reversible if and only if R is α- reversible. Hong et al [6], proved that an α- Armendariz ring satisfies the condition (C_α). Therefore if a ring R is α- Armendariz then it is reversible if and only if is α-reversible.

The following Theorem is an generalization of results of Rege and Chhawchharia [10], and Hong et al. [6].

Theorem 6. *Let R be an α- Armendariz ring. Then the following holds.*

1. *R is reversible if and only if $R[x;\alpha]$ is reversible.*
2. *R is symmetric if and only if $R[x;\alpha]$ is symmetric.*

Proof. Let $p = \sum_{i=0}^{m} a_i x^i$, $q = \sum_{j=0}^{n} b_j x^j$ and $h = \sum_{k=0}^{l} c_k x^k$ be elements of $R[x;\alpha]$.

1. Any subring of symmetric (respectively, reversible) ring is symmetric (respectively, reversible), therefore the "only if" part is obvious. Suppose that R is reversible ring and $pq = 0$. Since R is α- Armendariz ring one gets $a_i b_j = 0$ for all i and j. Since R is α- reversible we obtain $b_j\alpha(a_i) = 0$ for all i and j, then $\alpha(a_i)b_j = 0$ for all i and j (by the reversibility property). Hence $b_j\alpha^2(a_i) = 0$ for all i and j. Continuing this process, we can see that $b_j\alpha^j(a_i) = 0$ for all i and j. This implies that $qp = \sum_{t=0}^{m+n} \sum_{i+j=t} b_j\alpha^j(a_i)x^t = 0$. Therefore $R[x;\alpha]$ is reversible.

2. If $hpq = 0$, then $c_k a_i b_j = 0$ for all i, j and k, then $a_i c_k b_j = 0$ for all i, j and k (by the symmetricity property). Then $c_k b_j a_i = 0$ for all i, j and k (by the reversibility), and $a_i\alpha(c_k)\alpha(b_j) = 0$ (by the α- reversibility) then $\alpha(c_k)\alpha(b_j)a_i = 0$, hence $a_i\alpha^2(c_k)\alpha^2(b_j) = 0$ for all i, j and k. Continuing this process, we have $a_i\alpha^t(c_k)\alpha^t(b_j) = 0$ for any non-negative integer t, then $\alpha^{t+1}(b_j)a_i\alpha^t(c_k) = 0$ (by the α- reversibility), hence $a_i\alpha^t(c_k)\alpha^{t+1}(b_j) = 0$ (by the reversibility) for all i, j and k. Continuing this process, we obtain $a_i\alpha^t(c_k)\alpha^s(b_j) = 0$ for any $s \geq t$ and for all i, j and k. Then $phq = \sum_{e=0}^{m+n+l} \sum_{i+j+k=e} a_i\alpha^i(c_k)\alpha^{i+k}(b_j)x^e = 0$. Therefore $R[x;\alpha]$ is symmetric.

□

The semi-commutativity is not the Hilbert property, in general. There exist a skew polynomial ring $R[x;\alpha]$ over a semi-commutative ring R which is not semi-commutative. For instance, let R be the ring and α be the endomorphism of R in Example 2. Then R is semi-commutative, however $R[x;\alpha]$ is not. Indeed, $p^2 = 0$ but $p(a,0)xp \neq \bar{0}$.

Theorem 7. *Let R be an α- rigid ring. Then R is semi-commutative if and only if $R[x;\alpha]$ is semi-commutative.*

Proof. A subring of semi-commutative ring is semi-commutative, therefore the "only if" part is obvious. Let $p = \sum_{i=0}^{m} a_i x^i$, $q = \sum_{j=0}^{n} b_j x^j$ and $h = \sum_{k=0}^{l} c_k x^k$ be elements of $R[x;\alpha]$ with $pq = 0$. Since R is α- Armendariz ring one gets $a_i b_j = 0$ for all i and j, then $a_i c_k b_j = 0$ for all i, j, k (by the semi-commutativity). Moreover, R is reduced [5], and therefore it is reversible. So we have $c_k b_j a_i = 0$ for all i, j, k. Hence $a_i \alpha(c_k)\alpha(b_j) = 0$ for all i, j, k (by the α- reversibility). The similar arguments those in the proof of Theorem 6, we obtain $a_i \alpha^t(c_k)\alpha^s(b_j) = 0$ for any $s \geq t$ and for all i, j and k. Then $phq = \sum_{r=0}^{m+n+l} \sum_{i+j+k=r} a_i \alpha^i(c_k)\alpha^{i+k}(b_j)x^r = 0$. Therefore $R[x;\alpha]$ is semi-commutative. □

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