On the Cohomology with Inner Symmetry
of $A_{\infty}$-algebra

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Abstract. In this paper we study the first cohomology with inner symmetry, Hochschild (simplicial), cohomology of Hochschild complex of $A_{\infty}$-algebra with some homotopical properties. We get the relation between the set of twisted cochain of the complex $C_{\infty}(A, A)$ for this algebra and its cohomology. The extension of a fixed $A_{\infty}$-algebra and its cohomology group are studied.

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1 Introduction

By the (co)homology with inner symmetry we means the following (co)homology groups: simplicial (Hochschild), cyclic, reflexive, dihedral, symmetry, bisymmetry and Weil cohomology groups [1]. The relationship between a set of all $A_{\infty}$-algebra structures on fixed differential graded algebra and the Hochschild cohomology of that algebra has been studied in [6]. The Hochschild cohomology complex for $A_{\infty}$-modules over $A_{\infty}$-algebras has been studied in [4]. The triviality of Hochschild cohomology of dimension $(n, 2 - n)$-algebra is proved in [2]. In this article we are interested in the first cohomology with inner symmetry (Hochschild cohomology) of differential $A_{\infty}$-algebra. For a given
cochain Hochschild (simplicial) complex for $A_\infty$-algebra with finite number of non trivial high multiplication $\pi_i$ and differential $A_\infty$-algebra $A$, we show that the Hochschild cohomology is trivial.

2 Hochschild complex for differential algebra

In this part we recall the requisites definitions and results necessary for sequel. The main references are [4],[5],[7] and[8].

Definition 1 The graded algebra over field $K$ is defined to be a graded module $A = \oplus A_i$ over $K$ with the graded multiplication $\pi : A_i \otimes A_j \rightarrow A_{i+j}$ with the following associated condition $\pi(1 \otimes \pi) = \pi(\pi \otimes 1)$.

Definition 2 The differential graded algebra is a graded algebra $A$ over field $K$ with the differential $d : A_i \rightarrow A_{i-1}$ such that $d(\pi) = \pi(d \otimes 1) - \pi(1 \otimes d)$

Definition 3 Let $A$ be algebra. The triple $(A, d, \pi_i)$ is called $A_\infty$-algebra , where $(A, d)$ is graded module over field $K$, and $\{\pi_i\} : A^{\otimes i} \rightarrow A, i \geq 2$, such that for every $i$, we have:

$$\pi_i((A^{\otimes i})_q \subset A_{q+i-2}, i \geq 2,$$

$$\sum_{j=0}^{i} (-1)^{(j+1)(k+1)} \pi_{i-j+1}(1 \otimes \ldots \otimes \pi_j \otimes \ldots \otimes 1) = 0, \quad (1)$$

$$(A^{\otimes i})_q = \oplus (A_{q_1} \otimes A_{q_2} \ldots \otimes A_{q_i}), q = q_1 + q_2 + \ldots + q_i$$

Note that the summation in (1) is given by number $k$ of all possible place of $\pi_i$.

Examples of $A_\infty$-algebras [3]:

- Let $X$ be a topological space, $\Omega X$ is its loop space. On the singular chain complex $C_*(\Omega X)$, there is an $A_\infty$-algebras structure.
- If $A$ is a differential graded algebra $A$ over field, the homology $H_*(A)$ has a graded $A_\infty$-algebras structure.

Definition 4 The morphism between $A_\infty$-algebras $(A, \{\pi_i\})$ and $(A', \{\pi_i\})$ is defined by the family of morphisms $f_i : A^{\otimes i} \rightarrow A'$, $i \geq 1$, such that: $f_i((A^{\otimes i})_q \subset A'_{q+i-1}, i \geq 1,$

$$\sum_{j=0}^{i} (-1)\varepsilon f_{i-j+1}(1 \otimes \ldots \otimes \pi_j \otimes \ldots \otimes 1) = \sum_{k_1 + \ldots + k_\ell = i} (-1)^{\varepsilon'} \pi_\ell (f_{k_1} \otimes f_{k_2} \otimes \ldots \otimes f_{k_\ell}),$$

$$\varepsilon = (i-1)(j-1)(k+1), \quad \varepsilon' = k_2 + k_4 + \ldots \quad (2)$$

The summation in (2) are given in all possible place of $k$ that must be. The forms (1) and (2) are called Stasheff’s relation for $A_\infty$-algebra [8].
Remark 1 [5]

i- For given differential algebras \((A, d), (B, d)\), the homotopy between differential maps \(f, g : (A, d) \to (B, d)\) is a graded module \(h : A \to B_{*-1}\) of degree 1, such that \(dh + hd = f - g\).

ii- For a differential graded algebra \(A\), consider the homology \(H(A)\) of algebra \(A\). If \(H(A)\) is regarded as differential algebra, we can define the strong deformation retraction of \(A\) (SDR-data) by the relation (\(\eta : A \leftrightarrow H(A) : \xi, h\)), such that \(h : A \to A\) is a differential homotopy, \(dh + \eta h = \eta \xi - id\), \(hd = 0, h\xi = 0, hh = 0\).

