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On Square Root Closed Domains and Duality

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Abstract

In this note, we assume that R is an integral domain with quotient field K. We introduce the concept of square root closed domain and then we study when $I^{-1} = \{ x \in K \mid xI \subseteq R \}$ is a ring, for a nonzero ideal I of the square root closed domain.

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1 Introduction

Throughout this paper, R will be an integral domain, K will denote its quotient field and I will be a nonzero ideal of R. The R-submodule J of K is called fractional ideal if there exists an element $a \in R$ such that $aJ \subseteq R$. For a nonzero fractional ideal J of R, the fractional ideal $(R : J) = \{ x \in K \mid xJ \subseteq R \}$ is called the dual of J and we show with J^{-1}. In [7], Huckaba and Papick studied the question of when I^{-1} is a ring, and this question has received further attention in [1-6].

We note that while (I : I) is always an overring of R, I^{-1} need not be a ring at all. Our purpose in this paper is to determine when I^{-1} is a ring, where I is a nonzero ideal of the square root closed domain. But we must begin with the following definition:

Definition 1. *An integral domain R is called square root closed domain, whenever for every $x \in K$, if $x^2 \in R$ then $x \in R$.*

If R is an integrally closed domain, then R is a square root closed domain, but $\mathbf{Z}[i\sqrt{3}]$ is a square root closed domain which is not integrally closed.

Proposition 2. *Let R be a square root closed domain and S be a multiplicatively closed subset of R. Then $S^{-1}R$ is a square root closed domain.*

Proof. Let $x \in K$ and $x^2 \in S^{-1}R$. There exist $a \in R$ and $s \in S$ such that $x^2 = \frac{a}{s}$. Thus $sx^2 = a \in R$ and so $(sx)^2 = sa \in R$. Since R is a square root closed domain, then $sx \in R$. Therefore $x = \frac{sx}{s} \in S^{-1}R$. □

Theorem 3. *Let R be a square root closed domain and I be an ideal of R. Then*

$$(\sqrt{I} : \sqrt{I}) = \{ \ x \in K \ \ | \ \ x^n \in (R : I) \ for \ all \ n \geq 1 \ \}.$$

Proof. Suppose that $x \in (\sqrt{I} : \sqrt{I})$. Thus $x^n \in (\sqrt{I} : \sqrt{I})$ for every $n \geq 1$. Hence $x^n I \subseteq x^n \sqrt{I} \subseteq \sqrt{I} \subseteq R$ and consequently $x^n \in (R : I)$ for every $n \geq 1$.

Conversely, let $x \in K$ and $x^n \in (R : I)$ for all $n \geq 1$. If $t \in \sqrt{I}$, then $t^m \in I$ for some $m \geq 1$. Hence $x^n t^m \in R$ for each $n \geq 1$. Thus $(xt)^m \in R$. We can assume that $m = 2^k$ for some $k \geq 1$. Therefore $xt \in R$, because R is a square root closed domain. On the other hand, $x^n t^m \in R$ for all $n \geq 1$ implies that $(xt)^{m+1} = (x^{m+1}t^m)t \in \sqrt{I}$. Hence $xt \in \sqrt{I}$. Therefore $x \in (\sqrt{I} : \sqrt{I})$. □

Proposition 4. *For every ideal I of the square root closed domain R, the following statements are satisfied:*

1. $(\sqrt{I} : \sqrt{I}) \subseteq I^{-1}$.
2. $(\sqrt{I} : \sqrt{I})$ *is a square root closed domain.*
3. I^{-1} *is a ring if and only if* $I^{-1} = (\sqrt{I} : \sqrt{I})$.
4. *If I^{-1} is a ring, then I^{-1} is a square root closed domain.*
5. *If I is a radical ideal, then $(I : I)$ is a square root closed domain. Furthermore, $I^{-1} = (I : I)$ if and only if I^{-1} is a ring.*

Proof. 1. It is trivial by Theorem 3.
2. Let $x^2 \in (\sqrt{I} : \sqrt{I})$, for $x \in K$. Thus by Theorem 3, $x^{2n} \in I^{-1}$ for every $n \geq 1$. Hence $tx^{2n} \in R$ for each $t \in I$ and $n \geq 1$. Then $(tx^n)^2 \in R$ and so $tx^n \in R$, for all $n \geq 1$. Therefore $x^n \in (R : I)$ for every $n \geq 1$ and consequently $x \in (\sqrt{I} : \sqrt{I})$.

