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Zero-Divisor Graph of Triangular Matrix Rings over Commutative Rings[1]

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Abstract

Let R be a noncommutative ring. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the (directed) graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, there is an edge $x \to y$ if and only if $xy = 0$. In this paper we investigate the zero-divisor graph of triangular matrix rings over commutative rings.

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1. Introduction

Let R be a commutative ring with identity, and $Z(R)$ be its set of zero-divisors. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, there is an edge between vertices x and y if and only if $xy = 0$. Note that $\Gamma(R)$ is the empty graph if and only if R is an integral domain. The concept of the zero-divisor graph of a commutative was introduced by Beck [3]. However, he let all elements of R be vertices of the graph. The present definition of $\Gamma(R)$ was due to Anddeson and Livingston

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[1]. Redmond [7] further extended this concept to the noncommutative case. For a noncommutative ring R, the zero-divisor graph, also denoted by $\Gamma(R)$, is a directed graph with vertex set $Z(R)^*$ in which for any two vertices x and y, $x \to y$ is an edge if and only if $x \neq y$ and $xy = 0$. Note that if x and y are two distinct vertices and $xy = yx = 0$, then there are two directed edges between x and y. Also for a ring R, we define a simple undirect graph $\overline{\Gamma}(R)$ with vertex set $Z(R)^*$ in which for any two vertices x and y are adjacent if and only if $x \neq y$ or $xy = 0$. Note that for a commutative ring R, the definition of the zero-divisor graph of R coincides with that of $\overline{\Gamma}(R)$.

Let G be a graph. By declaring the length of each edge to be 1, G becomes a metric space. For vertices x and y in G, we define $d(x, y)$ to be the length of a shortest path between x and y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path.) The diameter of G is $diam(G) = sup\{d(x, y) | \ x \text{ and } y \text{ are vertices of } G\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. ($gr(G) = \infty$ if G contains no cycle.)

A graph G is said to be connected if there is a path between any two distinct vertices and G is said to be complete if there is an edge between any two distinct vertices in G. Obviously, the girth of a graph is at least 3. It was shown that $gr(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle(see [6]) and that $\Gamma(R)$ is always connected with $diam(\Gamma(R)) \leq 3$ for any commutative ring R. ([1], Theorem 2.3)

Let R be a noncommutative rings with $Z(R)^* \neq \emptyset$, Redmond [7] proved that $\overline{\Gamma}(R)$ is always connected with $\mathrm{diam}(\overline{\Gamma}(R)) \leq 3$ and $\mathrm{g}(\overline{\Gamma}(R)) \leq 4$, and $\Gamma(R)$ is connected if and only if $Z_L(R) = Z_R(R)$. If $\Gamma(R)$ is connected, then $diam(\Gamma(R)) \leq 3$.

Throughout, R is a commutative ring with $1 \neq 0$. $Q(R) = R_S$, where $S = R - Z(R)$, is the total quotient ring of R. $M_n(R)$ and $T_n(R)$ denote the rings of $n \times n$ matrices and $n \times n$ upper triangular matrices over R, respectively. Let E_{ij} be the matrix unit, which has a 1 in the (i, j) position as its only nonzero entry. For $A \subseteq R$, we use A^* to stands for $A \setminus \{0\}$ and $|A|$ to stand for the cardinal of A. As usual, the rings of integers and integers modulo n will be denoted by Z and Z_n, respectively.

2. **Main Results**

Anderson et al.[2] showed that $\Gamma(R)$ and $\Gamma(Q(R))$ were isomorphic as graphs, hence these two graph had the same diameter and girth. The authors in [4] proved that the same claim holds for $\Gamma(M_n(R))$ and $\Gamma(M_n(Q(R)))$. We will show that $\Gamma(T_n(R))$ and $\Gamma(T_n(Q(R)))$ are isomorphic as graphs. Recall that two graphs G and G' are isomorphic if there is a bijection $\phi : G \to G'$ of vertices such that x and y are adjacent in G if and only if $\phi(x)$ and $\phi(y)$ are adjacent in G'.

Theorem 2.1 Let R be a commutative ring with total quotient ring $Q(R)$. Then $\Gamma(T_n(R)) \simeq \Gamma(T_n(Q(R)))$.

Proof. We construct a proof similar to that of [Theorem 3.2, 4]. Let $A \in T_n(R)$, and let $ann_R(A)^L = \{X \in T_n(R) | XA = 0\}$ and $ann_R(A)^R = \{X \in T_n(R) | AX = 0\}$. For $A, B \in T_n(R)$, we define $A \sim B$ if and only if $ann_R(A)^L = ann_R(B)^L$ and $ann_R(A)^R = ann_R(A)^R$. Clearly $\sim$ is an equivalence relation on $T_n(R)$, and restricts to a equivalence relation on $\Gamma(T_n(R))$. Let $S = R - Z(R)$ and $Q(R) = Q$. Denote equivalence relations defined above on $Z(T_n(R))$ and $Z(T_n(Q))$ by $\sim_R$ and $\sim_Q$, and their respective equivalence classes by [A] and $[A]_Q$.

