International Journal of Algebra, Vol. 5, 2011, no. 7, 315 - 323

A Block Refinement of the Green–Puig Correspondences in Finite Group Modular Representation Theory

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Abstract. We present a block theoretic refinement of the fundamental Green–Puig Correspondences in Finite Group Modular Representation Theory. In particular, if B is a block of the finite group G and (Q, Z) is a possible vertex–source pair for an indecomposable B-module, we develop the Green–Puig Correspondence in this context using the Brauer homomorphism. Consequently there is an indecomposable B-module with (Q, Z) as a vertex–source pair. We also extend some related results on Heller Operators. This work was suggested by the well-known fact that the vertex of any indecomposable $(\mathcal{O}G)B$-module is a subgroup of a defect group of B.

Mathematics Subject Classification: 20C20

1. Introduction and Statements

Our notation and terminology are standard and tend to follow [3] and [6]. All rings have identities and are Noetherian and all modules over a ring are unitary and finitely generated left modules unless specified otherwise.

Let R be a ring. Then R-mod (resp. mod-R) will denote the abelian category of finitely generated left (resp. right) R-modules. Let U and V be left R-modules. Then $U|V$ in R-mod signifies that U is isomorphic to a direct summand of V in R-mod. Also if R has the unique decomposition property (cf. [3, p. 37]), then U is a component of V if U is indecomposable in R-mod and $U|V$ in R-mod.

Throughout this paper, G is a finite group, p is a prime integer and $(\mathcal{O}, K, k = \mathcal{O}/J(\mathcal{O}))$ is a p-modular system that is "large enough" for all subgroups of G (i.e., $\mathcal{O}$ is a complete discrete valuation ring, $k = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of characteristic p and the field of fractions K of $\mathcal{O}$ is of characteristic zero and is a splitting field for all subgroups of G).

Let A be an $\mathcal{O}$-algebra that is finitely generated as an $\mathcal{O}$-module. Then A has the unique decomposition property by the Krull–Schmidt Theorem ([3, I, Theorem 11.4] or [6, Theorem 4.4]). Also the natural ring epimorphism $-\colon \mathcal{O} \to \mathcal{O}/J(\mathcal{O}) = k$ induces an $\mathcal{O}$-algebra epimorphism $-\colon A \to A/(J(\mathcal{O})A) = \overline{A}$.

The statements of the main results of this paper are presented in the remainder of this section. The required proofs are presented in section 2.

Our results use the Brauer homomorphism to develop a block refinement of the fundamental Green–Puig Correspondences in Finite Group Modular Representation Theory. For example, our methods include a new proof of [2, Theorem 3]. Our last results concern the Heller operators in this context.

We begin by showing that the Puig Correspondence for modules, which was inspired by the Green Correspondence, implies the Green Correspondence.

Remark 1.1. [6, Corollary 20.9] *implies* [3, III, Theorem 5.6].

Proof. Let P, $N_G(P) \leq H \leq G$, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{A}$, etc. be as in [3, III,Theorem 5.6]. Suppose that $Q \in \mathfrak{A}$ and $x \in N_G(Q) - H$. Then $Q \leq ({}^xP) \cap P$ and [3, III, Lemma 5.1] implies that $Q \notin \mathfrak{A}$. This contradiction implies that $N_G(Q) \leq H$ and so [6, Corollary 20.9] yields the desired conclusion. □

Our next result, due to M. Linckelmann ([4]) developed out of his involvement in a discussion with the author related to the Green–Puig Correspondence.

Let Q be a p-subgroup of G, let A be an interior Q-algebra with a $Q \times Q$-stable $\mathcal{O}$-basis X and let V be an $\mathcal{O}Q$-module.

Proposition 1.2 (M. Linckelmann)**.** *Assume that there is an $x \in C_X(Q)$ such that $x(\mathcal{O}Q)$ is a free right $\mathcal{O}Q$-module. Then*

$$V \mid Res^A_Q(A \bigotimes_{\mathcal{O}Q} V)$$

in $\mathcal{O}Q$-mod.

For the remainder of this paper, Q is an <u>arbitrary</u> p-subgroup of G and Z is an <u>arbitrary</u> $\mathcal{O}Q$-module with vertex Q.

The next result is well-known and is presented for the convenience of the reader.

