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Essential Subsystems and Weakly Injective Systems

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Abstract. An S-system Q is called injective if for any monomorphism $f : A \to B$ of S-systems A, B and any homomorphism $g : A \to Q$, there exists a homomorphism $h : B \to Q$ such that $g = hf$.Also , an S-system A is called weakly injective if it is injective relative to all embedding of ideals in S (see[2]). If $\underline{\underline{A}}$ is a non-empty collection of ideals of a semigroup S, then we shall introduce an $\underline{\underline{A}}$-weakly injective S-system which is generalization of weakly injective S-system and we shall give some characterization of such S-systems .

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Introduction

Throughout this paper S is a commutative semigroup with identity and zero (Note that to every semigroup S without identity and zero, an identity 1 and a zero 0 can be adjoined by setting $1s = s = s1$, $11 = 1$ and $0s = 0 = s0$,$00 = 0$ for all $s \in S$.[see[2] page19]).A non-empty set X is called an unitary S-system if there exists a mapping $. : X \times S \to X$,$(x, s) \to xs$ such that $(a) x1 = x$ for every $x \in X$, $(b) x(st) = (xs)t$ for $x \in X, s, t \in S$.A non-empty subset $I \subseteq S$ is called an ideal of semigroup S if $SI \subseteq I$.Now let S be a semigroup and $\underline{\underline{A}}$ a non-empty collection of ideals of S. We say that an S-system X is $\underline{\underline{A}}$-weakly injective provided that every S-homomorphism $\phi : A \to X$ with A in $\underline{\underline{A}}$ can be lifted to an S-homomorphism $\theta : S \to X$. Recall that if $\underline{\underline{A}}$ is the collection of all ideals in S, then X is called weakly injective (see[2]). Also if $\underline{\underline{A}} = \{I\}$ where I is an ideal of S, then we apply the simple form I-weakly injective. An ideal I is called an essential ideal if $I \cap J \neq \phi$ for every ideal J of S. Recall that S is called reversible if every ideal of S is essential (see [2]).

Definition. The direct sum of two S-systems M and N is the disjoint union $M \sqcup N$ or $M \oplus N$ where S operates on $M \oplus N$ in the obvious way.If N is a subsystem of M , then N is called direct summand of M if there is subsystem K of M such that $K \cap N = \phi$ and $M = K \oplus N$.

Lemma 1. *If S is a group and M is an S-system, then every subsystem of M is a summand.*

Proof. Let N be a subsystem of M. Then $M \backslash N$ is also a subsystem and $N \cap (M \backslash N) = \phi$. Hence $M = N \oplus (M \backslash N)$.

Theorem 2. *Let S be a semigroup with zero, M be the only maximal ideal in S and $M^2 = 0$. Then M is the only essential ideal of S.*

Proof. Since M is an $\frac{S}{M}$-system ($[s].m = sm$ for $s \in S$ and $m \in M$ is well-defined) and $\frac{S}{M}$ is group, every subsystem of M is a summand. That is, if $J \leq M$ then $J \cap (M \backslash N) = \phi$. Hence J is not essential.

Definition. An S-system P is called projective if for any epimorphism $g : L \to M$ of S-systems L, M and any homomorphism $f : P \to M$, there exists an homomorphism $h : P \to L$ such that $f = gh$.

Theorem 3. *Let I be an ideal of S, and $\frac{S}{I}$ be a projective S-system. Then every S-system M with zero is I-weakly injective.*

Proof. Since $\frac{S}{I}$ is projective, the short exact sequence $I \xrightarrow{i} S \xrightarrow{\pi} \frac{S}{I}$ right splits. That is, $S \approx I \oplus \frac{S}{I}$. Now let $f : I \to M$ be an S-homomorphism where M is any S-system with zero. We define $h : S \to M$ by

$$h(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in S \backslash I \end{cases}$$

Hence the following diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & S \\ f \downarrow & \swarrow h & \\ M & & \end{array}$$

commutes. Therefore, M is I-weakly injective.

