On Square Root Closed Domains and Duality

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Abstract

In this note, we assume that $R$ is an integral domain with quotient field $K$. We introduce the concept of square root closed domain and then we study when $I^{-1} = \{ x \in K \mid xI \subseteq R \}$ is a ring, for a nonzero ideal $I$ of the square root closed domain.

Mathematics Subject Classification: 13B22, 13G05

Keywords: Square root closed domain; Strongly prime ideal; Dual of an ideal

1 Introduction

Throughout this paper, $R$ will be an integral domain, $K$ will denote its quotient field and $I$ will be a nonzero ideal of $R$. The $R$-submodule $J$ of $K$ is called fractional ideal if there exists an element $a \in R$ such that $aJ \subseteq R$. For a nonzero fractional ideal $J$ of $R$, the fractional ideal $(R : J) = \{ x \in K \mid xJ \subseteq R \}$ is called the dual of $J$ and we show with $J^{-1}$. In [7], Huckaba and Papick studied the question of when $I^{-1}$ is a ring, and this question has received further attention in [1-6].

We note that while $(I : I)$ is always an overring of $R$, $I^{-1}$ need not be a ring at all. Our purpose in this paper is to determine when $I^{-1}$ is a ring, where $I$ is a nonzero ideal of the square root closed domain. But we must begin with the following definition:

Definition 1. An integral domain $R$ is called square root closed domain, whenever for every $x \in K$, if $x^2 \in R$ then $x \in R$. 
Hence \(x\) is an integrally closed domain, then \(R\) is a square root closed domain, but \(\mathbb{Z}[i\sqrt{3}]\) is a square root closed domain which is not integrally closed.

**Proposition 2.** Let \(R\) be a square root closed domain and \(S\) be a multiplicatively closed subset of \(R\). Then \(S^{-1}R\) is a square root closed domain.

**Proof.** Let \(x \in K\) and \(x^2 \in S^{-1}R\). There exist \(a \in R\) and \(s \in S\) such that \(x^2 = \frac{a}{s}\). Thus \(sx^2 = a \in R\) and so \((sx)^2 = sa \in R\). Since \(R\) is a square root closed domain, then \(sx \in R\). Therefore \(x = \frac{sx}{s} \in S^{-1}R\).

**Theorem 3.** Let \(R\) be a square root closed domain and \(I\) be an ideal of \(R\). Then
\[
(\sqrt{I} : \sqrt{I}) = \{ x \in K \mid x^n \in (R : I) \text{ for all } n \geq 1 \}.
\]

**Proof.** Suppose that \(x \in (\sqrt{I} : \sqrt{I})\). Thus \(x^n \in (\sqrt{I} : \sqrt{I})\) for every \(n \geq 1\). Hence \(x^nI \subseteq x^n\sqrt{I} \subseteq \sqrt{I} \subseteq R\) and consequently \(x^n \in (R : I)\) for every \(n \geq 1\).

Conversely, let \(x \in K\) and \(x^n \in (R : I)\) for all \(n \geq 1\). If \(t \in \sqrt{I}\), then \(t^m \in I\) for some \(m \geq 1\). Hence \(x^nt^m \in R\) for each \(n \geq 1\). Thus \((xt)^m \in R\). We can assume that \(m = 2^k\) for some \(k \geq 1\). Therefore \(xt \in R\), because \(R\) is a square root closed domain. On the other hand, \(x^nt^m \in R\) for all \(n \geq 1\) implies that \((xt)^{m+1} = (x^{m+1}t^m)t \in \sqrt{I}\). Hence \(xt \in \sqrt{I}\). Therefore \(x \in (\sqrt{I} : \sqrt{I})\).

**Proposition 4.** For every ideal \(I\) of the square root closed domain \(R\), the following statements are satisfied:

1. \((\sqrt{I} : \sqrt{I}) \subseteq I^{-1}\).
2. \((\sqrt{I} : \sqrt{I})\) is a square root closed domain.
3. \(I^{-1}\) is a ring if and only if \(I^{-1} = (\sqrt{I} : \sqrt{I})\).
4. If \(I^{-1}\) is a ring, then \(I^{-1}\) is a square root closed domain.
5. If \(I\) is a radical ideal, then \((I : I)\) is a square root closed domain. Furthermore, \(I^{-1} = (I : I)\) if and only if \(I^{-1}\) is a ring.

