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Weakly Pure Submodules of Multiplication Modules

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Abstract

Let R be a commutative ring with non-zero identity and M a unital $R-module$. Then $R-submodule$ N of M is called weakly pure if for every Boolean ideal I of R, $IN = N \cap IM$. This paper is devoted to investigate some of properties of weakly pure submodules of multiplication modules.

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Introduction.

Throughout this paper all rings will be commutative with non-zero identity and all modules will be unitary. Pure submodules of multiplication modules have been investigated by Majid M. Ali and David J. smith (2004) and others.

A submodule N of $R-module$ M is called pure if $IN = N \cap IM$, for every ideal I of R. The aim of this paper is to prove for weakly pure submodules some of the results given in [1] for pure submodules of multiplication modules.

Now we define the concepts that we will use. If R is a ring and N is a submodule of an $R-module$ M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. An $R-module$ M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$. In this case $N = (N : M)M$. Also $N = ann(M/N)M$.

Main Results.

Definition 1. Let M be a module over a ring R. A proper submodule N of M is said to be prime if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in (N : M)$.

Definition 2. An ideal I of R is called Boolean ideal if every element of I is idempotent.

definition 3. $R-submodule$ N of $R-module$ M is called weakly pure if $IN = N \cap IM$, for every Boolean ideal I of R.

Theorem 1. Let R be a commutative ring, M a multiplication $R-module$ and N a prime $R-submodule$ of M. Then N is weakly pure.

Proof. Assume that I is a Boolean ideal of R. As $IN \subseteq N \cap IM$ is trivial, we shall prove the reverse inclusion. Let $x \in N \cap IM$. Then there exist $r \in I$ and $m \in M$ such that $x = rm$. But $x \in N$ and N is prime, so $m \in N$ (it follows that $rm \in IN$) or $r \in (N : M)$, Hence $rm = r^2m \in I(N : M)M = IN$. ■

Theorem 2. Let R be a commutative ring, M a free multiplication $R-module$ and N a weakly pure $R-submodule$ of M. Then $I(N : M) = I \cap (N : M)$ for every Boolean ideal I of R.

Proof. Assume that I is a Boolean ideal of R and let $\{x_i : i \in J\}$ be a basis of M. Also $IN = N \cap IM$. As $I(N : M) \subseteq I \cap (N : M)$ is trivial, we shall prove the reverse inclusion. Let $a \in (N : M) \cap I$. Then $ax_i \in (N : M)M \cap IM$, so $ax_i \in I(N : M)M$ because N is weakly pure. It follows that there exist $r \in I$, $b \in (N : M)$ such that $ax_i = rbx_i$; hence $a = rb \in I(N : M)$, as required. ■

Theorem 3. Let R be a domain, M a faithful cyclic multiplication $R-module$ and N a proper finitely generated weakly pure submodule of M. Then the ideal $(N : M)$ is idempotent.

Proof. Assume that $M = Rx$ and $N = Ra + Rb$ such that $a = r_1x$, $b = r_2x$ for some $r_1, r_2 \in R$. At the first we show that $(N : M)$ is Boolean. Let $r \in (N : M) = ann(M/N)$ (because M is cyclic), then $r(x+N) = r^2(x+N) = N = (r - r^2)(x + N) = (r - r^2)r(x + N)$, (because $ann(M/N)$ is the ideal of R). Hence $(r-r^2)(x+r_1x+r_2x) = (r-r^2)r(x+r_3x+r_4x)$ for $r_3, r_4 \in R$ such that $r_3x, r_4x \in N$. Then $(r - r^2)(1 + r_1 + r_2 - r - rr_3 - rr_4)x = 0$. Because M is faithful, $(r - r^2)(1 + r_1 + r_2 - r - rr_3 - rr_4) = 0$. But R is domain and $(1 + r_1 + r_2 - r - rr_3 - rr_4) \neq 0$ (if not $1 = -r_1 - r_2 + r + rr_3 + rr_4$ and $x = -r_1x - r_2x + rx + rr_3x + rr_4x \in N$, hence $M = N$, a contradiction.), so $r - r^2 = 0$, hence $r = r^2$ and $(N : M)$ is Boolean. Now by theorem 2, $(N : M)^2 = (N : M)(N : M) = (N : M) \cap (N : M) = (N : M)$. ■

Let N and K be submodules of a multiplication $R - moduleM$ with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [2, theorem 3.4], the product of N and K is independent of presentation of N and K. clearly, NK is a submodule of M and $NK \subseteq N \cap K$. Now we have the following result:

Corollary 4. Let R be a domain and M a faithful cyclic multiplication $R - module$. Then every proper finitely generated weakly pure submodule of M is idempotent and in this case if N is proper finitely generated weakly pure submodule of M, then $N = (N : M)N$.

Proof. Let N be a proper finitely generated weakly pure submodule of M. Then by theorem 3, $N^2 = (N : M)^2M = (N : M)M = N$. Also $N = (N : M)^2M = (N : M)N$. ■

Corollary 5. Let R be a domain, M a faithful cyclic multiplication $R - module$ and N a proper finitely generated weakly pure submodule of M. Then $annN = ann(N : M)$.

Proof. Assume that $M = Rx$ and $N = Ra + Rb$ such that $a = r_1x$ and $b = r_2x$ for $r_1, r_2 \in R$. At the first we show that $(annN)$ is Boolean. Let $r \in (annN)$, then $rN = r^2N = 0$. Also $(r - r^2)N = 0$ (because$(annN)$ is the ideal of R), hence $(r - r^2)(a + b) = 0$. It follows that $(r - r^2)(r_1 + r_2)x = 0$. Because M is faithful, $(r-r^2)(r_1+r_2) = 0$. But R is domain and $(r_1+r_2) \neq 0$ (because $a + b \neq 0$), so $r - r^2 = 0$, hence $r = r^2$ and $(annN)$ is Boolean.

As N is weakly pure, we have $IN = N \cap IM$ for every Boolean ideal I of R. Taking $I = annN$, then we get that $0 = N \cap (annN)M$, and hence $0 = [0 : M] = [(N \cap (annN)M) : M] = [N : M] \cap [(annN)M : M] = [N : M] \cap annN = [N : M]annN$. Hence $annN \subseteq ann(N : M)$.

($[(annN)M : M] = annN$. Because if $r \in [(annN)M : M]$, then $rM \subseteq (annN)M$, also $rM(N : M) \subseteq (annN)(N : M)M$, hence $rN = 0$ and $r \in annN$. If $r \in annN$, then $rM \subseteq (annN)M$).

Conversely if $x \in ann(N : M)$ then $x(N : M) = 0$, and hence $xN = x(N : M)N = 0$, so that $x \in annN$ and $ann(N : M) \subseteq annN$, and hence $annN = ann(N : M)$. ■

References

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