Rees Valuations for an Ideal
with Respect to a Module

Cornelia Naude

Department of Mathematics
University of Stellenbosch
Stellenbosch, South Africa
cnaude@sun.ac.za

Abstract

The Rees valuation rings of an ideal $I$ with respect to a module $M$ are defined and it is shown that these rings determine the integral closure of $I$ and all its powers with respect to $M$. Two applications of the results are also given.

1 Introduction.

Let $R$ be a Noetherian ring and $M$ be a finitely generated $R$ module. In [5] Rees proves that for an arbitrary ideal $I$ of $R$, there exist finitely many valuation rings, called the Rees valuation rings $V_i$ of $I$ that determine not just the integral closure of $I$, but also the integral closure of the powers of $I$, i.e. $\bar{I} = \cap_{i=1}^n V_i \cap R$. The asymptotic function $\bar{v}_I(x)$ defined by Samuel ([7]) and its relation to the integral closure of $I$ are used. This material is also covered in [1] and [9].

In [8], Sharp et. al. define the integral closure $I^{-M}$ of an ideal $I$ relative to a module $M$. In [4], the integral closure of an ideal $I$ with respect to a module $M$ is defined by means of a filtration $\bar{v}_{I,M}$ and it is shown that these two concepts of integral closure with respect to a module are identical.

In this paper Rees valuation rings of an ideal $I$ with respect to a module $M$ are defined and it is shown that these rings also determine the integral closure of $I$ and its powers with respect to $M$, i.e. we can give a characterization.
of \((I^k)^{(M)}\) involving valuation rings, similar to that of \(I\). Two applications of the results are also given. Firstly it is proved that \(\bigcap_n \text{Ass}(I^n)^{(M)}M\) is a finite set (cf [2]) and secondly, a definition of projective equivalence with respect to a module is given and the results of the paper are used to give a characterization of this (cf [1], 11.9).

2 Preliminaries.

Let \(R\) be a Noetherian ring, \(I\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module.

We review some of the definitions and propositions in [4] and [8] that we will need.

2.1. Definition ([4]) Define \(v_{I,M} : R \to \mathbb{N} \cup \{\infty\}; v_{I,M}(x) = \begin{cases} 0 & \text{if } xM \not\subseteq IM \\ \infty & \text{if } xM \subseteq I^kM \text{ for all } k \geq 1 \\ k & \text{if } xM \subseteq I^kM; xM \not\subseteq I^{k+1}M \text{ for some } k \geq 1 \end{cases}\)

2.2. Define \(\bar{v}_{I,M} : R \to \mathbb{R} \cup \{\infty\}; \bar{v}_{I,M}(x) = \lim_{n \to \infty} \frac{v_{I,M}(x^n)}{n}.\)

It follows from [6], lemma 2.11 that \(\lim_{n \to \infty} \frac{v_{I,M}(x^n)}{n}\) exists, and that it equals \(\sup\{\frac{v_{I,M}(x^n)}{n}\}\). (We allow \(\infty\) as a possible value for the limit.) \(\bar{v}_{I,M}\) is a homogeneous filtration on \(R\) with the following properties:

2.3(a) \(\bar{v}_{I,M}(x^n) = n\bar{v}_{I,M}(x)\)

(b) \(\bar{v}_{I^n,M}(x) = \frac{1}{n}\bar{v}_{I,M}(x)\)

2.4. Definition ([4]) \(x\) is defined to be integral over \(I\) with respect to \(M\) if \(\bar{v}_{I,M}(x) \geq 1\) and the integral closure of \(I\) with respect to \(M\) is defined as \(\{x \in R|\bar{v}_{I,M}(x) \geq 1\}\) and is denoted by \(\overline{I_M}\).

A result that will be used quite often in this paper is:

2.5. Lemma ([4, 2.2]): Suppose that \(\gamma \in \mathbb{R}^+ \cup \{\infty\}\). Then \(\bar{v}_{I,M}(x) \geq \gamma\) if and only if for every rational number \(\frac{p}{q}\), \(p, q \in \mathbb{N}\) with \(0 < \frac{p}{q} \leq \gamma\), there exists \(k \in \mathbb{N}\) such that \(x^q k M \subseteq Ip^k M\).