From [?] the Hochschild complex \(C \ast (A, A)\) for algebras \(A\) is an A-module over field with the multiplication \(\pi : A \otimes A \to A\) such that the associate law holds \((\pi \otimes 1)\pi = ((1 \otimes \pi)\pi)\).

The cochain Hochschild complex of algebra \(A\) over field is given by \((C^\ast (A, A), \delta)\) such that \(C^\ast (A, A) = \sum C^n (A^n, A), C^n (A, A) = \text{Hom} ((A^n, A))\) and \(\delta : C^n (A, A) \to C^{n+1} (A, A)\). The relation between operators \(\delta\) and \(\pi\) is given by:

\[
\delta f = \pi (1 \otimes f) + \sum (-1)^{n+1} f (1 \otimes \ldots \otimes 1) + (-1)^{n+1} \pi (f \otimes 1)
\]

where the summation is going on \(i\) where in the place of \(\pi\).

The homology of \((C^\ast (A, A), \delta)\) is Hochschild cohomology and defined by \(H^\ast (A, A)\).

In what follows, we consider that all algebras and over field \(Z_2\).

Definition 5 For a differential \(A_\infty\)-algebra \(A\) we can define the coalgebra \(BA\) which is called B-construction over \(A\). Consider the tensor algebra \(TA = \sum_{n \geq 1} A^{\otimes n}\) such that \(\text{deg}(a_1 \otimes \ldots \otimes a_k) = \text{deg}(a_1) + \ldots + \text{deg}(a_k) + k\).

The tensor algebra \(TX\) with the following differential \(d : BA_i \to A_{i-1}\) such that \(d((a_1 \otimes \ldots \otimes a_n)) = \sum_{j,k} a_1 \otimes \ldots \otimes \pi_k (a_j \otimes \ldots \otimes a_{j+k} \otimes \ldots \otimes a_n)\)

is called B-construction over \(A\) and denoted by \(BA\).

We consider the differential \(A_\infty\)-algebra \(A\) with finite integer nontrivial exterior multiplication \(\pi_i\), then there is \(A_\infty\)-algebra such that for \(n \in N, \pi_i = 0\) for \(i > n\).

Consider \(\text{Hom}(BA, A)\), then \(\text{Hom}^n (BA, A) = \{ f : BA_i \to A_{i+n}\}\).

Note that if \(f \in \text{Hom}^n (BA, A)\), then there is \(\{ f_i \}\) such that \(f_i : (A^{\otimes n})_q \to A_{q+i+n}\), the identity map \(Id_1 = id, Id_k = 0\) for \(k > 1\).

Define the differential \(\delta : \text{Hom}^n (BA, A) \to \text{Hom}^{n+1} (BA, A)\) such that:

\[
\delta f = \sum f (1 \otimes \ldots \otimes 1 \otimes \pi_i \otimes 1 \otimes \ldots \otimes 1) + \sum \pi_i (1 \otimes \ldots \otimes f \otimes \ldots \otimes 1)
\]

\((3)\)

The complex \(\text{Hom}(BA, A)\) with differential \(\delta\) (defined in relation \((3)\)) is called the Hochschild complex for \(A_\infty\)-algebra and denoted by \(C_\infty (A, A)\).
Definition 6: Homology of Hochschild complex is called Hochschild homology for $A_\infty$-algebra and denoted by $C_\infty(A, A)$.

Note that [2]:

- In $C_{\infty}^{-2}(A, A)$ there is an element $\pi$ such that:
  \[ \pi_i \neq 0 \text{ for } 2 < i < n + 1, \pi_i = 0 \text{ for } i > n \]  
  (4)

- An element $\pi$ is co cyclic, that is $\delta \pi = 0$.

Consider the following operations in Hochschild complex $C(A, A)$ from [2]:

\[ f \cup g = \pi(f \otimes g), \quad f \cup_1 g = \sum f(1 \otimes \cdots \otimes 1 \otimes g \otimes \cdots \otimes 1), \]

where $f \in C^m(A, A), g \in C^m(A, A)$.

