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Dual Basis Projective System

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Abstract. In this paper all semigroups are commutative with identity and all systems are unitary. In this paper we shall generalize the concept of dual basis from modules to S-systems and we shall give some characterization of such S-systems where S is a semigroup with identity.

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1. INTRODUCTION

A non-empty set M is called unitary an S-system if there exists mapping $. : M \times S \to M$ such that $m.1 = m$ and

$$m(st) = (ms).t$$

for any $m \in M$ and $s, t \in S$ and an S-system P is called projective if for every diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow \mu \\ L & \xrightarrow{\alpha} & M \end{array}$$

where α is onto can be completed (to yield a commutative diagram) by a homomorphism $\beta : P \to L$ such that $\alpha\beta = \mu$.

Definition. *Let S be a semigroup, M an S-system and $M^* = Hom(M, S)$. A subsystem $T = \{(x_i, f_i)\}_{i \in I}$ of $M \times M^*$ is called a dual basis for M if for every $x \in M$ there exists exactly one $(x_i, f_i) \in T$ such that $x = x_i f_i(x)$.*

Theorem 1. Let M be an S-system, then M is projective S-system if and only if M has a dual basis.

Proof. Let M be a projective S-system, then $M = \coprod_{i \in I} P_i \approx \coprod_{i \in I} e_i S$ [see [2], Theorem 17.8] where $P_i \approx e_i S$ for some $e_i^2 = e_i \in S$. Since $P_i \approx e_i S$, there exists an Isomorphism $f_i : e_i S \to P_i$ such that $P_i = f_i(e_i)S$. Now by ([2] Theorem 17.10) there exists $g_i \in Hom(f_i(e_i)S, S)$ such that $f_i(e_i) = f_i(e_i)g_i(f_i(e_i))$. We shall show that the set $T = \{(f_i(e_i), g_i)\}_{i \in I}$ is a Dual Basis for M. Let $x \in M$ then there exists $i \in I$ such that $x \in P_i$. Hence $x = f_i(e_i)s_i$ for some $s_i \in S$, thus $x = f_i(e_i)g_i(f_i(e_i))s_i = f_i(e_i)g_i(f_i(e_i)s_i) = f_i(e_i)g_i(x)$.

Conversely let M have a dual basis as $T = \{(x_i, f_i)\}_{i \in I} \subseteq M \times M^*$, i.e. if $x \in M$ then there exists $i \in I$ such that $x = x_i f_i(x)$. Now let F be the free S-system $\{x_i\}_{i \in I} \times S$ by multiplication, $(x_i, s)t = (x_i, st)$ and the basis $\{(x_i, 1)\}_{i \in I}$. We define $\beta : F \to M$, such that $\beta(e_i) = x_i$ where $e_i = (x_i, 1)$. Also we define $\phi : M \to F$, such that $\phi(x) = \phi(x_i f_i(x)) = (x_i, f_i(x)) = (x_i, 1)f_i(x) = e_i f_i(x)$. We have $\beta \circ \phi(x) = \beta(\phi(x)) = \beta(e_i f_i(x) = \beta(e_i) f_i(x) = x_i f_i(x) = x$. Hence M is a retract of F. Since F is free, M is projective.

Definition. *An S-system M is called a multiplication system if for every subsystem N of M there exists an ideal of S such that $N = IM$.*

Theorem 2. If J is a projective ideal of a semigroup S, then J must be multiplication. i.e. for any ideal $I \subseteq J$, there exists an ideal K of S such that $I = JK$.

Proof. Let $I \subseteq J$. By Theorem 1, J have a dual basis as $\{(b_\alpha, f_\alpha)\}_{\alpha \in \Lambda} \subseteq J \times J^*$, i.e. if $b \in J$ then there exist $\alpha \in \Lambda$ such that $b = b_\alpha f_\alpha(b)$. Let K be the ideal generated by the elements of the form $f_\alpha(x)$ for $x \in I$. We claim that $I = JK$. If $x \in I \subseteq J$, then $x = f_\alpha(x)b_\alpha$, for some $\alpha \in \Lambda$, and hence x is in JK. Conversely, if $x \in I, b \in J$, and α is some index, then $bf_\alpha(x) = f_\alpha(bx) = f_\alpha(b)x$ which is in I.

As a result, $JK \subseteq I$ and we obtain equality. This completes the proof.

References

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