Lemma 1.3. *Let $Q \leq H \leq G$, let V be an indecomposable $\mathcal{O}H$-module with vertex–source pair (Q, Z) and let W be an indecomposable $\mathcal{O}G$-module. If $W | Ind^G_H(V)$ in $\mathcal{O}G$-mod and $V | Res^G_H(W)$ in $\mathcal{O}H$-mod, then (Q, Z) is a vertex–source pair of W.*

Clearly this result implies that if $N_G(Q) \leq H$ and V is an indecomposable $\mathcal{O}H$-module with vertex–source pair (Q, Z), then $Pu^G_H(V) \cong \mathcal{G}r^G_H(V)$ in $\mathcal{O}G$-mod. (Here $\mathcal{G}r^G_H(V)$ denotes the Green correspondent of V in $\mathcal{O}G$-mod ([3, III, Theorem 5.6]) and $Pu_H^G(V)$ denotes the Puig correspondent of V in $\mathcal{O}G$-mod ([6, Proposition 20.8])).

Let $M \trianglelefteq G$ and let V be an indecomposable $\mathcal{O}M$-module with vertex–source pair (Q, Z). Let $H = Stab_G(V)$ so that $M \leq H \leq G$ and let

$$Ind_M^H(V) = \bigoplus_{j \in J} \mathcal{V}_j$$

be a direct sum decomposition in $\mathcal{O}H$-mod where J is a finite set and $\mathcal{V}_j$ is an indecomposable $\mathcal{O}H$-module for each $j \in J$. Clearly $Res_M^H(\mathcal{V}_j) \cong n_j V$ for a positive integer n_j for each $j \in J$.

Our next result extends [1, Theorem 9.6]) to our context:

Lemma 1.4. (a) *$Ind_H^G(\mathcal{V}_j)$ is an indecomposable $\mathcal{O}G$-module and (Q, Z) is a vertex–source pair for $\mathcal{V}_j$ in $\mathcal{O}H$-mod and for $Ind_H^G(\mathcal{V}_j)$ in $\mathcal{O}G$-mod for each $j \in J$;*

(b) *If $j, j' \in J$, then $Ind_H^G(\mathcal{V}_j) \cong Ind_H^G(\mathcal{V}_{j'})$ in $\mathcal{O}G$-mod if and only if $\mathcal{V}_j \cong \mathcal{V}_{j'}$ in $\mathcal{O}H$-mod; and*

(c) *If $j \in J$ and $\mathcal{V}_j$ is an irreducible $\mathcal{O}H$-module, then V is irreducible in $\mathcal{O}M$-mod and $Ind_H^G(\mathcal{V}_j)$ is irreducible in $\mathcal{O}G$-mod.*

Let Q and Z be as above. Set $N = N_G(Q)$ and $H = N_G(Q, Z)$ so that $QC_G(Q) \leq H \leq N$.

Let

$$Ind_Q^H(Z) = \bigoplus_{i=1}^{n} \mathcal{V}_i$$

be a direct sum decomposition of $Ind_Q^H(Z)$ into indecomposable $\mathcal{O}H$-modules. Thus, by Lemma 1.4, $Res_Q^H(\mathcal{V}_i) \cong n_i Z$ for some positive integer n_i and (Q, Z) is the unique vertex–source pair for $\mathcal{V}_i$ for all $1 \leq i \leq n$. Also, by Lemma 1.4, $Ind_H^N(\mathcal{V}_i)$ is an indecomposable $\mathcal{O}N$-module with (Q, Z) as a vertex–source pair. Here

$$Ind_Q^N(Z) = \bigoplus_{i=1}^{n} Ind_H^N(\mathcal{V}_i)$$

in $\mathcal{O}N$-mod and if $1 \leq i, j \leq n$, then $\mathcal{V}_i \cong \mathcal{V}_j$ in $\mathcal{O}H$-mod if and only if $Ind_H^N(\mathcal{V}_i) \cong Ind_H^N(\mathcal{V}_j)$ in $\mathcal{O}N$-mod.