3. Suppose that I^{-1} is a ring and $x \in I^{-1}$. Then $x^n \in I^{-1}$ for all $n \geq 1$, and so $x \in (\sqrt{I} : \sqrt{I})$, by Theorem 3. Therefore $I^{-1} \subseteq (\sqrt{I} : \sqrt{I})$ and consequently $I^{-1} = (\sqrt{I} : \sqrt{I})$ by 1. The other implication is clear.
4 and 5. It is obvious, by 2 and 3. □

Corollary 5. *Let R be a square root closed domain and I and J be ideals of R. Then the following statements are hold:*

1. $(\sqrt{I} : \sqrt{I})$ *is the largest subring of* $(R : I)$.
2. *If* $I \subseteq J$, *then* $(\sqrt{J} : \sqrt{J}) \subseteq (\sqrt{I} : \sqrt{I})$.
3. *If* I^{-1} *is a ring, then* $\sqrt{I}$ *is an ideal of* I^{-1}.

Proof. **1.** It is clear, by 1 and 3 of Proposition 4.
2. Let $x \in (\sqrt{J} : \sqrt{J})$. Thus $x^n \in (R : J)$ for every $n \geq 1$, by Theorem 3. $I \subseteq J$ implies that $(R : J) \subseteq (R : I)$, and so $x^n \in (R : I)$ for all n. Therefore $x \in (\sqrt{I} : \sqrt{I})$.
3. It follows from 3 of Proposition 4. □

Proposition 6. *Let R be a square root closed domain and $I \subseteq J$ be ideals of R with the same radical. If I^{-1} is a ring, then $I^{-1} = J^{-1} = (\sqrt{I} : \sqrt{I})$.*

Proof. By 1 and 3 of Proposition 4, we have

$$I^{-1} = (\sqrt{I} : \sqrt{I}) = (\sqrt{J} : \sqrt{J}) \subseteq (R : J) = J^{-1} \subseteq I^{-1}. \quad \square$$

If I is an ideal of integral domain R, then $I^n \subseteq I$ and $\sqrt{I^n} = \sqrt{I}$, for each $n \geq 1$. Therefore we have the following:

Corollary 7. *For every ideal I of the square root closed domain R, if $(R : I^n)$ is a ring, for some $n > 1$, then I^{-1} is a ring.* □

We recall that, a prime ideal P of the integral domain R is said to be *strongly prime* if whenever $xy \in P$, for $x, y \in K$, then either $x \in P$ or $y \in P$.

Proposition 8. *Let R be an integral domain and I be an ideal of R such that I^{-1} is a ring. If P is a strongly prime ideal of R containing I, then I^{-1} is a square root closed domain.*

Proof. Let $x^2 \in I^{-1}$, for $x \in K$. Then $x^2 I \subseteq R$. Hence $(xI)^2 = (x^2 I)I \subseteq I \subseteq P$. Thus $xI \subseteq P$, because P is a strongly prime. Therefore, $x \in (P : I) \subseteq (R :$

$I) = I^{-1}$. □

We note that, if I is an ideal of the integral domain R and P is a minimal prime ideal of I, then $\sqrt{IR_P} = \sqrt{PR_P} = PR_P$. For every element $a \in P$, we have $\frac{a}{1} \in PR_P = \sqrt{IR_P}$ which implies that $\frac{a^n}{1} \in IR_P$, for some integer n. Hence there exists an element $s \in R \setminus P$ such that $sa^n \in I$ and so $sa \in \sqrt{I}$. Therefore, we conclude that for every $a \in P$ there is an element $s \in R \setminus P$ such that $sa \in \sqrt{I}$.

An element $a \in K$ is *almost integral* over R, if there exists a nonzero element $r \in R$ such that $ra^n \in R$, for all $n \geq 1$. We say that R is *completely integrally closed*, if $a \in K$ is almost integral over R, then $a \in R$.