We can rewrite the operations $\cup$ and $\cup_1$ on the Hochschild complex $C_\infty(A, A)$ as follows:

\[ \cup : (C_\infty(A, M) \otimes C_\infty(A, A))^i \rightarrow (C_\infty(A, A))^i, \]

\[ f \cup g = f(1 \otimes \cdots \otimes 1 \otimes g), \]

\[ \cup_1 : (C_\infty(A, A) \otimes C_\infty(A, A))^i \rightarrow C_\infty^{i+1}(A, A), \]

\[ f \cup_1 g = \sum f(1 \otimes \cdots \otimes 1 \otimes g \otimes \cdots \otimes 1). \]  
  (5)

where $f, g \in C_\infty(A, A)$.

For some $(g_1, g_2, \ldots, g_k)$ we can generalize the operation $\cup_1$ in relation (5) to be:

\[ \cup_1^k : (C_\infty(A, A))^\otimes k+1 \rightarrow C_\infty^{i+k}(A, A) \]

\[ f \cup_1^k (g_1, g_2, \ldots, g_k) = \sum f(1 \otimes \cdots \otimes 1 \otimes g_1 \otimes \cdots \otimes 1 \otimes \]

\[ \otimes g_2 \otimes \cdots \otimes 1 \otimes \cdots \otimes 1 \otimes 1), \]  
  (6)

where $f, g \in C_\infty(A, A)$ and the summation will be in all place of elements $g_1, g_2, \ldots, g_k$.

The relation between operators $\cup_1, \delta$ and $\pi$ is given by:

\[ \delta f = f \cup_1 \pi + f \cup_1 \pi \]

But the relation between operators $\cup_1^k, \delta$ and $\pi$ is given the following assertion:

**Proposition 1** The following relation holds

\[ \delta(f \cup_1^k (g_1, g_2, \ldots, g_k)) = \delta f \cup_1^k (g_1, g_2, \ldots, g_k) + \sum_{i=1}^k f \cup_1^k (g_1, g_2, \ldots, g_k) + \]

\[ + \sum_{i=2}^k \sum_{s=1}^{k-2} f \cup_1^{k-i+1} (g_1, \ldots, \pi \cup_1^i (g_s, \ldots, g_{s+i}), \ldots, g_k) + \]

\[ + \sum_{i=0}^{k-1} \sum_{s=1}^{k-2} \pi \cup_1^{k-i+1} (g_1, \ldots, f \cup_1^i (g_s, \ldots, g_{s+i}), \ldots, g_k) \]  
  (7)
Proof: Directly can be got from relation (3), (4), (5), (6).

Proposition 2 The relation (7), when \( k = 1 \), can be written in the form: \( \delta(f \cup_1 g) = (\delta f) \cup_1 g + f \cup_1 (\delta g) + f \cup g + g \cup f \), since \( \cup_1^1 = \cup \), and \( \pi \cup_1^2 (f, g) = f \cup g \) in (7) and \( f \cup_1^0 = f \) in (5).

3 Twisted cochain Hochschild complex for \( A_\infty \)-algebra and related cohomology.

Recall the definition of twisted cochain on Hochschild complex from [2]. The twisted cochain is an element \( a = a_{3,-1} + a_{4,-2} + \ldots + a^i_{1,-i} + \ldots \), where \( a^i_{1,-i} \in C^i_{1,-i}(A, A) \), such that \( \delta a = a \cup_1 a \), \( \cup_1 \) is defined above. The set of twisted cochains is denoted by \( TW(A, A) \).

Definition 6 Two twisted cochain \( a \) and \( \dot{a} \) are equivalent if there exist an element \( p = p_{2,-1} + p_{3,-2} + \ldots + p^i_{1,-i} = p_{1,-i} \in C^i_{1,-i}(A, A) \) such that

\[
\delta a - \delta \dot{a} = \delta p + p \cup_1 a + \dot{a} \quad \cup_1 (p \otimes p) + \dot{a} \quad \cup_1 (p \otimes p \otimes p) + \ldots
\]

The set \( TW(A, A)/\sim \), where \( \sim \) is equivalent relation will denoted by \( D(A, A) \).

In [2] is proved that if \( Hoch^{n,n-2}(A, A) = 0 \) for \( n > 2 \), then \( D(A, A) = 0 \).

We define a new concept of twisted cochain on Hochschild complex \( C_\infty(A, A) \) for \( A_\infty \)-algebra.