As in [6, Proposition 20.8], $Pu_H^N(\mathcal{V}_i) \cong Ind_H^N(\mathcal{V}_i)$ in $\mathcal{O}N$-mod, and so

$$Pu_H^G(\mathcal{V}_i) \cong \mathcal{G}r_N^G(Ind_H^N(\mathcal{V}_i)) \cong Pu_N^G(Pu_H^N(\mathcal{V}_i))$$

in $\mathcal{O}G$-mod and (Q, Z) is a vertex–source pair of $Pu_H^G(\mathcal{V}_i)$ for all $1 \leq i \leq n$.

Let Q and Z be as above (that is, Q is an arbitrary p-subgroup of G and Z is an arbitrary indecomposable $\mathcal{O}Q$-module with vertex Q) and for each subgroup L of G such that $Q \leq L$ and for each idempotent E of $Z(\mathcal{O}L)$, let $\mathcal{IT}(L, Q, Z, E)$ denote a set of representatives for the isomorphism types of indecomposable $(\mathcal{O}L)E$-modules with vertex–source pair (Q, Z).

As above, $QC_G(Q) \leq H = N_G(Q,Z) \leq N = N_G(Q)$. Assume that $\mathcal{IT}(H,Q,Z,1)$ is chosen so that $\mathcal{IT}(H,Q,Z,1) \subseteq \{\mathcal{V}_i \mid 1 \leq i \leq n\}$.

Proposition 1.5. (a) $\{Ind_H^N(\mathcal{V}) \mid \mathcal{V} \in \mathcal{IT}(H,Q,Z,1)\}$ *is a set of representatives for the isomorphism types of indecomposable $\mathcal{O}N$-modules with vertex–source pair (Q,Z) and*

$$Ind_H^N\colon \mathcal{IT}(H,Q,Z,1) \to \{Ind_H^N(\mathcal{V}) \mid \mathcal{V} \in \mathcal{IT}(H,Q,Z,1)\}$$

is a bijection;

(b) $\{\mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) \mid \mathcal{V} \in \mathcal{IT}(H,Q,Z,1)\}$ *is a set of representatives for the isomorphism types of indecomposable $\mathcal{O}G$-modules with vertex–source pair (Q,Z) and* $\mathcal{G}r_N^G\colon \{Ind_H^N(\mathcal{V}) \mid \mathcal{V} \in \mathcal{IT}(H,Q,Z,1)\} \to \{\mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) \mid \mathcal{V} \in \mathcal{IT}(H,Q,Z,1)\}$ *is a bijection; and*

(c) *If $e \in B\ell(\mathcal{O},H)$ and $\mathcal{V} \in \mathcal{IT}(Q,H,Z,1)$ is such that $e\mathcal{V} = \mathcal{V}$, then e^N is defined,*

$$e^N Ind_H^N(\mathcal{V}) = Ind_H^N(\mathcal{V}),$$

e^G is defined and

$$e^G\, \mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) = \mathcal{G}r_N^G(Ind_H^N(\mathcal{V})).$$

Let $\mathcal{S}(B\ell(\mathcal{O}C_G(Q)))$ denote the set of subsets of $B\ell(\mathcal{O}C_G(Q))$ and let

$$\alpha\colon \mathcal{S}(B\ell(\mathcal{O}C_G(Q))) \to Z(\mathcal{O}C_G(Q))$$

be defined by:

$$\alpha(\omega) = \sum_{\beta\in\omega} \beta$$

for each $\omega \in \mathcal{S}(B\ell(\mathcal{O}C_G(Q)))$. Thus α is an N-equivariant injection. Let $\Omega(H), \Omega(N)$, denote the sets of orbits of H, N on $B\ell(\mathcal{O}C_G(Q))$, resp. Thus, by restriction, $\alpha\colon \Omega(H) \to B\ell(\mathcal{O}H)$ and $\alpha\colon \Omega(N) \to B\ell(\mathcal{O}N)$ are bijective maps. Clearly if $\delta \in \Omega(N)$, then

$$\alpha(\delta) = \sum_{\{\omega\in\Omega(H) \mid \omega\subseteq\delta\}} \alpha(\omega)$$

is an orthogonal sum and

$$1 = \sum_{\delta\in\Omega(N)} \alpha(\delta)$$

is an orthogonal sum.

Fix $B \in B\ell(\mathcal{O}G)$ and assume that Q is contained in a defect group of B. Here $0 \neq Br_Q(B) \in Z(kC_G(Q))^N$. Let b_Q be the unique idempotent of $Z(\mathcal{O}C_G(Q))^N$ such that $Br_Q(B) = \overline{b}_Q$.