If $A = (a_{ij}) \in T_n(R)$ and $s \in R - Z(R)$, we denote by A/s the matrix $(a_{ij}/s) \in T_n(Q)$. Clearly $A \in Z(T_n(Q))$ if and only if $A = B/s$ for some $B \in Z(T_n(R))$ and $s \in R - Z(R)$. Since $(a_{ij}/s) \sim_Q (a_{ij}/t)$ for any $s, t \in S$, we have $Z(T_n(R))^* = \bigcup_{\lambda \in \Lambda} [A_\lambda]_R$ and $Z(T_n(Q))^* = \bigcup_{\lambda \in \Lambda} [A_\lambda/1]_Q$ (both disjoint unions) for some index set Λ. There is a bijection between sets of equivalence classes of $\sim_R$ and $\sim_Q$, given by $[A_\lambda]_R \to [A_\lambda/1]_Q$, By using the analogous to the proof of the corresponding fact from [Theorem 2.2, 2], we can prove that $|[A_\lambda]_R| = |[A_\lambda/1]_Q|$ for all $A \in Z(T_n(R))^*$. Therefore, there is a bijection $\phi_\lambda [A_\lambda]_R \to [A_\lambda/1]_Q$ for each $\lambda \in \Lambda$.

We define $\phi : Z(T_n(R))^* \to Z(T_n(Q))^*$ by $\phi(X) = \phi_\lambda(X)$ for any $X \in [A_\lambda]_R$. The map ϕ is a bijection from $\Gamma(T_n(R))$ to $\Gamma(T_n(Q))$. We show that $XY = 0$ in $Z(T_n(R))^*$ if and only if $\phi(X)\phi(Y) = 0$ in $Z(T_n(Q))^*$. Let $X \in [A]_R, Y \in [B]_R, W \in [A/1]_Q, Z \in [B/1]_Q$. Note that $ann_Q(X)^R = ann_Q(A)^R = ann_Q(W)^R$ and $ann_Q(Y)^L = ann_Q(B)^L = ann_Q(Z)^L$. Thus $XY = 0 \Leftrightarrow Y \in ann_Q(X)^R = ann_Q(W)^R \Leftrightarrow WY = 0 \Leftrightarrow W \in ann_Q(Y)^L = ann_Q(Z)^L \Leftrightarrow WZ = 0$. We conclude that $\Gamma(T_n(R)) \simeq \Gamma(T_n(Q(R)))$.

Unlike the zero-divisor graphs of commutative rings, zero-divisor graphs of noncommutative rings need not to be connected. The author in [4] proved that $\Gamma(M_n(R))$ is connected for any commutative ring R. We show that this also holds for $\Gamma(T_n(R))$.

Theorem 2.2 Let R be a commutative ring. Then $\Gamma(T_n(R))$ is connected and $diam(\Gamma(T_n(R))) \leq 3$.

Proof By [Theorem 2.2, 4], a zero-divisor graph of a noncommutative ring R is connected if and only if $Z_L(R) = Z_R(R)$. Since $\Gamma(T_n(R)) \simeq \Gamma(T_n(Q(R)))$, it is equal to show that $\Gamma(T_n(Q(R)))$ is connected and $diam(\Gamma(T_n(Q(R)))) \leq 3$. Recall that a matrix A is either a left or a right zero-divisor in $M_n(R)$ if and only if det(A)$\in Z(R)$. Note that elements in $Q(R)$ are either invertible or zero-divisors. For $A \in Z_L(T_n(Q(R))$, we have $a_{11}a_{22} \cdots a_{nn} \in Z(R)$, hence $a_{ii} \in Z(Q(R))$ for some $i \in \{1, 2, \cdots, n\}$. If $a_{11} \in Z(Q(R))$, then we can find $b \in Q(R)^*$ such that $a_{11}b = 0$. Let $B = bE_{11}$, then $AB = 0$, hence $A \in Z_R(T_n(Q(R))$. If $a_{11}, a_{22} \cdots a_{kk} \in U(R)$ and $a_{k+1,k+1} \in Z(R)$ for some

$k < n$, then there exists some $b \in Q(R)^*$ such that $a_{k+1,k+1}b = 0$. By using the elementary transformation of matrix, we can find some invertible matrices $P_1, P_2, \cdots, P_S$ in $T_n(Q(R))$ such that

$$AP_1P_2\cdots P_s = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & a_{22} & \cdots & 0 & 0 & \cdots & 0 \\ & & \ddots & \vdots & \vdots & & \vdots \\ & & & a_{kk} & 0 & \cdots & 0 \\ & & & & a_{k+1,k+1} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & a_{nn} \end{pmatrix}$$

Set $C = bE_{(k+1)n} \neq 0$, then it is easy to verify that $AP_1P_2\cdots P_sC = 0$ and $P_1P_2\cdots P_sC \neq 0$. Hence $A \in Z_R(T_n(Q(R))$. By [Theorem2.2, 7], $\Gamma(T_n(Q(R)))$ is connected and $diam(\Gamma(T_n(Q(R)))) = diam(\Gamma(T_n(R))) \leq 3$, as asserted.