Definition 7 Any element \( g \in C^{-2}_\infty(A, A) \) is called twisted cochain if the following hold:

1. \( g_i = 0 \), if \( i < n + 1 \)
2. \( \delta g = g \cup_1 g \) \hspace{1cm} (8)

The set of all twisted cochain in Hochschild complex is denoted by \( TW(C_\infty(A, A)) \).

Definition 8 Two twisted cochains \( g, \dot{g} \) are equivalent and denoted by \( g \sim \dot{g} \) if there is \( f \in C^{-1}_\infty(A, A) \), such that:

1. \( f_1 = id \),
2. \( \delta f + \sum_{i=2} \pi \cup_1^i (f, \ldots f) + f \cup_1 \dot{g} + \sum_{i=1} g \cup_1^i (f, \ldots f) = 0 \) \hspace{1cm} (9)

Where \( \cup_1 \) and \( \cup_1^i \) are defined by formula (5), (6).

Suppose \( D(A, A) = TW(C_\infty(A, A)) / \sim \), that where \( \sim \) is an equivalent relation, then the following holds.
Theorem 3 Let \( g \in TW(C_{\infty}(A, A)) \) be an arbitrary twisted cochain and \( f \in C_{\infty}^{-1}(A, A) \), such that \( f_1 = id \), \( f_i = 0 \) for \( i > n + 1 \), then there exist twisted cochain \( \bar{g} \) such that:

1. \( g_i = \bar{g}_i \), \( i < k + 1 \), \( k > n \),
2. \( \bar{g}_{k+1} = g_{k+1} + (\delta f)_{k+1} \),
3. \( \bar{g} \sim g \)

Proof. We use the method of constructing element \( \bar{g} \sim g \). Note that, to use the condition of the theorem 1 we have the relation \( (\delta f)_{n+1} = \delta(f_n) \). For every element in definition 9, which make the equivalent relation \( \bar{g} \sim g \), we consider it as an element satisfies condition of theorem 1. Define \( \bar{g}_i \), \( i < k + 1 \) from condition 1 of theorem 1. For elements \( g \) and \( \bar{g} \), the first nontrivial elements in right hand side of relation (9) is given in \( (k + 1) \)-dimension, such that \( \delta f + f_1(g) + f_1(\bar{g}) = 0 \), this relation is true if \( f_1 = id \) ( all remain \( f_i = 0 \) for \( 1 < i < k + 1 \) ).

The following theorem relate the cohomology of Hochschild for algebra and set of twisted cochain \( D(A, A) \) of this complex.

Theorem 4 If \( H^{-2}(C_{\infty}(A, A)) = 0 \), then \( D(A, A) = 0 \).

Proof: we must prove that the arbitrary twisted cochain, given condition, is equal zero. The formula (8), for element \( g \), in \( (n+1) \)-dimension has the form \( \delta g = 0 \), that is \( g \) is acyclic. By considering the condition \( H^{-2}(C_{\infty}(A, A)) = 0 \) there exist \( f^{-1} \) such that \( g_{n+1} = (\delta f^{-1})_{n+1} \) or \( g - 0 = \delta f \). Following theorem 1 we can get a twisted cochain \( g_1 \) such that \( g^1_{n+1} = 0 \) and \( g \sim g^1 \). Hence the formula (8) in \( (n+2) \)-dimension, for element \( g^1 \) is given by \( \delta g^1 = 0 \), that is \( g^1 \) is acyclic. since \( H^{-2}(C_{\infty}(A, A)) = 0 \), then there is \( f^2 \) such that \( g_{n+1} = (\delta f^1)_{n+1} \) or \( g^1 - 0 = \delta f^2 \) and so on. Repeating this process we get a sequence of twisted cochain such that \( g^i_{n+k} = 0 \), \( k < i + 1 \). The extension of this process to infinity get trivial twisted cochain with the element \( f \) with components \( f_i = 0 \) and \( f_i = 0 \) for \( i < n + 1 \), \( f_i = f_i^{n+1} \), \( i > n \).

In the end of this part we consider a commutative \( A_{\infty} \)-algebra and related theorems.

Definition 9 An \( A_{\infty} \)- algebra is commutative , if the following condition is satisfied ( add to conditions in definition 3 ) \( \sum_{\sigma} (-1)^{\varepsilon} \pi_\sigma(i, n-i) = 0 \), where the summation is given on all \( (i, n-i) \)- permutations \( \sigma \) and \( \varepsilon \) is permutation’s sign.

Definition 10 : If \( A \) is commutative, then its Hochschild complex is called Harssan complex \( (C^\ast(A, A), \delta) \).