Clearly, by conjugation, N permutes $B\ell(\mathcal{O}C_G(Q)b_Q)$. Let $\mathcal{S}(B\ell(\mathcal{O}C_G(Q)b_Q))$ denote the set of subsets of $B\ell(\mathcal{O}C_G(Q)b_Q)$. Thus, by restriction,

$$\alpha\colon \mathcal{S}(B\ell(\mathcal{O}C_G(Q)b_Q)) \to Z(\mathcal{O}C_G(Q)b_Q)$$

is an N-equivariant injection. Let $\Omega(H, b_Q), \Omega(N, b_Q)$ denote the set of orbits of H, N on $B\ell(\mathcal{O}C_G(Q)b_Q)$, resp. Thus $B\ell((\mathcal{O}H)b_Q) = \{\alpha(\omega) \mid \omega \in \Omega(H, b_Q)\}$, $B\ell((\mathcal{O}N)b_Q) = \{\alpha(\delta) \mid \delta \in \Omega(N, b_Q)\}$ and α induces bijective maps $\alpha\colon \Omega(H, b_Q) \to B\ell((\mathcal{O}H)b_Q)$ and $\alpha\colon \Omega(N, b_Q) \to B\ell((\mathcal{O}N)b_Q)$. Also

$$b_Q = \sum_{\omega\in\Omega(H,b_Q)} \alpha(\omega) = \sum_{\delta\in\Omega(N,b_Q)} \alpha(\delta)$$

and if $\delta \in \Omega(N, b_Q)$, then $\delta = \bigcup_{\{\omega\in\Omega(H,b_Q)\mid\omega\subseteq\delta\}} \omega$. Thus, with Z as above,

$$\mathcal{IT}(H, Q, Z, b_Q) = \bigcup_{\omega\in\Omega(H,b_Q)} \mathcal{IT}(H, Q, Z, \alpha(\omega))$$

and

$$\mathcal{IT}(N, Q, Z, b_Q) = \bigcup_{\delta\in\Omega(N,b_Q)} \mathcal{IT}(H, Q, Z, \alpha(\delta))$$

are disjoint unions.

We may and shall assume that $\mathcal{IT}(H, Q, Z, \alpha(\omega)) \subseteq \{\mathcal{V}_i \mid 1 \leq i \leq n\}$ for each $\omega \in \Omega(H, b_Q)$.

Let $\omega \in \Omega(H, b_Q)$. Then $\alpha(\omega)^N$ is defined and $\alpha(\omega)^N = \alpha(\delta)$ where δ is the unique element of $\Omega(N, b_Q)$ such that $\omega \subseteq \delta$. Also $\alpha(\delta)^G = B = \alpha(\omega)^G$ for each $\delta \in \Omega(N, b_Q)$ and each $\omega \in \Omega(H, b_Q)$.

Our main result is:

Theorem 1.6. (a) *$\mathcal{IT}(H, Q, Z, \alpha(\omega)) \neq \emptyset$ for each $\omega \in \Omega(H, b_Q)$;*

(b) *If $\delta \in \Omega(N, b_Q)$, then*

$$\mathcal{IT}(N, Q, Z, \alpha(\delta)) = \bigcup_{\{\omega\in\Omega(H,b_Q)\mid\omega\subseteq\delta\}} \{Ind_H^N(\mathcal{V}) \mid \mathcal{V} \in \mathcal{IT}(H, Q, Z, \alpha(\omega))\}$$

and the union is disjoint;

(c) *$Ind_H^N\colon \mathcal{IT}(H, Q, Z, b_Q) \to \mathcal{IT}(N, Q, Z, b_Q)$ is a bijection; and*

(d) *$\{\mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) \mid \mathcal{V} \in \mathcal{IT}(H, Q, Z, b_Q)\}$ is a set of representatives for the isomorphism classes of indecomposable $(\mathcal{O}G)B$-modules with vertex–source pair (Q, Z) and $\mathcal{G}r_N^G\colon \{Ind_H^N(\mathcal{V}) \mid \mathcal{V} \in \mathcal{IT}(H, Q, Z, b_Q)\} \to \{\mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) \mid \mathcal{V} \in \mathcal{IT}(H, Q, Z, b_Q)\}$ is a bijection.*

Our next result generalizes and extends [3, III, Theorem 7.8].

