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On the Cohomology with Inner Symmetry of A_∞-algebra

Y. Gh. Gouda

Dept. of Mathematics, Faculty of Science
Aswan and Quia South Valley University, Egypt
yasiengouda@yahoo.com

S. A. Omran

Department of Mathematics, Faculty of Science
TAIF University, Saudia Arabia
salehomran@yahoo.com

Abstract. In this paper we study the first cohomology with inner symmetry ,Hoschschild (simplicial), cohomology of Hoschschild complex of A_∞-algebra with some homotopical properties . We get the relation between the set of twisted cochain of the complex $\mathrm{C}_\infty(A, A)$ for this algebra and its cohomology. The extension of a fixed A_∞-algebra and its cohomology group are studied.

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1 Introduction

By the (co)homology with inner symmetry we means the following (co)homology groups: simplicial (Hochschild), cyclic, reflexive, dihedral, symmetry, bisymmetry and Weil cohomology groups [1]. The relationship between a set of all A_∞-algebra structures on fixed differential graded algebra and the Hochschild cohomology of that algebra has been studied in [6]. The Hochschild cohomology complex for A_∞ -modules over A_∞ -algebras has been studied in [4]. The triviality of Hochschild cohomology of dimension $(n, 2-n)$ -algebra is proved in[2]. In this article we are interested in the first cohomology with inner symmetry (Hochschild cohomology) of differential A_∞-algebra. For a given cochain Hochschild (simplicial) complex for A_∞- algebra with finite number of non trivial high multiplication π_i and differential A_∞ -algebra A, we show that the Hochschild cohomology is trivial.

2 Hochschild complex for differential algebra

In this part we recall the requisites definitions and results necessary for sequel. The main references are [4],[5],[7] and[8] .

Definition 1 *The graded algebra over field K is defined to be a graded module $A = \oplus A_i$ over K with the graded multiplication $\pi : A_i \otimes A_j \to A_{i+j}$ with the following associated condition $\pi(1 \otimes \pi) = \pi(\pi \otimes 1)$.*

Definition 2 *The differential graded algebra is a graded algebra A over field K with the differential $d : A_i \to A_{i-1}$ such that $d(\pi) = \pi(d \otimes 1) - \pi(1 \otimes d)$*

Definition 3 *Let A be algebra. The triple (A, d, π_i) is called A_∞-algebra , where (A, d) is graded module over field K, and $\{\pi_i\} : A^{\otimes i} \to A$, $i \geq 2$,such that for every i, we have:*

$\pi_i((A^{\otimes i}))_q \subset A_{q+i-2}, i \geq 2,$

$$\sum_{j=0}^{i}(-1)^{(j+1(k+1)+i}\pi_{i-j+1}(1\otimes ... \otimes\pi_j\otimes ... \otimes 1) = 0, \tag{1}$$

$(A^{\otimes i})_q = \oplus(A_{q1}\otimes A_{q2}\otimes ... \otimes A_{qi}), q = q_1+q_2+...+q_i$

Note that the summation in (1) is given by number k of all possible place of π_i .

Examples *of A_∞ -algebras [3]:*

- *Let X be a topological space, ΩX is its loop space. On the singular chain complex $C_*(\Omega X)$, there is an A_∞-algebras structure.*
- *If A is a differential graded algebra A over field, the homology $H_*(A)$ has a graded A_∞ -algebras structure.*

Definition 4 *The morphism between A_∞ -algebras $(A, \{\pi_i\})$ and $(A', \{\acute{\pi}_i\})$ is defined by the family of morphisms$\{f_i : A^{\otimes i} \to A'\}, i \geq 1$, such that: $f_i((A^{\otimes i}))_q \in A'_{q+i-1}$, $i \geq 1$,*

$$\begin{aligned}\sum_{j=0}(-1)^{\varepsilon} f_{i-j+1}(1 \otimes ... \otimes \pi_j \otimes ... \otimes 1) &= \sum_{k_1+...+k_\ell=i}(-1)^{\varepsilon'} \pi'_\ell\, (f_{k_1} \otimes f_{k_2} \otimes ... \otimes f_{k_\ell}),\\ \varepsilon &= (i-1)(j-1)(k+1),\ \ \varepsilon' = k_2 + k_4 + ...\end{aligned} \tag{2}$$

The summation in (2) are given in all possible place of k that must be .The forms (1) and (2) are called Stasheff's relation for A_∞-algebra [8].

