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The Maximal Subgroups of the Symplectic Group PSp(10, 2)

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Abstract

In this paper, we will find the maximal subgroups of the symplectic group PSp(10, 2) by Aschbacher's Theorem ([1]).

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1. Introduction

The general linear group GL(n, q), consisting of the set of all invertible n×n matrices, so in the matrix form**,** *the symplectic group* Sp(2n, q) = {g ∈ GL(2n, q): $g^t P g = P$, where g^t is the transpose matrix of the matrix g and $P = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ or $P = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$}. Since, the determinant of any skew-symmetric matrix $\{A^t = -A\}$ of odd size is zero, thus in the symplectic case, the dimension must be even. If $g = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, then g ∈ Sp(2n, q) if and only if $X_1^t X_3 - X_3^t X_1 = 0 = X_2^t X_4 - X_4^t X_2$ and $X_1^t X_4 - X_3^t X_2 = I_n$. Thus, $\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$, $\begin{pmatrix} A & 0 \\ 0 & \mathrm{inv}(A^t) \end{pmatrix}$, $\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$, $\begin{pmatrix} Q & I_n\text{-}Q \\ Q\text{-}I_n & Q \end{pmatrix}$ are in Sp(2n, q), where A is an invertible $n \times n$ matrix, B is $n \times n$ symmetric matrix, Q is a diagonal matrix of 0's and 1's, so that $Q^2 = Q$ and $(Q - I_3)^2 = I_3 - Q$ {see [3] and [7]}.

The projective symplectic group PSp(2n, q) is the quotient group PSp(2n, q) $\cong$ Sp(2n, q) / (Sp(2n, q) $\cap$ Z), where Z is the group of non-zero scalar matrices. The group PSp(2m, q) (= Sp(2m, q)) is simple, except for PSp(2, 2), PSp(2, 3) and PSp(4, 2).

An element $T \in GL(n, q)$ is called *a transvection* if T satisfies $\text{rank}(T - I_n) = 1$ and $(T - I_n)^2 = 0$. The collineation of projective space induced by a transvection is called *elation*. *The axis* of the transvection is the hyperplane $\text{Ker}(T - I_n)$; this subspace is fixed elementwise by T, Dually, *the centre* of T is the image of $(T - I_n)$.

A split extension (a semidirect product) **A:B** is a group G with a normal subgroup A and a subgroup B such that G = AB and A∩B = 1. *A non-split extension* **A.B** is a group G with a normal subgroup A and G/A ≅ B, but with no subgroup B satisfying G = AB and A∩B = 1. A group $G = A \circ B$ is *a central product* of its subgroups A and B if G = AB and [A, B], the commutator of A and B = {1}, in this case A and B are normal subgroups of G and A∩B ≤ Z(G). If A∩B = {1}, then $A \circ B$ = AB.

Through this article, G will denote PSp(10, 2), unless otherwise stated. G is a simple group of order $2^{25}.3^6.5^2.7.11.17.31$ = 24815256521932800 and G acts primitively on the points of the projective space PG(9, 2) which is a rank 3 permutation group on PG(9, 2) {see [2]}.

The main theorem of this research is the following theorem:

Theorem 1.1. Let G = PSp(10, 2). If H is a maximal subgroup of G, then H isomorphic to one of the following subgroups:

1. $G_{(p)}$, a group stabilizing a point. This is isomorphic to a group of form 2^9**:**(PGL(1, 2)×PSp(8, 2)).
2. $G_{(l)}$, a group stabilizing a line. This are isomorphic to a group of form 2^{15}**:**(PGL(2, 2)×PSp(6, 2)).
3. $G_{(2\text{-}\pi)}$, a group stabilizing a plane. This are isomorphic to a group of form 2^{18}**:**(PGL(3, 2)×PSp(4, 2)).
4. $G_{(3\text{-}\pi)}$, a group stabilizing a 3-space. This are isomorphic to a group of form 2^{18}**:**(PGL(4, 2)×PSp(2, 2)).
5. $G_{(4\text{-}\pi)}$, a group stabilizing a 4-space. This are isomorphic to a group of form 2^{15}**:**PGL(4, 2).
6. PSp(2, 2)$\circ$PSp(8, 2);
7. PSp(4, 2)$\circ$PSp(6, 2).
8. H_1 = PSp(2, 2)**:**S_5;
9. H_2 = PSp(2, 2^5)**.**5;
10. $PSGO^+(10, 2)$;
11. $PSGO^-(10, 2)$;
12. SGU(5, 2);