Let L be a subgroup of G such that $H = N_G(Q, Z) \leq L$. Let $E \in B\ell(\mathcal{O}L)$ and let W be an indecomposable $(\mathcal{O}L)E$-module with vertex–source pair (Q, Z). Thus $Pu_L^G(W)$ is an indecomposable $(\mathcal{O}G)$-module with vertex–source pair (Q, Z).

Corollary 1.7. *Assume that $BPu_L^G(W) = Pu_L^G(W)$. Then E^G is defined and $E^G = B$.*

Let $B \in B\ell(\mathcal{O}G)$.

Our next result greatly extends the well-known fact that if V is an indecomposable $(\mathcal{O}G)B$-module, then a vertex of V is contained in a defect group of B:

Corollary 1.8. *Let Q be any subgroup of a defect group of B and let Z be any indecomposable $\mathcal{O}Q$-module with vertex Q. Then there is an indecomposalbe $(\mathcal{O}G)B$-module with (Q, Z) as a vertex–source pair.*

Corollary 1.9. *If Q is any subgroup of a defect group of B, then there is an indecomposable $(\mathcal{O}G)B$-module with vertex Q and trivial source.*

Our final results concern the Heller Operators in this context.

Let W be a non-projective indecomposable $(\mathcal{O}G)B$-lattice with vertex–source pair (Q, Z). Thus $Q \neq 1$ and Z is a non-projective $\mathcal{O}Q$-lattice with vertex Q. Hence $\Omega(W)$ is a non-projective indecomposable $\mathcal{O}Q$-lattice and $\Omega(Z)$ is a non-projective indecomposable $\mathcal{O}Q$-lattice by [6, Proposition 6.7].

The proof of [5, Chapter 4, Theorem 4.10(i)], in which $\Omega(W) \mid Ind_Q^G(\Omega(Z))$ in $\mathcal{O}G$-mod and it is shown that Q is a vertex of $\Omega(X)$, applies in this situation to yield:

Lemma 1.10. $(Q, \Omega(Z))$ *is a vertex–source pair for* $\Omega(W)$.

Note that $N_G(Q, \Omega(Z)) = N_G(Q, Z)$; let $N_G(Q, Z) \leq L \leq G$ and let X be an indecomposable $\mathcal{O}L$-lattice with vertex–source pair (Q, Z) such that $Pu_L^G(X) \cong W$ in $(\mathcal{O}G)B$-mod. Thus $(Q, \Omega(Z))$ is a vertex–source pair for the indecomposable $\mathcal{O}L$-lattice $\Omega(X)$. Let Y be an indecomposable $\mathcal{O}L$-lattice with vertex–source pair $(Q, \Omega(Z))$ such that $Pu_L^G(Y) \simeq \Omega(W)$ in $(\mathcal{O}G)B$-mod.

Our final result records that the proof of [5, Chapter 4, Theorem 4.10(ii)] yields:

Lemma 1.11. $\Omega(X) \cong Y$ *in $\mathcal{O}L$-mod.*

2. Required Proofs

2.1. **A Proof of Proposition 1.2** (M. Linckelmann)**.** Assume the situation of Proposition 1.2. Let $\{x_i \mid 1 \leq i \leq n\}$ be a set of representatives for the orbits of Q acting on the right of X such that $x_1 = x$. Set

$$Y_i = \sum_{q \in Q} (\mathcal{O}(x_i q)) = x_i(\mathcal{O}Q)$$

for each $1 \leq i \leq n$. Thus

$$A = \bigoplus_{i=1}^{n} Y_i$$

in mod-$\mathcal{O}Q$. Since $qx_1 = x_1 q$ for all $q \in Q$,

$$Y_1 \bigotimes_{\mathcal{O}Q} V \text{ and } \bigoplus_{i=2}^{n} (Y_i \bigotimes_{\mathcal{O}Q} V)$$