**Proof.**
1. It is trivial by Theorem 3.
2. Let \(x^2 \in (\sqrt{I} : \sqrt{I})\), for \(x \in K\). Thus by Theorem 3, \(x^{2n} \in I^{-1}\) for every \(n \geq 1\). Hence \(tx^{2n} \in R\) for each \(t \in I\) and \(n \geq 1\). Then \((tx^n)^2 \in R\) and so \(tx^n \in R\), for all \(n \geq 1\). Therefore \(x^n \in (R : I)\) for every \(n \geq 1\) and consequently \(x \in (\sqrt{I} : \sqrt{I})\).
3. Suppose that $I^{-1}$ is a ring and $x \in I^{-1}$. Then $x^n \in I^{-1}$ for all $n \geq 1$, and so $x \in (\sqrt{I} : \sqrt{I})$, by Theorem 3. Therefore $I^{-1} \subseteq (\sqrt{I} : \sqrt{I})$ and consequently $I^{-1} = (\sqrt{I} : \sqrt{I})$ by 1. The other implication is clear.

4 and 5. It is obvious, by 2 and 3.

Corollary 5. Let $R$ be a square root closed domain and $I$ and $J$ be ideals of $R$. Then the following statements are hold:

1. $(\sqrt{I} : \sqrt{I})$ is the largest subring of $(R : I)$.

2. If $I \subseteq J$, then $(\sqrt{J} : \sqrt{J}) \subseteq (\sqrt{I} : \sqrt{I})$.

3. If $I^{-1}$ is a ring, then $\sqrt{I}$ is an ideal of $I^{-1}$.

Proof. 1. It is clear, by 1 and 3 of Proposition 4.

2. Let $x \in (\sqrt{J} : \sqrt{J})$. Thus $x^n \in (R : J)$ for every $n \geq 1$, by Theorem 3. $I \subseteq J$ implies that $(R : J) \subseteq (R : I)$, and so $x^n \in (R : I)$ for all $n$. Therefore $x \in (\sqrt{I} : \sqrt{I})$.

3. It follows from 3 of Proposition 4. □

Proposition 6. Let $R$ be a square root closed domain and $I \subseteq J$ be ideals of $R$ with the same radical. If $I^{-1}$ is a ring, then $I^{-1} = J^{-1} = (\sqrt{I} : \sqrt{I})$.

Proof. By 1 and 3 of Proposition 4, we have

$$I^{-1} = (\sqrt{I} : \sqrt{I}) = (\sqrt{J} : \sqrt{J}) \subseteq (R : J) = J^{-1} \subseteq I^{-1}. \quad \square$$

If $I$ is an ideal of integral domain $R$, then $I^n \subseteq I$ and $\sqrt{I^n} = \sqrt{I}$, for each $n \geq 1$. Therefore we have the following:

Corollary 7. For every ideal $I$ of the square root closed domain $R$, if $(R : I^n)$ is a ring, for some $n > 1$, then $I^{-1}$ is a ring. □

We recall that, a prime ideal $P$ of the integral domain $R$ is said to be strongly prime if whenever $xy \in P$, for $x, y \in K$, then either $x \in P$ or $y \in P$.

Proposition 8. Let $R$ be an integral domain and $I$ be an ideal of $R$ such that $I^{-1}$ is a ring. If $P$ is a strongly prime ideal of $R$ containing $I$, then $I^{-1}$ is a square root closed domain.

Proof. Let $x^2 \in I^{-1}$, for $x \in K$. Then $x^2 I \subseteq R$. Hence $(xI)^2 = (x^2 I)I \subseteq I \subseteq P$. Thus $xI \subseteq P$, because $P$ is a strongly prime. Therefore, $x \in (P : I) \subseteq (R :
We note that, if $I$ is an ideal of the integral domain $R$ and $P$ is a minimal prime ideal of $I$, then $\sqrt{IP} = \sqrt{PR} = PR$. For every element $a \in P$, we have $a^n I \subseteq PR = \sqrt{IP}$ which implies that $\frac{a^n}{1} \in IR$, for some integer $n$. Hence there exists an element $s \in R \setminus P$ such that $sa^n \in I$ and so $sa \in \sqrt{I}$. Therefore, we conclude that for every $a \in P$ there is an element $s \in R \setminus P$ such that $sa \in \sqrt{I}$.