We will prove this theorem by Aschbacher's theorem (Result 2.8) {see [1]}:

2. Aschbacher's theorem

A classification of the maximal subgroups of GL(n, q) by Aschbacher's theorem {see [1]}, is a very strong tool in the finite groups for finding the maximal subgroups of finite linear groups. There are many good works in finite groups which simplify this theorem, see for example {[12] and [15]}. But before starting a brief description of this theorem, we will give the following definitions:

Definition 2.1: Let V be a vector space of dimensional n over a finite field q, a subgroup H of GL(n, q) is called *reducible* if it stabilizes a proper nontrivial subspace of V. If H is not reducible, then it is called *irreducible*. If H is irreducible for all field extension F of F_q, then H is *absolutely irreducible*. An irreducible subgroup H of GL(n, q) is called *imprimitive* if there are subspaces $V_1, V_2, \ldots, V_k$, $k \geq 2$, of V such that $V = V_1 \oplus \ldots \oplus V_k$ and H permutes the elements of the set $\{ V_1, V_2, \ldots, V_k\}$ among themselves. When H is not imprimitive then it is called *primitive*.

Definition 2.2: A group $H \leq GL(n, q)$ is a *superfield group* of degree s if for some s divides n with $s > 1$, the group H may be embedded in $GL(n/s, q^s)$.

Definition 2.3: If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes V_2$ with $\dim(V_1) \neq \dim(V_2)$, then H is a *tensor product group*.

Definition 2.4: Suppose that $n = r^m$ and $m > 1$. If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes \ldots \otimes V_m$ with $\dim(V_i) = r$ for $1 \leq i \leq m$, then H is a *tensor induced* group.

Definition 2.5: A group $H \leq GL(n, q)$ is a *subfield group* if there exists a subfield $F_{q_o} \subset F_q$ such that H can be embedded in $GL(n, q_o).Z$, where Z is the centre group of H.

Definition 2.6: A p-group H is called a *special group* if $Z(H) = H'$ and is called *an extraspecial group* if also $|Z(H)| = p$.

Definition 2.7: Let Z denote the centre group of H. Then H is *almost simple modulo scalars* if there is a non-abelian simple group T such that $T \leq H/Z \leq Aut(T)$, the automorphism group of T.

A classification of the maximal subgroups of GL(n, q) by Aschbacher's theorem {see [1]}, can be summarized as follows:

Result 2.8. (Aschbacher's theorem):

Let H be a subgroup of GL(n, q), $q = p^e$ with the centre Z and let V be the underlying n-dimensional vector space over a field q. If H is a maximal subgroup of GL(n, q), then one of the following holds:
C_1:- H is a reducible group.
C_2:- H is an imprimitive group.
C_3:- H is a superfield group.
C_4:- H is a tensor product group.
C_5:- H is a subfield group.
C_6:- H normalizes an irreducible extraspecial or symplectic-type group.
C_7:- H is a tensor induced group.
C_8:- H normalizes a classical group in its natural representation.
C_9:- H is absolutely irreducible and $H/(H\cap Z)$ is almost simple.