are $\mathcal{O}Q$-submodules of

$$Res_Q^A(A \bigotimes_{\mathcal{O}Q} V)$$

and

$$Res_Q^A(A \bigotimes_{\mathcal{O}Q} V) = (Y_1 \bigotimes_{\mathcal{O}Q} V) \oplus (\bigoplus_{i=2}^{n}(Y_i \bigotimes_{\mathcal{O}Q} V))$$

in $\mathcal{O}Q$-mod. Since the $\mathcal{O}$-linear map $\alpha\colon \mathcal{O}Q \to Y_1$ such that $q \mapsto qx_1 = x_1q$ for all $q \in Q$ is an $\mathcal{O}Q$-mod-$\mathcal{O}Q$-isomorphism,

$$Y_1 \bigotimes_{\mathcal{O}Q} V \cong V$$

in $\mathcal{O}Q$-mod and we are done.

2.2. **A Proof of Lemma 1.3.** Assume the situation of Lemma 1.3. Let $R \leq Q$ be a vertex of W in $\mathcal{O}G$-mod. Thus [3, III, Lemma 4.1] implies that there is an $x \in G$ such that V is $\mathcal{O}(H \cap ({}^xR))$-projective. Thus $|Q| \leq |R|$, $R = Q$ and we are done.

2.3. **A Proof of Lemma 1.4.** Assume the situation of Lemma 1.4 and apply the proof of [1, Theorem 9.6]. Fix $j \in J$ and let T be a left transversal of H in G with $1 \in T$ so that

$$G = \bigcup_{t \in T} tH$$

is disjoint. Let $Ind_H^G(\mathcal{V}_j) = W \oplus X$ be a direct sum in $\mathcal{O}G$-mod where W is indecomposable and $\mathcal{V}_j \mid Res_H^G(W)$. Note that

$$\{t \bigotimes_{\mathcal{O}} V \mid t \in T\}$$

is a set of $|T|$ non-isomorphic indecomposable $\mathcal{O}M$-modules,

$$Res_M^G(Ind_H^G(\mathcal{V}_j)) \cong \bigoplus_{t \in T} n_j(t \bigotimes_{\mathcal{O}} V)$$

in $\mathcal{O}M$-mod and $tW \cong W$ in $\mathcal{O}G$-mod. Since $n_jV \cong Res_M^H(\mathcal{V}_j) \mid Res_M^G(W)$ in $\mathcal{O}M$-mod, we conclude that

$$(\bigoplus_{t \in T} n_j(t \bigotimes_{\mathcal{O}} V)) \mid Res_M^G(W)$$

in $\mathcal{O}M$-mod. Thus $X = (0)$, $Ind_H^G(\mathcal{V}_j)$ is indecomposable in $\mathcal{O}G$-mod and two applications of Lemma 1.3 complete a proof of (a).

The proof of [1, Theorem 9.6(c)] directly yields (b), (c) is [1, Theorem 9.6(b)] and we are done.

2.4. **A Proof of Proposition 1.5.** Here (a) is clear and (b) follows from [6, Corollary 20.9]. For (c), let $e \in B\ell(\mathcal{O}H)$ and $\mathcal{V} \in \mathcal{IT}(Q,H,Z,1)$ be such that $e\mathcal{V} = \mathcal{V}$. There is a unique $\omega \in \Omega(H)$ such that $e = \alpha(\omega)$.

If δ is the unique element of $\Omega(N)$ such that $\omega \subseteq \delta$, then $e^N = \alpha(\delta)$ is defined and $e^N Ind_H^N(\mathcal{V}) = Ind_H^N(\mathcal{V})$. Also e^G is defined and $(e^N)^G = e^G$ by [3, III, Lemma 9.2] and [3, III, Theorem 7.8] implies that $e^G\, \mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) = \mathcal{G}r_N^G(Ind_H^N(\mathcal{V}))$. This concludes our proof of Proposition 1.5.