Remark 1 *[5]*

i- For given differential algebras $(A,d),(B,d)$, the homotopy between differential maps$f,g:(A,d)\to(B,d)$ is a graded module $h:A_\bullet\to B_{\bullet+1}$ of degree 1, such that $dh+hd=f-g$.

ii- For a differential graded algebra A , consider the homology $H(A)$ of algebra A. If $H(A)$ is regarded as differential algebra, we can define the strong deformation retraction of A $(SDR-data)$by the relation $(\eta:A\rightleftarrows H(A):\xi,h)$, such that $h:A\to A$ is a differential homotopy, $dh+\eta h=\xi\eta-id$, $hd=0,h\xi=0,hh=0$.

*from [**?**] the Hochschild complex $C*(A,A)$ for algebras A is a A-module over field with the multiplication $\pi:A\otimes A\to A$ such that the associate law holds $(\pi\otimes 1)\pi=((1\otimes\pi)\pi..$*

The cochain Hochschild complex of algebra A over field is given by $(C^*(A,A),\delta)$ such that $C^*(A,A)=\sum C^n(A^n,A)$, $C^n(A,A)=Hom\ ((A^n,A))$ and $\delta: C^n(A,A)\to C^{n+1}(A,A)$. The relation between operators δ and π is given by:

$$\delta f=\pi(1\otimes f)+\sum(-1)^{n+i}f(1\otimes...\otimes..\otimes 1)+(-1)^{n+1}\pi(f\otimes 1)$$

where the summation is going on i where in the place of π .

The homology of $(C^*(A,A),\delta)$ is Hochschild cohomology and defined by $H^*(A,A)$.

In what follows, we consider that all algebras and over field Z_2 .

Definition 5 *For a differential A_∞-algebra A we can define the coalgebra BA which is called B-construction overA. Consider the tensor algebra $TA=\sum_{n\geq 1}A^{\otimes n}$ such that $\deg(a_1\otimes...\otimes a_k)=\deg(a_1)+...+\deg(a_k)+k$.*

The tensor algebra TX with the following differential $d:BA_i\to A_{i-1}$, such that $d((a_1\otimes...\otimes a_n)=\sum_{j,k}a_1\otimes...\otimes\pi_k((a_j\otimes...\otimes a_{j+k}\otimes...\otimes a_n)$

is called B-construction over A and denoted by BA .

We consider the differential A_∞-algebra A with finite integer nontrivial exterior multiplication π_i , then there is A_∞-algebra such that for $n\in N,\pi_i=0$,for $i>n$.

Consider $Hom(BA,A)$, then $Hom^n(BA,A)=\{f:BA_i\to A_{i+n}\}$.

Note that if $f\in Hom^n(BA,A)$, then there is $\{f_i\}$ such that $f_i:(A^{\otimes i})_q\to A_{q+i+n}\}$, .The identity map $Id_1=id$, $Id_k=0$ for $k>1$.

Define the differential $\delta:Hom^n(BA,A)\to Hom^{n+1}(BA,A)$ such that:

$\delta f=\sum_i f(1\otimes...\otimes 1\otimes\pi_i\otimes 1\otimes...\otimes 1)+\sum_i\pi_i(1\otimes...\otimes f\otimes...\otimes 1)$ (3)

The complex $Hom(BA,A)$ with differential δ (defined in relation (3)) is called the Hochschild complex for A_∞-algebra and denoted by $C_\infty(A,A)$.

Definition 6 : Homology of Hochschild complex is called Hochschild homology for A_∞-algebra and denoted by $C_\infty(A, A)$.

Note that [2]:

- *In $C_\infty^{-2}(A, A)$ there is an element π such that:*

$$\pi_i \neq 0 \text{ for } 2 < i < n+1, \pi_i = 0 \text{ for } i > n \tag{4}$$

- *An element π is co cyclic, that is $\delta\pi = 0$.*

Consider the following operations in Hochschild complex $C(A, A)$ from [2] : $f \cup g = \pi(f \otimes g)$, $f \cup_1 g = \sum_k f(1 \otimes ... \otimes 1 \otimes g \otimes ... \otimes 1)$,

where $f \in C^m(A, A)$, $g \in C^n(A, A)$.