3. Classes $C_1 - C_8$ of Result 2.8

In this section, we will find the maximal subgroups in the classes $C_1 - C_8$ of Result 2.8:

Lemma 3.1: There are three reducible maximal subgroups of C_1 in G which are:

1. $G_{(p)}$, a group stabilizing a point. This is isomorphic to a group of form 2^9**:**(PGL(1, 2)×PSp(8, 2));
2. $G_{(l)}$, a group stabilizing a line. This are isomorphic to a group of form 2^{15}**:**(PGL(2, 2)×PSp(6, 2));
3. $G_{(2-\pi)}$, a group stabilizing a plane. This are isomorphic to a group of form 2^{18}**:**(PGL(3, 2)×PSp(4, 2));
4. $G_{(3-\pi)}$, a group stabilizing a 3-space. This are isomorphic to a group of form 2^{18}**:**(PGL(4, 2)×PSp(2, 2));
5. $G_{(4-\pi)}$, a group stabilizing a 4-space. This are isomorphic to a group of form 2^{15}**:**PGL(4, 2);
6. PSp(2, 2)$\circ$PSp(8, 2);
7. PSp(4, 2)$\circ$PSp(6, 2).

Proof:
Let H be a reducible subgroup of the symplectic group Sp(2n, q) and W be an invariant subspace of H. Let r = dim (W), $1 \le r \le n$ and let $G_r = G_{(W)}$ denote the subgroup of Sp(2n, q) containing all elements fixing W as a whole and $H \subseteq G_{(W)}$. with a suitable choice of a basis, $G_{(W)}$ consists of all matrices of the form $\begin{pmatrix} B & . & A \\ . & GL(r,q) & . \\ . & . & Sp(2m,q) \end{pmatrix}$ where n = r + m, A is an elementary abelian group of order q^{2rm} and B is a p- group of lower triangular matrix of order $q^{\frac{r(r+1)}{2}}$. Thus, G_r is

isomorphic to a group of the form $q^{\frac{r(r+1)}{2}+2rm}$**:**(GL(r, q)×Sp(2m, q)). Also, H is a maximal reducible subgroup of the symplectic group Sp(2n, q) if H = Sp(2a, q)∘Sp(2b, q) and n = a + b, a ≠ b. Thus, for PSp(10, 2), there are three reducible maximal subgroups of G:

1. If r = 1 and m = 4, then we get a group $G_{(p)}$, stabilizing a point. This is isomorphic to a group of form 2^9**:**(PGL(1, 2)×PSp(8, 2));
2. If r = 2 and m = 3, then we get a group $G_{(l)}$, stabilizing a line. This are isomorphic to a group of form 2^{15}**:**(PGL(2, 2)×PSp(6, 2));
3. If r = 3 and m = 2, then we get a group $G_{(2-\pi)}$, stabilizing a plane. This are isomorphic to a group of form 2^{18}**:**(PGL(3, 2)×PSp(4, 2));
4. If r = 4 and m = 1, then we get a group $G_{(3-\pi)}$, stabilizing a 3-space. This are isomorphic to a group of form 2^{18}**:**(PGL(4, 2)×PSp(2, 2));
5. If r = 5 and m = 0, then we get a group $G_{(4-\pi)}$, stabilizing a 4-space. This are isomorphic to a group of form 2^{15}**:**PGL(4, 2);
6. PSp(2, 2)∘PSp(8, 2);
7. PSp(4, 2)∘PSp(6, 2).

Which proves the points (1), (2), (3), (4), (5), (6) and (7) of the main theorem 1.1.

Lemma 3.2: There is one imprimitive group of C_2 in G which is H_1 = PSp(2, 2)**:**S_5.
Proof:
If H is imprimitive of the symplectic group Sp(2n, q), then H preserves a decomposition of V as a direct sum $V = V_1 \oplus_{\dots} \oplus V_t$, $t \geq 2$, into subspaces of V, each of dimension m = n/t, which are permuted transitively by H, thus H are isomorphic to Sp(2m, q)**:**S_t with 0 < m < n = mt, t ≥ 2. Consequently, there is one imprimitive group of C_2 in PSp(10, 2) which is H_1 = PSp(2, 2)**:**S_5, a group preserving five mutually lines of projective plane PG(9, 2) and H_1 interchanges them. This proves the point (8) of the main theorem 1.1.