2.5. **A Proof of Theorem 1.6.** Let $\omega \in \Omega(H, b_Q)$ so that $\alpha(\omega) \in B\ell((\mathcal{O}H)b_Q)$. Proposition 1.2, with $A = (\mathcal{O}H)\alpha(\omega)$ implies that

$$Z \mid Res_{\mathcal{O}Q}^{(\mathcal{O}H)\alpha(\omega)}((\mathcal{O}H)\alpha(\omega) \bigotimes_{\mathcal{O}Q} Z)$$

in $\mathcal{O}Q$-mod. Thus there is an indecomposable component V of

$$((\mathcal{O}H)\alpha(\omega) \bigotimes_{\mathcal{O}Q} Z)$$

in $(\mathcal{O}H)\alpha(\omega)$-mod such that $Z \mid Res_{\mathcal{O}Q}^{(\mathcal{O}H)\alpha(\omega)}(V)$. Now Lemma 1.3 implies that (Q,Z) is a vertex–source pair for V and (a) holds. Clearly if $\omega \subseteq \delta$, where $\delta \in \Omega(N, b_Q)$ and $\mathcal{V} \in \mathcal{IT}(H,Q,Z,\alpha(\omega))$, then $\alpha(\delta)Ind_H^N(\mathcal{V}) \neq 0$ and so $\alpha(\delta)Ind_H^N(\mathcal{V}) = Ind_H^N(\mathcal{V})$. Conversely, let $\delta \in \Omega(N, b_Q)$ and $\mathcal{V} \in \mathcal{IT}(H,Q,Z,1)$ be such that $\alpha(\delta)Ind_H^N(\mathcal{V}) = Ind_H^N(\mathcal{V})$. Thus $\alpha(\delta)\mathcal{V} = \mathcal{V}$ and so there is an $\omega \in \Omega(H, b_Q)$ with $\omega \subseteq \delta$ such that $\alpha(\omega)\mathcal{V} = \mathcal{V}$. Now proposition 1.5(a) completes a proof of (b) and (c).

Assume that $\mathcal{V} \in \mathcal{IT}(H,Q,Z,1)$ is such that

$$B\, \mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) = \mathcal{G}r_N^G(Ind_H^N(\mathcal{V})).$$

Then [3, III, Theorem 7.8] implies that $b_Q Ind_H^N(\mathcal{V}) = Ind_H^N(\mathcal{V})$. But then $b_Q\mathcal{V} = \mathcal{V}$ and $\mathcal{V} \in \mathcal{IT}(H,Q,Z,b_Q)$. Conversely let $\mathcal{V} \in \mathcal{IT}(H,Q,Z,b_Q)$ and let $\mathcal{L} \in B\ell(\mathcal{O}G)$ be such that $\mathcal{L}\, \mathcal{G}r_N^G(Ind_H^N(\mathcal{V})) = \mathcal{G}r_N^G(Ind_H^N(\mathcal{V}))$. Thus Q is contained in a defect group of $\mathcal{L}$ and if b_Q^* is the unique idempotent of $Z(\mathcal{O}C_G(Q))^N$ such that $Br_Q(\mathcal{L}) = \overline{b_Q^*}$, then $b_Q^* Ind_H^N(\mathcal{V}) = Ind_H^N(\mathcal{V})$ by [3, III, Theorem 7.8]. Since $b_Q Ind_H^N(\mathcal{V}) = Ind_H^N(\mathcal{V})$ by (b), $b_Q b_Q^* \neq 0$. Thus $\mathcal{L} = B$, (d) holds and we are done.

2.6. **A Proof of Corollary 1.7.** Let $\mathcal{V} \in \mathcal{IT}(H,Q,Z,1)$ be such that

$$Pu_H^L(\mathcal{V}) \cong W$$

in $\mathcal{O}L$-mod. Let b_Q^* be the idempotent of $Z(\mathcal{O}H)$ such that $Br_Q(E) = \overline{b_Q^*}$. By Theorem 1.6 there is an orbit ω^* of H on $B\ell(\mathcal{O}C_G(Q))b_Q^*$ such that $\alpha(\omega^*)\mathcal{V} = \mathcal{V}$. Note that $Pu_H^G(\mathcal{V}) \cong Pu_L^G(Pu_H^L(\mathcal{V})) \cong Pu_L^G(W)$ in $\mathcal{O}G$-mod. Thus there is an orbit ω of H on $B\ell(\mathcal{O}C_G(Q)b_Q)$ such that $\alpha(\omega)\mathcal{V} = \mathcal{V}$. Thus $\omega = \omega^*$ and $\alpha(\omega) = \alpha(\omega^*)$. Here $\alpha(\omega)^L = E$, $\alpha(\omega)^G = B$ and [3, III, Lemma 9.2] completes our proof.

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Received: October, 2010