Let \(R[u, It]\) be the Rees ring of \(R\) with respect to \(I\), with \(t\) an indeterminate and \(u = t^{-1}\). The filtration \(\bar{v}_{I,M}\) can be used to define the following graded module:
2.6. \( M(\bar{v}) \) is defined to be the module consisting of all the finite sums \( m_{-r}t^{-r} + \ldots + m_0 + a_1mt + \ldots + a_sm_s t^s \) where \( \bar{v}_{I,M}(a_k) \geq k \).

Another definition of the integral closure of an ideal with respect to a module is introduced by [8]:

2.7. \( x \in R \) is called integrally dependant on \( I \) relative to \( M \) if there exists an \( n \in \mathbb{N} \) such that \( x^nM \subseteq \sum_{i=1}^n x^{n-i}I^iM \). The set of all such elements is called the integral closure of \( I \) with respect to \( M \) and is denoted by \( I^{-(M)} \).

In [8] the following theorem is proved by using the “determinant trick”:

2.8. \( x \in I^{-(M)} \) if and only if \( x + \text{Ann}M \) is in the integral closure of \( I + \text{Ann}M/\text{Ann}M \) in \( R/\text{Ann}M \).

The relationship between \( I^{-(M)} \) and \( \bar{v}_{I,M} \) is proved in [4]:

2.9 Proposition:

Let \( R \) be a Noetherian ring, \( I \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module. Let \( k \in \mathbb{N}, k \geq 1 \). \( x \in (I^k)^{-(M)} \) if and only if \( \bar{v}_{I,M}(x) \geq k \).

3 Rees valuations for an ideal with respect to a module.

We will follow the trends in [1], Chapter 11.

We first consider the case where \( R \) is a Noetherian domain, \( I \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module with \( \text{Ann}M = 0 \). Let \( \mathcal{R} \) denotes the Rees ring \( R[u, It] \) of \( R \) with respect to \( I \) and let \( \bar{\mathcal{R}} \) be the integral closure of \( \mathcal{R} \). Write \( \mathcal{M} \) for \( M[u, It] \).

3.1 Proposition

Let \( R, M, \mathcal{R}, \mathcal{M} \) be as above. Let \( p_1, p_2, \ldots, p_r \) be the height 1 primes of \( \bar{\mathcal{R}} \) which contain \( u \), and let \( v_i \) be the valuation associated with the discrete valuation ring \( \bar{\mathcal{R}}_{p_i} \). Then for \( x \in R \), \( \bar{v}_{I,M}(x) = \min \left\{ \frac{v_i(x)}{v_i(u)} \mid i = 1, \ldots, r \right\} \).

We will need the following two lemmas.

3.2 Lemma.

Let \( S \) be a Krull domain with \( I = aS \) and let \( N \) be a finitely generated \( S \)-module with \( \text{Ann}N = 0 \). Let \( p_1, p_2, \ldots, p_r \) be the height 1 primes of \( S \)
Let $T$ be a ring with $N$ and thus $\alpha xN$ then $\bar{S}$ ring holds at all primes $\bar{S}$.

$\alpha$ generated $S$.

Lemma generated $S$.

Let $t$ be a Noetherian domain and let $T$ be a Noetherian domain.

Proof: Let $N_i = N_{pi}$. Since $x = r_i p_i^{v_i(x)}; r_i$ a unit in $S_{pi}$, we have $x N_i \subseteq p_i^{v_i(x)} N_i$ and thus $v_{pi,N_i}(x) \geq v_i(x)$. If $x N_i \subseteq p_i^{v_i(x)+1} N_i$, then $p_i^{v_i(x)+1} N_i \subseteq p_i^{v_i(x)+1} N_i$ and this implies that $p_i^{v_i(x)} N_i = 0$, contradicting $AnnN = 0$. Thus $v_{pi,N_i}(x) = v_i(x)$. If we use the notation $v_{a,N}(x) = k$; i.e. $x N \subseteq a^k N$.

Thus $x N_i \subseteq a^k N_i = p_i^{v_i(a)k} N_i$ for $i = 1, \ldots, r$ and therefore $v_{pi,N_i}(x) \geq v_i(a)k$.

Let $k \leq \frac{v_{pi,N_i}(x)}{v_i(a)} = \frac{v_i(x)}{v_i(a)}$ for $i = 1, \ldots, r$, since $AnnN = 0$. Therefore $k \leq \alpha$. Let $t$ be any integer such that $\alpha - 1 < t \leq \alpha$. Then, for $i = 1, \ldots, r$, we have $v_i(t^a) = tv_i(a) \leq \alpha v_i(a) \leq v_{pi,N_i}(x)$ and thus $x N_i \subseteq p_i^{v_i(a)} N_i = a^t N_i$. Since this holds at all primes $p$, we have $x N \subseteq a^t N$ and therefore $v_{a,N}(x) = \alpha - 1 < k \leq \alpha$ and thus $\alpha - 1 < k \leq \alpha$.