Definition 11 The cohomology of Harssan complex \( (C^\ast(A, A), \delta) \) is called Harssan cohomology of \( A_{\infty} \)- algebra and denoted by \( Harr(C^\ast(A, A), \delta) \).
The following result is the same in theorem 12 for commutative $A_\infty$-algebra.

**Theorem 5** If $\text{Harr}^{n,2-n}(C^*(A, A)) = 0$, then $D(A, A) = 0$.

4 Extension of $A_\infty$-algebra and cohomology of Hochschild of $C_\infty(A, A)$ for $A_\infty$-algebra

**Definition 12** Let $A$ be $A_\infty$-algebra $(A, \pi_i)$ with nontrivial finite number of the multiplication $\pi_i$ i.e. $(\pi_i \neq 0, 0 \leq i \leq n, \pi_i = 0, i > n$.

The extension of $A_\infty$-algebra is an $A_\infty$-algebra $\bar{A}$ such that $A$ and $\bar{A}$ coincided and the high multiplication $\bar{\pi}_i = \pi_i$ for $i < n + 1$.

In [2] is proved that there is a bijection between the set of structure $A_\infty$-algebra on fixed graded algebra, such that $\pi_1 = 0, \pi_2 = \pi, \pi$ is multiplication in algebra, denoted by $(A, \pi)(\infty)$ and the set of twisted cochains Hochschild complex factored by the equivalent relation $\sim$.

Here we give an extension of this fact between the set of all extension of a fixed $A_\infty$-algebra, denoted by $(A, \pi_i)(\infty)$, where $\pi_i$ is the structure on a fixed $A_\infty$-algebra, and the set of twisted cochains Hochschild complex factored by the equivalent relation $\sim$ is $(D(A, A))$.

The following theorem is the main result in this part.

**Theorem 6** There is a bijection between sets $(A, \pi_i)(\infty)$ and $(D(A, A))$.

We must show that there is a bijection between twisted cochain of Hochschild complex $C\ast(A, A)$ and $A$-structure. That is there is two equivalent cochain, under this bijection, send to one equivalent class set of $(A, \pi)(\infty)$.

**Proof:** In $A_\infty$-algebra $A \in (A, \pi_i)(\infty)$, beginning of $n + 1$ dimension, we can divide the stasheff relation (1) as follow:

$$\sum_{i, j=1}^{n} \pi_i(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes 1 \otimes \ldots \otimes 1) + \sum_{i=1, j=n+1}^{n} \pi_i(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes 1 \otimes \ldots \otimes 1) +$$

$$+ \sum_{i=n+1, j=1}^{n} \pi_i(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes 1 \otimes \ldots \otimes 1) + \sum_{i, j=n+1}^{n} \pi_i(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes 1 \otimes \ldots \otimes 1) = 0 \quad (10)$$

Clearly that the first term of (10) is equal zero, following stasheff relation for fixed algebra $A$. The second and third terms of (10) , following (5), (6) can be written in the form $\delta g$. The fourth term in form $g \cup_1 g$ where $g_i = 0, i < n + 1$ and $g_i = \pi_i, i > n$.

Therefore the stasheff’s relation (1) can be written in the form $\delta g + g \cup_1 g$, and hence $g$ is twisted cochain. This means that every $A_\infty$-structure from
\((A, \pi_i)(\infty)\) defines a twisted cochain from Hochschild complex \(C_\infty(A, A)\). The inverse is every twisted cochain defines \(A_\infty\)-structure.

From theorem 18 and definition 10 we get the following assertion

**Theorem 7** If \(H^{-2}(C_\infty(A, A)) = 0\), then any structure of extension of a fixed \(A_\infty\)-algebra is trivial.

**Proof:** If \(H^{-2}(C_\infty(A, A)) = 0\), then the factor set belong to zero equivalent class, that is all twisted cochains in in Hochschild complex for \(A_\infty\)-algebra equivalent zero. Considering theorem 18 the theorem is proved.

**Theorem 8** There is a bijection between set of class of equivalent twisted cochain in complex Harrsan and the set of commutative \(A_\infty\)-algebra structure \(((A, \pi_i)(\infty))\).

**Proof:** directly from theorems 18 and 19.

**Theorem 9** If for graded commutative algebra \(A\) with multiplication \(\pi\), the \(\text{Harr}^{n,2-n}(C_\infty(A, A)) = 0, n \geq 3\), then any \(A_\infty\)-algebra structure on \(A\), such that \((\pi_1 = 0, \pi_2 = \pi)\) is trivial.

**References**


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