We can rewrite the operations $\cup$ and $\cup_1$ on the Hochschild complex $C_\infty(A, A)$ as follows :

$$\begin{aligned} \cup &: (C_\infty(A, M) \otimes C_\infty(A, A))^i \to (C_\infty(A, A))^i, \\ f \cup g &= f(1 \otimes ... \otimes 1 \otimes g), \\ \cup_1 &: (C_\infty(A, A) \otimes C_\infty(A, A))^i \to C_\infty^{i+1}(A, A), \\ f \cup_1 g &= \sum f(1 \otimes ... \otimes 1 \otimes g \otimes ... \otimes 1). \end{aligned} \tag{5}$$

where $f, g \in_\infty C_\infty(A, A)$.

For some $(g_1, g_2, ..., g_k)$ we can generalize the operation $\cup_1$ in relation (5) to be:

$$\begin{aligned} \cup_i^k &: (C_\infty(A, A)^{\otimes k+1})^i \to C_\infty^{i+k}(A, A) \\ f \cup_1^k (g_1, g_2, ..., g_k) &= \sum f(1 \otimes ... \otimes 1 \otimes g_1 \otimes 1 \otimes ... \otimes 1 \otimes \\ &\otimes g_2 \otimes 1 \otimes ... \otimes 1 \otimes g_k \otimes 1 \otimes ... \otimes 1), \end{aligned} \tag{6}$$

where $f, g \in C_\infty(A, A)$ and the summation will be in all place of elements $g_1, g_2, ..., g_k$.

The relation between operators $\cup_1, \delta$ and π is given by: $\delta f = f \cup_1 \pi + \pi \cup_1 f$.

But the relation between operators $\cup_1^k, \delta$ and π is given the following assertion:

Proposition 1 *The following relation holds*

$$\begin{aligned} \delta(f \cup_1^k (g_1, g_2, ..., g_k)) &= \delta f \cup_1^k (g_1, g_2, ..., g_k) + \sum_{i=1}^{k} f \cup_1^k (g_1, g_2, ..., g_k) + \\ &+ \sum_{i=2}^{k} \sum_{s=1}^{k-2} f \cup_1^{k-i+1} (g_1, ..., \pi \cup_1^i (g_s, ..., g_{s+i}), ..., g_k) + \\ &+ \sum_{i=0}^{k-1} \sum_{s=1}^{k-2} \pi \cup_1^{k-i+1} (g_1, ..., f \cup_1^i (g_s, ..., g_{s+i}), ..., g_k) \end{aligned} \tag{7}$$

Proof: Directly can be got from relation (3), (4), (5), (6).

Proposition 2 *The relation (7), when $k = 1$, can be written in the form:$\delta(f \cup_1 g) = (\delta f) \cup_1 g + f \cup_1 (\delta g) + f \cup g + g \cup f$, since $\cup_1^1 = \cup$,and $\pi \cup_1^2 (f, g) = f \cup g$ in (7) and $f \cup_1^0 = f$ in (5).*

3 Twisted cochain Hochschild complex for A_∞-algebra and related cohomology.

Recall the definition of twisted cochain on Hochschild complex from [2]. The twisted cochain is an element $a = a^{3,-1} + a^{4,-2} + ... + a^{i,2-i} + ...$ where $a^{i,2-i} \in C^{i,2-i}(A, A)$, such that $\delta a = a \cup_1 a$, $\cup_1$ is defined above. The set of twisted cochains is denoted by $TW(A, A)$.

Definition 6 *Two twisted cochain a and $\acute{a}$ are equivalent if there exist an element $p = p^{2,-1} + p^{3,-2} + ... + p^{i,1-i}$, $p^{i,1-i} \in C^{i,1-i}(A, A)$ such that*

$$a - \acute{a} = \delta p + p \cup_1 a + \acute{a} \cup_1 (p \otimes p) + \acute{a} \cup_1 (p \otimes p \otimes p) + ...$$

The set $TW(A, A)/ \sim$, where $\sim$ is equivalent relation will denoted by $D(A, A)$.

In [2] is proved that if $Hoch^{n.n-2}(A, A) = 0$ for $n > 2$, then $D(A, A) = 0$.

We define a new concept of twisted cochain on Hochschild complex $C_\infty(A, A)$ for A_∞-algebra.