An element $a \in K$ is almost integral over $R$, if there exists a nonzero element $r \in R$ such that $ra^n \in R$, for all $n \geq 1$. We say that $R$ is completely integrally closed, if $a \in K$ is almost integral over $R$, then $a \in R$.

**Lemma 9.** Let $R$ be an completely integrally closed domain and $I$ be an ideal of $R$. If $P$ is a minimal prime ideal of $I$, then $(\sqrt{I} : \sqrt{I}) \subseteq (P : P)$.

**Proof.** Let $x \in (\sqrt{I} : \sqrt{I})$. For each $a \in P$, by above note, $sa \in \sqrt{I}$, for some $s \in R \setminus P$. Then $sax^n \in \sqrt{I}$, for all $n \geq 1$, because $x^n \in (\sqrt{I} : \sqrt{I})$. Hence $s(ax)^n = a^{n-1}(sax^n) \in \sqrt{I}$, for each $n \geq 1$ and so $ax \in R$, because $R$ is completely integrally closed. Since $sax \in \sqrt{I} \subseteq P$ and $s \not\in P$, then $ax \in P$, it follows that $x \in (P : P)$. □

**Proposition 10.** Let $R$ be a square root closed domain, $I$ be an ideal of $R$ and $P$ is a minimal prime ideal of $I$. If $R$ is also completely integrally closed, then the following statements are hold:

1. $(\sqrt{I} : \sqrt{I}) = (P : P)$.

2. If $I^{-1}$ is a ring, then $I^{-1} = P^{-1} = (P : P)$.

**Proof.** 1. It follows from 2 of Corollary 5 and Lemma 9.

2. Since $I \subseteq P$, then $P^{-1} \subseteq I^{-1}$. On the other hand, $I^{-1}$ is a ring, then $I^{-1} = (\sqrt{I} : \sqrt{I})$, by 3 of Proposition 4. Therefore by 1, we have

$$I^{-1} = (\sqrt{I} : \sqrt{I}) = (P : P) \subseteq P^{-1} \subseteq I^{-1}. \quad \square$$

For every ideal $I$ of the integral domain $R$, we have

$$(R : I) \subseteq (R : I^2) \subseteq (R : I^3) \subseteq \cdots \subseteq (R : I^n) \subseteq \cdots$$

Therefore we can state the following result:

**Proposition 11.** Let $R$ be a square root closed domain and $I$ be an ideal of $R$. If $(R : I^{n+1}) = (R : I^n)$, for some $n \geq 1$, then
1. \((R : I^m) = (R : I^n)\), for each \(m \geq n\).

2. \(I^{-1}\) is a ring.

3. \((R : I^2) = (R : I)\).

**Proof.** 1. We can use induction on \(m \geq n\) and the equality

\[(R : I^{m+1}) = ((R : I^m) : I) = ((R : I^n) : I) = (R : I^{n+1}) = (R : I^n)\.

2. We first show that \((R : I^n)\) is a ring. For every element \(x, y \in (R : I^n)\), we have \(xI^n \subseteq R\) and \(yI^n \subseteq R\). Thus \(xyI^{2n} \subseteq R\) and so \(xy \in (R : I^{2n}) = (R : I^n)\), by 1. Since \((R : I^n)\) is an additive subgroup, \((R : I^n)\) is a subring of \(K\). It follows from Corollary 7, that \(I^{-1}\) is ring.

3. It is clear if \(n = 1\). On the otherwise, \(I^n \subseteq I^2\) and so \((R : I^2)\) is a ring, by Corollary 7. Now, for every element \(x \in (R : I^2)\), \(x^2 \in (R : I^2)\). Then \((xI)^2 = x^2I^2 \subseteq R\). Hence \(xI \subseteq R\), because \(R\) is a square root closed domain. Therefore \(x \in (R : I)\). \(\square\)

**Corollary 12.** For every ideal \(I\) of the square root closed domain \(R\), the following statements are hold:

1. If \(I^{-1}\) is not ring, then we have

\[R \subseteq (R : I) \subseteq (R : I^2) \subseteq (R : I^3) \subseteq \cdots \subseteq (R : I^n) \subseteq \cdots\]

2. If \(I^{n+1} = I^n\), for some \(n \geq 1\), then \(I^{-1}\) is a ring.

3. If \(R\) is Noetherian and \(\bigcap_{n=1}^{\infty} I^n \neq 0\), then \(I^{-1}\) is a ring. \(\square\)

**References**


Received: October, 2010