It is well known that even if R does not contain zero-divisors, $T_n(R)$ does. We can always find distinct $A, B \in Z(T_n(R))^*$ such that $AB \neq 0$. Therefore, $diam(\Gamma(T_n(R))) \geq 2$ for all commutative ring R.

Proposition 2.3 Let R be a commutative ring. Then $diam(\Gamma((R)) \leq diam(\Gamma(T_n(R)))$.

Proof. We define $\phi : Z(R)^* \to Z(T_n(R))^*$ by $\phi(a) = aE_{11}$ for any $a \in Z(R)^*$. The map ϕ is an injection from $\Gamma(R)$ to $\Gamma(T_n(R))$. It is easy to verify that $ab = 0$ if and only if $\phi(a)\phi(b) = 0$. Hence $\Gamma(R)$ is isomorphic to a subgraph of $\Gamma(T_n(R))^*$. Since $\Gamma(R)$ is connected, we conclude that $diam(\Gamma((R)) \leq diam(\Gamma(T_n(R)))$.

Theorem 2.4 Let R be a commutative ring and for any $a, b \in Z(R)$, (a, b) has a nonzero annihilator. Then $diam(\Gamma(T_n(R))) = 2$.

Proof. Since $\Gamma(T_n(R)) \simeq \Gamma(T_n(Q(R)))$ and the diameter is a graph invariant, it surffices to show that $diam(\Gamma(T_n(Q(R)))) = 2$. For any $A = (a_{ij}), B = (b_{ij}) \in Z(T_n(Q(R)))^*$, $det(A) = a_{11}a_{22}\cdots a_{nn}, det(B) = b_{11}b_{22}\cdots b_{nn} \in Z(Q(R))$. Hence there exist some $i, j \in \{1, 2, \cdots, n\}$ such that $a_{ii}, b_{jj} \in Z(Q(R))$. Let $a_{ss} \in Z(Q(R))$ and $a_{jj} \in U(Q(R))$ for all $j < s$ and $b_{tt} \in Z(Q(R))$ and $b_{jj} \in U(Q(R))$ for all $j > t$. Therefore, we can find some $P_1, \cdots, P_r, Q_1, \cdots, Q_l \in$

$U(T_n(Q(R)))$ such that

$$AP_1P_2\cdots P_r = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & a_{22} & \cdots & 0 & 0 & \cdots & 0 \\ & & \ddots & \vdots & \vdots & & \vdots \\ & & & a_{s-1,s-1} & 0 & \cdots & 0 \\ & & & & a_{s,s} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & a_{nn} \end{pmatrix},$$

$$Q_1Q_2\cdots Q_lB = \begin{pmatrix} b_{11} & \cdots & b_{1t} & \cdots & \cdots & b_{1n} \\ & \ddots & \vdots & & & \vdots \\ & & b_{tt} & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ & & & & \ddots & \vdots \\ & & & & & b_{nn} \end{pmatrix}.$$

For $a_{ss}, b_{tt} \in Z(Q(R))$, there exist nonzero element $c \in Q(R)$ such that $a_{ss}c = b_{tt}c = 0$ by assumption. Let $C = cE_{st}$, then $0 \neq P_1P_2\cdots P_rCQ_1Q_2\cdots Q_l \in Z(T_n(Q(R)))$ and one can check that $A \to C \to B$ is a direct path from A to C in $\Gamma(Q(R))$, as asserted.

Corollary 2.5 If R is an integral domain or $\Gamma(R)$ is a star graph. Then $diam(\Gamma(T_n(R))) = 2$.

Proposition 2.6 Let R be a commutative ring. Then $g(\overline{\Gamma}(T_n(R)) = g(\Gamma(T_n(R)) = 3$.

Proof. It suffices to show that $g(\Gamma(T_n(R)) = 3$. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

A, B, C are distinct matrices in $Z(T_n(R))^*$ such that $AB = BC = CA = 0$. So $A \to B \to C \to A$ is a directed cycle of length 3, as asserted.

References

[1] Anderson D. F., Livingston P. S., The zero-divisor graph of a commutative rings, J. Algebra, 1999; 217: 434-447.

[2] Anderson D. F., Ron Levy, Jay Shapiro, Zero-divisor graph, von Neumann regular rings, and Boolean algebras, J. Pure Appl. Algebra, 2003; 180: 221-241.

[3] Beck I., coloring of a commutative ring, J. Algebra, 1988; 116: 208-226.

[4] Ivanna Boziv, Zoran Petrovic, Zero-divisor graphs of matrices over commutative rings, Comm. Alg. 2009; 37: 1186-1192.

[5] Lucas, T., The diameter of a zero-divisor graph, J. Algebra 2006; 301: 174-193.

[6] Mulay, S.B., Cycles and symmetries of zero-divisors, Comm. Alg. 2002; 30: 3533-3558.

[7] Redmond S., The zero-divisor graph of a noncommutative ring. Int. J. Commutative rings 2002; 4: 203-211.

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