We can repeat the above for $x^n$ and noting that $v_{pi,N_i}(x^n) = v_i(x^n) = n v_{pi,N_i}(x)$ and thus $n \alpha - 1 < v_{a,M}(x^n) \leq n \alpha$. We now have

$$\alpha = \lim_{n \to \infty} \frac{n \alpha - 1}{n} < \lim_{n \to \infty} \frac{v_{a,N}(x^n)}{n} \leq \lim_{n \to \infty} \frac{n \alpha}{n} = \alpha$$

so that $\bar{v}_{I,N}(x) = \alpha = \min \left\{ \frac{v_i(x)}{v_i(a)} \mid i = 1, \ldots, r \right\}$. 

Let $S$ be a Noetherian domain and let $K$ be the quotient field of $S$. Let $T$ be a ring with $S \subseteq T \subseteq K$ and let $N = \langle n_1, \ldots, n_k \rangle$ be a finitely generated $S$ module. If we use the notation $N_K = \{ \frac{n}{s} \mid n \in N; s \in S; s \neq 0 \} (N_K \cong N \otimes_S K)$, then $N_T$ is the module obtained from $N_K$ by the restriction of scalars.

3.3 Lemma.

Let $S$ be a Noetherian domain, $I$ an ideal of $S$. Let $T$ be an integral extension of $S$ contained in the quotient field $K$ of $S$ and let $J = IT$. Let $N$ be a finitely generated $S$ module with $AnnN = 0$. Then for $x \in S$, $\bar{v}_{I,N}(x) = \bar{v}_{I,N_T}(x)$.

Proof: Let $N = \langle n_1, \ldots, n_k \rangle$ and suppose that $\bar{v}_{I,N_T}(x) = l$, i.e. for every rational number $\frac{a}{q}$ with $0 \leq \frac{a}{q} \leq l$, there exits a $k \in N$ such that $x^{\frac{a}{q}} N_T \subseteq J^{pk} N_T$ (cf 2.5). Then for $i = 1, \ldots, k$ we have

$$x^{q_i n_i} = \sum_{j=1}^{k} a_{ij} \frac{n_j}{t_j} t_j; a_{ij} \in J^{pk}; t_j \in T.$$

Let $T_1 = S[t_{11}, \ldots, t_{kk}]$; $T_1$ is a finitely generated $S$ module, say $T_1 = \langle w_1, \ldots, w_r \rangle$. 

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and $S \subseteq T_1 \subseteq T$ and from the above we get

$$x^{qk} \frac{n_j}{1} \in I^{pk} N_{T_1}.$$

Since $N_{T_1}$ is a finitely generated $S$ module, generated by the products $\frac{n_j}{1} w_j, i = 1, \ldots, k, j = 1, \ldots, r$, it follows from the above that

$$x^{qk} N_{T_1} \subseteq I^{pk} N_{T_1}.$$

Let $r \in Ann_S(N_{T_1})$, then $\frac{rn_j}{1} w_j = 0, i = 1, \ldots, k, j = 1, \ldots, r$. Say $w_j = \frac{w'_j}{s_j}$ where $s_j \in S; w'_j, s_j \neq 0$, then there exists a $s \in S, s \neq 0$, such that $srn_s w'_j = 0$ for $i = 1, \ldots, k$. Thus $srw'_j \in Ann_S N = 0$, but $sw'_j \neq 0$ and it follows that $r = 0$. Now $Ann_S(N_{T_1}) = 0$ and we can use the “determinant trick” to show that

$$x^{qk} \in \frac{I^{pk}}{R} = (I^{pk})^{-(N)}$$

Thus $\tilde{v}_{I^{pk}, N}(x^{qk}) \geq 1$ (by 2.9) and it follows from 2.3, 2.4 that $\tilde{v}_{I,N}(x) \geq \frac{p}{q}$ and thus $\tilde{v}_{I,N}(x) \geq l$ (2.5).