Definition 7 *Any element $g \in C_\infty^{-2}(A, A)$ is called twisted cochain if the following hold:*

1. $g_i = 0, if \ i < n + 1$
2. $\delta g = g \cup_1 g$ (8)

The set of all twisted cochain in Hochschild complex is denoted by $TW(C_\infty(A, A))$.

Definition 8 *.Two twisted cochains g, $\acute{g}$ are equivalent and denoted by $g \sim \acute{g}$ if there is $f \in C_\infty^{-1}(A, A)$, such that:*

1. $f_1 = id$,
2. $\delta f + \sum\limits_{i=2} \pi \cup_1^i (f, ...f) + f \cup_1 \acute{g} + \sum\limits_{i=1} g \cup_1^i (f, ...f) = 0$ (9)

Where $\cup_1$ and $\cup_1^i$ are defined by formula (5), (6).

Suppose $D(A, A) = TW(C_\infty(A, A) \ / \sim$, that where $\sim$ is an equivalent relation, then the following holds.

Theorem 3 *Let $g \in TW(C_\infty(A,A))$ be an arbitrary twisted cochain and $f \in C_\infty^{-1}(A,A)$, such that $f_1 = id$, $f_i = 0$ for $i > n+1$, then there exist twisted cochain $\bar{g}$ such that:*

1. $g_i = \bar{g}_i$, $i < k+1, k > n$,
2. $\bar{g}_{k+1=} g_{k+1} + (\delta f)_{k+1}$,
3. $\bar{g} \sim g$

Proof. We use the method of constructing element $\bar{g} \sim g$. Note that , to use the condition of the theorem 1 we have the relation $(\delta f)_{n+1} = \delta(f_n)$. For every element in definition 9, which make the equivalent relation $\bar{g} \sim g$, we consider it as an element satisfies condition of theorem 1. Define $\bar{g}_i$, $i < k+1$ from condition 1 of theorem 1. For elements g and $\bar{g}$, the first nontrivial elements in right hand side of relation (9) is given in $(k+1)$-dimension, such that $\delta f + f_1(g) + f_1(\bar{g}) = 0$, this relation is true if $f_1 = id$ (all remain $f_i = 0$ for $1 < i < k+1$).

The following theorem relate the cohomology of Hochschild for algebra and set of twisted cochain $D(A,A)$ of this complex.

Theorem 4 *If $H^{-2}(C_\infty(A,A)) = 0$, then $D(A,A) = 0$.*

Proof: *we must prove that the arbitrary twisted cochain, given condition, is equal zero. The formula (8), for element g, in $(n+1)$-dimension has the form $\delta g = 0$, that is g is acyclic. By considering the condition $H^{-2}(C_\infty(A,A)) = 0$ there exist f^{-1} such that $g_{n+1} = (\delta f^{-1})_{n+1}$ or $g - 0 = \delta f$. Following theorem 1 we can get a twisted cochain g_1 such that $g^1_{n+1} = 0$ and $g \sim g^1$. Hence the formula (8) in $(n+2)$-dimension, for element g^1 is given by $\delta g^1 = 0$, that is g^1 is acyclic. since $H^{-2}(C_\infty(A,A)) = 0$, then there is f^2 such that $g_{n+1} = (\delta f^1)_{n+1}$ or $g^1 - 0 = \delta f^2$ and so on. Repeating this process we get a sequence of twisted cochain such that $g^i_{n+k} = 0$, $k < i+1$. The extension of this process to infinity get trivial twisted cochain with the element f with components $f_1 = 0$ and $f_i = 0$ for $i < n+1$, $f_i = f_i^{n-i}$, $i > n$.*

In the end of this part we consider a commutative A_∞-algebra and related theorems.

Definition 9 *An A_∞- algebra is commutative , if the following condition is satisfied (add to conditions in definition 3) $\sum\limits_{\sigma} (-1)^\varepsilon \pi_i \sigma(i, n-i) = 0$, where the summation is given on all $(i, n-i)$- permutations σ and ε is permutation's sign.*

Definition 10 : *If A is commutative, then its Hochschild complex is called Harssan complex $(C^*(A,A), \delta)$.*

Definition 11 *The cohomology of Harssan complex $(C^*(A,A), \delta)$ is called Harssan cohomology of A_∞- algebra and denoted by $Harr(C^*(A,A), \delta)$.*

The following result is the same in theorem 12 for commutative A_∞- algebra.