Theorem 4. *Let I be an ideal of S and every S-system is I-weakly injective. Then the short exact sequence $I \xrightarrow{i} S \xrightarrow{\pi} \frac{S}{I}$ left splits. That is $S \approx I \times \frac{S}{I}$.*

Proof. Since I is I-weakly injective as an S-system, there exists an homomorphism $h : S \to I$ such that the following diagram commutes.

$$\begin{array}{ccccc} I & \xrightarrow{i} & S & \xrightarrow{\pi} & \frac{S}{I} \\ i \downarrow & \swarrow h & & & \\ I & & & & . \end{array}$$

Hence the short exact sequence $I \xrightarrow{i} S \xrightarrow{\pi} \frac{S}{I}$ left splits. That is, $S \simeq I \times \frac{S}{I}$.

Theorem 5. *Let S be a semigroup, I an ideal of S and X an S-system with zero. If X is not I weakly injective, then there is a proper essential ideal J_0 containing I such that X is not J_0 weakly injective.*

Proof. Since X is not I-weakly injective, there is a homomorphism $f : I \to X$, which does not extend to S. Consider $P = \{(J, h) | J$ is an ideal of S, $I \subseteq J \subseteq S$, $h : J \to X$ is a homomorphism extending $f\}$.
Define a relation $\leq$ on P as follows:

$$(J_1, h_1) \leq (J_2, h_2) \longleftrightarrow J_1 \subseteq J_2 \text{ and } h_2|_{J_1} = h_1.$$

It is easy to check that $\leq$ is a partial order on P. By Zorn's Lemma we can find an ideal (J_0, h_0) of S containing I and maximal with respect to the property that h_0 extends to f.

Of course X is not J_0 weakly-injective. Also J_0 is essential in S for if $J_0 \cap K = \phi$ for some ideal K of S, then f trivially extends to $J_0 \oplus K$. That is a contradiction with maximality of J_0.

Theorem 6. *Let S be a semigroup and let $\underline{\underline{A}}$ be the collection of proper essential ideals of S. The every $\underline{\underline{A}}$-weakly injective S-system with zero is weakly injective.*

Proof. Let I be an ideal of S, X an S-system with zero, and $f : I \to X$ be a homomorphism. By Zorn's Lemma we can find an ideal J_0 of S containing I and maximal with respect to the property that f extends to J_0. That is there exists $\bar{f} : J_0 \to X$, such that $\bar{f}|_I = f$ and J_0 is maximal. Also J_0 is essential in S for if $J_0 \cap K = \phi$ for some ideal K of S, then f trivially extends to $J_0 \oplus K$, that is a contradiction with maximality of J_0. Since J_0 is essential we can lift $\bar{f} : J_0 \to X$ to S (by assumption). Hence X is I-weakly injective.

Corollary 7. *Let S be a semigroup with zero. Then the set of proper essential ideals of S is a test set for weakly injectivity, of S-systems with zero.*

Proof. Let I be an ideal of S, and $f : I \to S$ be a homomorphism. Since zero ideal belongs to $\sum = \{J | J \text{ is an ideal and } J \cap I = \{0\}\}$, by Zorn's Lemma there exists J_0 maximal for $\sum$. We define $g : I \oplus J_0 \to M$. Where M is an S-system with zero as follow:

$$g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in J_0 \end{cases} .$$

Since $I \oplus J_0$ is essential, there exists $h : S \to M$, such that the following diagram commutes,

$$\begin{array}{lll} I \to & I \oplus J_0 \to & S \\ f \downarrow & \swarrow g & \\ & & \\ M & & \swarrow \end{array}$$

Hence M is I-weakly injective.

References

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York 1992.

[2] M. Kilp, U. Knaure and A. V. Mikhalev, Monoids Acts and Categories, Walter de Grugter. Berlin. New York 2000.

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