Conversely, suppose that $\tilde{v}_{I,N}(x) = l$; then for every rational number $\frac{p}{q}$ with $0 < \frac{p}{q} \leq l$, there exits a $k \in \mathbb{N}$ such that $x^{qk} N \subseteq I^{pk} N$ and it follows as above that $x^{qk} N_T \subseteq I^{pk} N_T \subseteq J^{pk} N_T$. Therefore $\tilde{v}_{I,N_T}(x) \geq l$ and the result follows.

The proof of 3.1 follows now easily.

Proof of 3.1: Since $u^m M \cap M = I^m M$ for all $m > 0$, we have $v_{I,M}(x^n) = v_{u,M}(x^n)$ for all $n \geq 0$ and thus $\tilde{v}_{I,M}(x) = \tilde{v}_{u,M}(x) = \tilde{v}_{u,\bar{R},M_R}(x)$ (by 3.3). The result follows now from 3.2. ■

We now drop the assumption that $R$ is a domain and that $Ann M = 0$.

### 3.4 Proposition.

Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$ module, $I \not\in Ann M$. Let $q_1, \ldots, q_s$ be the minimal primes of $\bar{R} = R/Ann M$ and let $\bar{T} = I + Ann M/Ann M$. For $i = 1, \ldots, s$ let $\bar{R}_i = \bar{R}/q_i; \bar{T}_i = \bar{T} + q_i/q_i M_i = M/q_i M_i$ and $\bar{x}_i = \bar{x} + q_i$. Then $\tilde{v}_{I,M}(x) = \min \{ \tilde{v}_{\bar{I}_i,M_i}(\bar{x}_i) \mid i = 1, \ldots, s \}$.

Proof: Let $\beta = \min \{ \tilde{v}_{I_i,M_i}(\bar{x}_i) \mid i = 1, \ldots, s \}$. (Note that if $\bar{T}_i = \bar{R}_i$, then $\tilde{v}_{I_i,M_i}(\bar{x}_i) = \infty$ and if $\bar{T} \subseteq q_i$ then $\tilde{v}_{I_i,M_i}(\bar{x}_i) = 0$.) Let $\frac{p}{q}$ be a rational number with $0 \leq \frac{p}{q} \leq \beta$. Then $p \leq \tilde{v}_{I_i,M_i}(\bar{x}_i^q)(2.3(\alpha))$. Since $\bar{R}/q_i$ is a domain and
Ann(\(M_i\)) = 0, it follows from 2.9 that \(\tilde{x}_i^q \in (\overline{I}_i)^{(M_i)}\) and thus \(\overline{I}_i^p\) is a reduction of \(\overline{I}_i^p + \tilde{x}_i^q\overline{R}_i\) relative to \(M_i\) ([8], 1.5) and therefore \(\overline{P}^p\) is a reduction of \(\overline{P}^p + \tilde{x}_i^q\overline{R}\) relative to \(M\) ([3], 3.10). By using [8], 1.5 again, it follows that \(I^p\) is a reduction of \(I^p + x^qR\) relative to \(M\) and thus \(x^q \in (I)^{(M)}\). Thus \(\bar{v}_{I,M}(x^q) \geq p\) (2.9) and it follows from 2.3 that \(\bar{v}_{I,M}(x) \geq \frac{p}{q}\) and therefore \(\bar{v}_{I,M}(x) \geq \beta\).

Conversely; suppose that \(\bar{v}_{I,M}(x) \geq l\), then, by 2.5, for every rational number \(\frac{p}{q}\) with \(0 < \frac{p}{q} \leq l\), there exists \(k \in \mathbb{N}\) such that \(x^{pk}M \subseteq I^{pk}M\), giving \(\tilde{x}_i^{pk}M_i \subseteq I_i^{pk}M_i\) and therefore \(\bar{v}_{I_i,M_i}(\tilde{x}_i) \geq l\) and the result follows. □

The proof of the following corollary follows easily from 3.4 and 3.1.

3.5 Corollary.

Let \(R\), \(I\) and \(M\) be as in 3.5. Then there exist finitely many discrete valuation rings \(V_1, ..., V_n\) such that for \(x \in R\), we have \(\bar{v}_{I,M}(x) = \min\left\{\frac{v_i(x)}{v_i(u)} \mid i = 1, ..., r\right\}\), where \(v_i\) is the valuation associated with \(V_i\) and \(u\) is the generator of \(IV_i\).