Lemma 3.3: There is one semilinear group of C_3 in G which is H_2 = PSp(2, 2^5)**.**5.
Proof:
Let H is (superfield group) a semilinear groups of PSp(2n, q) over extension field F_r of GF(q) of prime degree r > 1 where r prime number divide n. Thus V is an F_r-vector space in a natural way, so there is an F-vector space isomorphism between 2n-dimensional vector space over F and the m-dimensional vector space over F_r, where m = n/r, thus H embeds in PSp(2m, q^r)**.**r. Consequently, there is one C_3 group in PSp(10, 2) which is H_2 = PSp(2, 2^5)**.**5.
This proves the point (9) of the main theorem 1.1.

Lemma 3.4: There is no tensor product group in G.
Proof:
If H is a tensor product group of Sp(2n, q), then H preserves a decomposition of V as a tensor product $V_1 \otimes V_2$, where $\dim(V_1) \neq \dim(V_2)$ of spaces of dimensions 2k and

2m over GF(q) and 2n = 4km, $k \neq m$. So, H stabilize the tensor product decomposition $F^{2k} \otimes F^{2m}$. Thus, H is a subgroup of the central product of Sp(2k, q)$\circ$Sp(2m, q). Consequently, there is no tensor product group in PSp(10, 2), since 5 is a prime number.

Lemma 3.5: There is no subfield group of C_5 in G.
Proof:
If H is a subfield group of the symplectic group Sp(2n, q) and $q = p^k$, then H is the symplectic group over subfield of GF(q) of prime index. Thus H can be embedded in Sp(2n, p^f), where f is prime number divides k. Consequently, since 2 is a prime number, then there is no subfield group of C_5 in G.

Lemma 3.6: There are no C_6 groups in G.
Proof:
For the dimension $2n = r^m$ and r is prime of the symplectic group Sp(2n, q). If r is odd prime and r divides q-1, then $H = 2^{2m+1}$ **.** $\Omega^-(2m, 2)$ normalizes an extraspecial r-group which fixes the symplectic form. Otherwise if r = 2 and 4 divides q - 1, then $H = 2^{2m}$**.**$O^-(2m, 2)$ normalizes an extraspecial 2-group which fixes the symplectic form {see [15]}. Consequently, there are no C_6 groups in PSp(10, 2) since 10 is not a prime power.

Lemma 3.7: There is no tensor induced group of C_7 in G.
Proof:
If H is a tensor induced of the symplectic group Sp(2n, q), then H preserves a decomposition of V as $V_1 \otimes V_2 \otimes \ldots \otimes V_r$, where V_i are isomorphic, each V_i has dimension 2m, dim $V = 2n = (2m)^r$, and the set of V_i is permuted by H, so H stabilize the tensor product decomposition $F^{2m} \otimes F^{2m} \otimes \ldots \otimes F^{2m}$, where $F = F_q$. Thus, $H/Z \leq$ PSp(2m, q)**:**S_r. Consequently, there is no tensor induced group in PSp(10, 2), since n = 10 is not prime power.

Lemma 3.8: There are two maximal C_8 groups in G which are $PSGO^+(10, 2)$ and $PSGO^-(10, 2)$.
Proof:
The groups in this class are stabilizers of forms, this means H is the normalizers of one classical groups PSL(2n, q), $PO^{\epsilon}(2n, q)$ or PSU(2n, q) as a subgroup of PSp(2n, q). But from [5] and [10], if q is even, then the normalizers of $PO^+(2n, q)$ and $PO^-(2n, q)$ are maximal subgroups of PSp(2n, q) except when n = 2 and ϵ = -. Consequently, in the class C_8, there are two irreducible maximal subgroups in PSp(10, 2) that are $PSGO^+(10, 2)$ and $PSGO^-(10, 2)$. Which prove the points (10) and (11) of theorem 1.1.