Lemma 9. *Let R be an completely integrally closed domain and I be an ideal of R. If P is a minimal prime ideal of I, then $(\sqrt{I} : \sqrt{I}) \subseteq (P : P)$.*

Proof. Let $x \in (\sqrt{I} : \sqrt{I})$. For each $a \in P$, by above note, $sa \in \sqrt{I}$, for some $s \in R \setminus P$. Then $sax^n \in \sqrt{I}$, for all $n \geq 1$, because $x^n \in (\sqrt{I} : \sqrt{I})$. Hence $s(ax)^n = a^{n-1}(sax^n) \in \sqrt{I}$, for each $n \geq 1$ and so $ax \in R$, because R is completely integrally closed. Since $sax \in \sqrt{I} \subseteq P$ and $s \notin P$, then $ax \in P$, it follows that $x \in (P : P)$. □

Proposition 10. *Let R be a square root closed domain, I be an ideal of R and P is a minimal prime ideal of I. If R is also completely integrally closed, then the following statements are hold:*

1. $(\sqrt{I} : \sqrt{I}) = (P : P)$.
2. *If I^{-1} is a ring, then $I^{-1} = P^{-1} = (P : P)$.*

Proof. 1. It follows from 2 of Corollary 5 and Lemma 9.
2. Since $I \subseteq P$, then $P^{-1} \subseteq I^{-1}$. On the other hand, I^{-1} is a ring, then $I^{-1} = (\sqrt{I} : \sqrt{I})$, by 3 of Proposition 4. Therefore by 1, we have

$$I^{-1} = (\sqrt{I} : \sqrt{I}) = (P : P) \subseteq P^{-1} \subseteq I^{-1}. \quad \square$$

For every ideal I of the integral domain R, we have

$$(R : I) \subseteq (R : I^2) \subseteq (R : I^3) \subseteq \cdots \subseteq (R : I^n) \subseteq \cdots$$

Therefore we can state the following result:

Proposition 11. *Let R be a square root closed domain and I be an ideal of R. If $(R : I^{n+1}) = (R : I^n)$, for some $n \geq 1$, then*

1. $(R : I^m) = (R : I^n)$, *for each* $m \geq n$.
2. I^{-1} *is a ring.*
3. $(R : I^2) = (R : I)$.

Proof. **1.** We can use induction on $m \geq n$ and the equality

$$(R : I^{m+1}) = ((R : I^m) : I) = ((R : I^n) : I) = (R : I^{n+1}) = (R : I^n).$$

2. We first show that $(R : I^n)$ is a ring. For every element $x, y \in (R : I^n)$, we have $xI^n \subseteq R$ and $yI^n \subseteq R$. Thus $xyI^{2n} \subseteq R$ and so $xy \in (R : I^{2n}) = (R : I^n)$, by 1. Since $(R : I^n)$ is an additive subgroup, $(R : I^n)$ is a subring of K. It follows from Corollary 7, that I^{-1} is ring.
3. It is clear if $n = 1$. On the otherwise, $I^n \subseteq I^2$ and so $(R : I^2)$ is a ring, by Corollary 7. Now, for every element $x \in (R : I^2)$, $x^2 \in (R : I^2)$. Then $(xI)^2 = x^2I^2 \subseteq R$. Hence $xI \subseteq R$, because R is a square root closed domain. Therefore $x \in (R : I)$. □

Corollary 12. *For every ideal I of the square root closed domain R, the following statements are hold:*

1. *If* I^{-1} *is not ring, then we have*

$$R \subset (R : I) \subset (R : I^2) \subset (R : I^3) \subset \cdots \subset (R : I^n) \subset \cdots$$

2. *If* $I^{n+1} = I^n$, *for some* $n \geq 1$, *then* I^{-1} *is a ring.*
3. *If R is Noetherian and* $\bigcap_{n=1}^{\infty} I^n \neq 0$, *then* I^{-1} *is a ring.* □

References

[1] D. F. Anderson, When the dual of an ideal is a ring? Houston J. Math **9** no.3 (1983) 325 - 332

[2] M. Fontana, J. A. Huchaba and I. J. Papick, Divisorial ideals in Prüfer domains, Canad. Math. Bull. **27** (1984) 324 - 328

[3] M. Fontana, J. A. Huchaba and I. J. Papick, Some properties of divisorial prime ideals in Prüfer domains, J. Pure Appl. Algebra **39** (1986) 95 - 103

[4] M. Fontana, J. Huchaba and I. Papick, Domains satisfying the trace property, J. Algebra **107** (1987) 169 - 182

[5] W. Heinzer and I. Papick, The radical trace property, J. Algebra **112** (1988) 110 - 121

[6] E. Houston, S. Kabbaj, T. Lucas, and A. Mimouni, When is the dual of an ideal a ring ?, J. Algebra **225** (2000) 429 - 450.

[7] J. A. Huckaba and I. J. Papick, When the dual of an ideal is ring? Manuscripta Math. **37** (1982) 67 - 85

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