Theorem 5 *If $Harr^{n,2-n}(C^*(A,A)) = 0$, then $D(A,A) = 0$.*

4 Extension of A_∞-algebra and cohomology of Hochschild of $C_\infty(A, A)$ for A_∞-algebra

Definition 12 *Let A be A_∞-algebra (A, π_i) with nontrivial finite number of the multiplication π_i i.e. $(\pi_i \neq 0,\ 0 \leq i \leq n,\ \pi_i = 0,\ i > n$.*

The extension of A_∞-algebra is an A_∞-algebra $\bar{A}$ such that A and $\bar{A}$ coincided and the high multiplication $\bar{\pi}_i = \pi_i$ for $i < n+1$.

In [2] is proved that there is a bijection between the set of structure A_∞-algebra on fixed graded algebra, such that $\pi_1 = 0$, $\pi_2 = \pi$, π is multiplication in algebra, denoted by $(A,\pi)(\infty)$ and the set of twisted cochains Hochschild complex factored by the equivalent relation $\sim$.

Here we give an extension of this fact between the set of all extension of a fixed A_∞-algebra , denoted by $(A,\pi_i)(\infty)$, where π_i is the structure on a fixed A_∞-algebra, and the set of twisted cochains Hochschild complex factored by the equivalent relation $\sim$ is $(D(A,A))$.

The following theorem is the main result in this part.

Theorem 6 *There is a bijection between sets $(A,\pi_i)(\infty)$ and $(D(A,A))$.*

We must show that there is a bijection between twisted cochain of Hochschild complex $C*(A,A)$ and A- structure. That is there is two equivalent cochian , under this bijection, send to one equivalent class set of $(A,\pi)(\infty)$.

Proof: In A_∞-algebra $A \in (A,\pi_i)(\infty)$, beginning of $n+1$ dimension, we can divide the stasheff relation (1) as follow:

$$\sum_{i,\ j=1}^{n} \pi_i(1\otimes ...\otimes 1 \otimes \pi_j \otimes 1 \otimes ... \otimes 1) + \sum_{i=1,\ j=n+1}^{n} \pi_i(1\otimes ...\otimes 1 \otimes \pi_j \otimes 1 \otimes ... \otimes 1) + \sum_{i=n+1,\ j=1}^{n} \pi_i(1 \otimes ... \otimes 1 \otimes \pi_j \otimes 1 \otimes ... \otimes 1) + \sum_{i,\ j=n+1}^{n} \pi_i(1 \otimes ... \otimes 1 \otimes \pi_j \otimes 1 \otimes ... \otimes 1) = 0 \quad (10)$$

Clearly that the first term of (10) is equal zero, following stasheff relation for fixed algebra A. The second and third terms of (10) , following (5), (6) can be written in the form δg . The fourth term in form $g \cup_1 g$ where $g_i = 0$, $i < n+1$ and $g_i = \pi_i$, $i > n$.

Therefore the stasheff's relation (1) can be written in the form $\delta g + g \cup_1 g$, and hence g is twisted cochain. This means that every A_∞-structure from

$(A, \pi_i)(\infty)$ defines a twisted cochain from Hochscihld complex $C_\infty(A, A)$. The inverse is every twisted cochain defines A_∞- structure.

From theorem 18 and definition 10 we get the following assertion

Theorem 7 *If $H^{-2}(C_\infty(A, A)) = 0$, then any structure of extension of a fixed A_∞-algebra is trivial.*

Proof: *If $H^{-2}(C_\infty(A, A)) = 0$, then the factor set belong to zero equivalent class, that is all twisted cochains in in Hochschild complex for A_∞-algebra equivalent zero. Considering theorem 18 the theorem is proved.*

Theorem 8 *There is a bijection between set of class of equivalent twisted cochain in complex Harrsan and the set of commutative A_∞-algebra structure $((A, \pi_i)(\infty))$.*

Proof: directly from theorems 18 and 19.

Theorem 9 *If for graded commutative algebra A with multiplication π , the $Harr^{n,2-n}(C_\infty(A, A)) = 0, n \geq 3$, then any A_∞-algebra structure on A,such that $(\pi_1 = 0, \pi_2 = \pi)$ is trivial.*

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