In the following, we will find the maximal subgroups of class C_9 of Result 2.8:

4. The maximal subgroups of C_9

In Corollary 4.1, we will find the primitive non abelian simple subgroups of G. In Theorem 4.2, we will find the maximal primitive subgroups H of G which have the property that the minimal normal subgroup M of H is not abelian group and simple. We will prove this Theorem 4.2 by finding the normalizers of the groups of Corollary 4.1 and determine which of them are maximal.

Corollary 4.1: If M is a non abelian simple group of a primitive subgroup H of G, then M is isomorphic to one of the following groups:

(i) $PSO^-(10, 2)$;
(ii) $PSO^+(10, 2)$;
(iii)PSU(5, 2);
(iv)$P\Omega^-(6, 2)$;
(v) $P\Omega^+(8, 2)$;
(vi)$P\Omega^+(10, 2)$.

Proof:
Let H be a primitive subgroup of G with a minimal normal subgroup M of H which is not abelian and simple. So, we will discuss the possibilities of M of H according to:

(I) M contains transvections, {section 4.1}.
(II) M is a finite primitive subgroup of rank three, {section 4.2 }.

4.1 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections:

To find the primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections, we will use the following result of Kantor {see [9]}:

Result 4.1.1:
Let H be a proper irreducible subgroup of $Sp(2n, q^i)$ generated by transvections. Then H is one of:

1. Sp(2n, q);
2. $O^{\pm}(n, q^i)$ for q even;
3. S_{2n} or $S_{2n+1;}$
4. $SL(2, 5) < Sp(2, 9^i)$;
5. Dihedral subgroups of $Sp(2, 2^i)$.

In the following, we will discuss the different possibilities of Result 4.1.1:

Corollary 4.1.2:
If M is a primitive subgroup of PSp(10, 2) generated by transvections which is not abelian and simple, then M isomorphic to orthogonal groups $PSO^-(10, 2)$ and $PSO^+(10, 2)$.

Proof:
From Result 4.1.1, M is isomorphic to one of the following groups:

1. From Lemma 3.8, $PSO^-(10, 2)$ and $PSO^+(10, 2)$ are maximal subgroups of PSp(10, 2).
2. $S_{10} \not\subset G$, since, the irreducible 2-modular characters for S_{10} by GAP are:
   [[1, 1], [8, 1], [16, 1], [26, 1], [48, 1], [128, 1], [160, 1], [198, 1], [200, 1], [768, 1]]
   (gap> CharacterDegrees(CharacterTable("S10")mod 2);)
3. $S_{11} \subset G$, since the irreducible 2-modular characters for S_{11} by GAP are:
   [[1, 1], [10, 1], [32, 1], [44, 1], [100, 1], [144, 1], [164, 1], [186, 1], [198, 1], [416, 1], [848, 1], [1168, 1]]
   (gap> CharacterDegrees(CharacterTable("S11")mod 2);)
   But S_{11} is not a simple group.
4. SL(2, 5) $\not\subset$ G, since the irreducible 2-modular characters for SL(2, 5) by GAP are:
   [[1, 1], [2, 2], [4, 1]]
   (gap> CharacterDegrees(CharacterTable("L2(5)") mod 2);)
   And non of these of degree 8.
5. If M is a Dihedral subgroups of $Sp(2, 2^i)$, then M $\not\subset$ G, since M is not a simple group.

4.2 Primitive subgroups H of G which have the property that a minimal normal subgroup M of H which is not abelian is a finite primitive subgroup of rank three:

A group G has rank 3 in its permutation representation on the cosets of a subgroup K if there are exactly 3 (K, K)-double cosets {see [8] }. Indeed, the rank of a transitive permutation group is the number of orbits of the stabilizer of a point, thus if we consider PSp(2m, q), $m \geq 2$ and q is of a prime power, as group of permutations of the absolute points of the corresponding projective space, then PSp(2m, q) is a transitive group of rank 3. Ideed, the pointwise stabilizer of PSp(2m, q) has 3 orbits of lengths 1, $q(q^{2m-2} - 1)/(q - 1)$ and q^{2m-1} {see [17]}. So, we will consider the minimal normal subgroup M of H is not abelian and a finite primitive subgroup of rank three, so will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three {see [8]}. The following Corollary is the main result of this section:
Corollary 4.2.1: If M is a non abelian simple group which is a finite primitive subgroup of rank three group of H, then M is isomorphic to one of the following groups:

1. PSU(5, 2);
2. $P\Omega^-(6, 2)$;
3. $P\Omega^+(8, 2)$;
4. $P\Omega^+(10, 2)$.

Proof:
Let M is not an abelian and is a finite primitive subgroup of rank three of H, and will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three {see [8]}. So, we will prove Corollary 4.2.1 by series of Lemmas 4.2.3 through Lemmas 4.2.20 and Result 4.2.2.

Result 4.2.2:
If Y acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of Sp(2n-2, q), $\Omega^{\pm}(2n, q)$, $\Omega(2n\text{-}1, q)$ or SU(n, q). Then for n ≥3, Y has a simple normal subgroup M^*, and $M^* \subseteq Y \subseteq \mathrm{Aut}(M^*)$, where M^* as follows:

- (i) M = Sp(4, q), SU(4, q), SU(5, q), $\Omega^-(6, q)$, $\Omega^+(8, q)$ or $\Omega^+(10, q)$.
- (ii) M = SU(n, 2), $\Omega^{\pm}(2n, 2)$, $\Omega^{\pm}(2n, 3)$ or $\Omega(2n\text{-}1, 3)$.
- (iii) M = $\Omega(2n\text{-}1, 4)$ or $\Omega(2n\text{-}1, 8)$;
- (iv) M = SU(3, 3);
- (v) SU(3, 5);
- (vi) SU(4, 3);
- (vii) Sp(6, 2);
- (viii) $\Omega(7, 3)$;
- (ix) SU(6, 2);

In the following, we will discuss the different possibilities of Result 4.2.2;

Lemma 4.2.3: If M = PSp(4, q), then $M \not\subset G$.
Proof:
In our case q = 2, But PSp(4, 2) is not simple.

Lemma 4.2.4: If M = PSU(4, q), then $M \not\subset G$.
Proof:
In our case q = 2, But PSU(4, 2) $\not\subset$ G since the irreducible 2-modular characters for PSU(4, 2) by GAP:
[[1, 1], [4, 2], [6, 1], [14, 1], [20, 2], [64, 1]].
(gap> CharacterDegrees(CharacterTable("U4(2)")mod 2);)
and non of these of degree 10.

Lemma 4.2.5: PSU(5, 2) ⊂ G.
Proof:
Since, the irreducible 2-modular characters for PSU(5, 2) by GAP are:
[[1, 1], [5, 2], [10, 2], [24, 1], [40, 4], [74, 1], [160, 2], [280, 2], [1024, 1]]
(gap> CharacterDegrees(CharacterTable("U5(2)")mod 2);)

Lemma 4.2.6: $P\Omega^-(6, 2) \subset G$.
Proof:
Since the irreducible characters for $P\Omega^-(6, 2)$ by GAP are:
[1, 2, 4, 4, 2, 4, 2, 8, 4, 4, 4, 3, 6, 12, 5, 10, 20, 20, 2, 2, 4, 4, 8, 8, 8, 8, 6, 6]

(gap> g:= GO(-1,6,2);;
gap> c:=CharacterTable("g");;
gap> OrdersClassRepresentatives(c);)

Lemma 4.2.7: $P\Omega^+(8, 2) \subset G$.
Proof:
Since the irreducible characters for $P\Omega^+(8, 2)$ by GAP are:
[1, 2, 4, 4, 2, 4, 2, 8, 4, 4, 4, 3, 6, 12, 5, 10, 20, 20, 2, 2, 4, 4, 8, 8, 8, 8, 6, 6]
(gap> g:= GO(+1,8,2);;
gap> c:=CharacterTable("g");;
gap> OrdersClassRepresentatives(c);)

Lemma 4.2.8: $P\Omega^+(10, 2) \subset G$.
Proof:
See lemma 3.8.