Proof: For each of the minimal primes \(q_i\), \(i = 1, ..., s\), of \(\overline{R} = R/AnnM\), we have \(\bar{v}_{I_i,M_i}(\tilde{x}_i) = \min\left\{\frac{v_{ij}(x)}{v_{ij}(u)} \mid j = 1, ..., r\right\}\) where \(v_{ij}\) is the valuation associated with a discrete valuation ring \(V_{ij}\) (cf 3.1). The result now follows from 3.4 □

We can now give a characterization of \((I^k)^{(M)}\) involving valuation rings:

3.6 Proposition.

Let \(R\) be a Noetherian ring, \(I\) an ideal of \(R\) and \(M\) a finitely generated \(R\) module, \(I \not\subseteq AnnM\). Then for each of the minimal primes \(q_i\), \(i = 1, ..., s\) of \(\overline{R} = R/AnnM\), there exist finitely many discrete valuation rings \(V_{i1}, ..., V_{in}\) such that

\[(I^k)^{(M)}M = \bigcap_{i=1}^s \bigcap_{j=1}^n \varphi_{ij}^{-1}(\varphi_{ij}(I^k))M\]

where \(\varphi_{ij} : R \to V_{ij}\) is the natural ring homomorphism.

Proof: (the “\(\subseteq\)” part): Suppose that \(x \in (I^k)^{(M)}\); then, from 2.9 \(\bar{v}_{I^k,M}(x) \geq k\). Now 3.4 implies that \(\bar{v}_{I_i,M_i}(\tilde{x}_i) \geq k\). For each \(i\), there exists discrete valuation rings \(V_{i1}, V_{i2}, ..., V_{in}\) with corresponding valuations \(v_{i1}, v_{i2}, ..., v_{in}\) such that \(\bar{v}_{I_i,M_i}(\tilde{x}_i) = \min\left\{\frac{v_{ij}(\tilde{x}_i)}{v_{ij}(u_{ij})} \mid j = 1, ..., n_i\right\}\), \((\overline{I}_iV_{ij} = u_{ij}V_{ij})\) and thus \(v_{ij}(\tilde{x}_i) \geq kv_{ij}(u_{ij})\), i.e. \(\tilde{x}_i \in u_{ij}^kV_{ij}\) and the result follows.

The proof of the “\(\supseteq\)” part is similar. □

3.7 Definition. The valuation rings that appear in 3.6 are called the Rees valuation rings of \(I\) with respect to \(M\) and the corresponding valuations are called the Rees valuations of \(I\) with respect to \(M\).
4 Two applications of Rees valuations.

It is shown in [2] that $\cup_n \text{Ass}(I^n)^{(M)} M$ is a finite set. The following proposition follows easily from the work done in section 3.

4.1 Proposition (cf [9], 10.2.4)
Let $R$, $I$ and $M$ be as in 3.4. Then $\cup_n \text{Ass}(M/(I^n)^{(M)} M) \subseteq \text{Ass}(M/(I^{n+1})^{(M)} M)$ ([2], 2.10).

Proof: We first note that $\text{Ass}(M/(I^n)^{(M)} M) \subseteq \text{Ass}(M/(I^{n+1})^{(M)} M)$ ([2], 2.10).

Let $V_1, ..., V_n$ be the finitely many Rees valuation rings of $I$ with respect to $M$ and let $m_{V_i}$ be the maximal ideal of $V_i$. Then $I^n V_i$ is a $m_{V_i}$ primary ideal of $V_i$. It follows from 3.6 that the associated primes of $(I^n)^{(M)} M$ are contained in the set $\{ p_i = m_{V_i} V_i \cap R | i = 1, ..., n \}$. The proof follows.

In [1], Corollary 11.9 gives a characterization of projective equivalence. We can obtain a characterization of the projective equivalence of an ideal with respect to a module in a similar way.

4.2 Definition. Let $I$, $J$ be ideals in a Noetherian ring $R$ and let $M$ be a finitely generated $R$ module. We say that $I$ and $J$ are projectively equivalent with respect to $M$ if $(I^n)^{(M)} = (J^k)^{(M)}$ for some $n, k \in \mathbb{N}$.

4.3 Proposition.
(i) $(I)^{(M)} = (J)^{(M)}$ if and only if $\bar{v}_{I,M}(x) = \bar{v}_{J,M}(x)$
(ii) $(I^n)^{(M)} = (J^k)^{(M)}$ if and only if $k \bar{v}_{I,M}(x) = n \bar{v}_{J,M}(x)$.

References


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