Lemma 4.2.9: if M = PSU(n, 2), then $M \not\subset G$.
Proof:
In our case, 2n-2 = 10, thus n = 6 and PSU(6, 2) $\not\subset$ G, since the irreducible 2-modular characters for PSU(6, 2) by GAP are: [[1, 1], [20, 1], [34, 1], [70, 2], [154, 1], [400, 1], [896, 2], [1960, 1], [3114, 1], [32768, 1]]
(gap> CharacterDegrees(CharacterTable("U6(2)")mod 2);)
and non of these of degree 10.

Lemma 4.2.10: If $M = P\Omega^{\pm}(2n, 2)$, then $M \not\subset G$.
Proof:
In our case n = 6, thus we need to consider $P\Omega^{\pm}(12, 2)$:

- $P\Omega^+(2n, q)$, $n \geq 4$, q = 2 has no projective representation in G of degree less than $q^{n-2}(q^{n-1} - 1)$, {see [13] and [14]}, but this bound is greater than 10, thus $P\Omega^+(12, 2) \not\subset G$.
- $P\Omega^-(2n, q)$, $n \geq 4$, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [14]}, but this bound is greater than 10, thus $P\Omega^-(12, 2) \not\subset G$.

Lemma 4.2.11: If $M = P\Omega^{\pm}(2n, 3)$, then $M \not\subset G$.
Proof:
In our case n = 6, thus we need to consider $P\Omega^{\pm}(12, 3)$:

- $P\Omega^+(2n, q)$, $n \geq 4$, q = 2 has no projective representation in G of degree less than $q^{n-2}(q^{n-1}-1)$, {see [13] and [14]}, but this bound is greater than 10, thus $P\Omega^+(12, 3) \not\subset G$.
- $P\Omega^-(2n, q)$, $n \geq 4$, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [14]}, but this bound is greater than 10, thus $P\Omega^-$

(12, 3) $\not\subset$G.

Lemma 4.2.12: If M = PΩ(2n-1, 3), then M $\not\subset$ G.
Proof:
In our case n = 6, thus, we have PΩ(11, 3) $\not\subset$ G, since PΩ(2n+1, q), n ≥ 3, q = 3, has no projective representation in G of degree less than $q^{n-1}(q^{n-1} - 1)$, {see [13] and [14]}, which is greater than 10.

Lemma 4.2.13: If M = PΩ(2n-1, 4), then M $\not\subset$ G.
Proof:
In our case n = 6, thus we have PΩ(11, 4) $\not\subset$ G. since, PΩ(2n+1, q) ≅ PSp(2n, q) for q even, then PΩ(11, 4) ≅ PSp(10, 4), and it is clear that PSp(10, 4) has a projective representation in G of degree 8, but the order of PSp(10,4) is equal to 1211875293642881119668928512000000. Which is not dividing the order of G.
(gap> G:= PSp(10, 4) ;;
gap> Order(G);)

Lemma 4.2.14: If M = PΩ(2n-1, 8), then M $\not\subset$ G.
Proof:
In our case n = 6, thus we have PΩ(11, 8) $\not\subset$ G. since, PΩ(2n+1, q) ≅ PSp(2n, q) for q even, then PΩ(11, 8) ≅ PSp(10, 8). It is clear that PSp(10, 8) has a projective representation in G of degree 10, but the order of PSp(10, 8) is not dividing the order of G, since the order of PSp(10, 8) is greater than the order of G .

Lemma 4.2.15: PSU(3, 3) $\not\subset$ G.
Proof:
since the irreducible 2-modular characters for PSU(3, 3) by GAP are:
[[1, 1], [6, 1], [14, 1], [32, 2]],
{gap> CharacterDegrees(CharacterTable("U3(3)")mod 2);}
and non of these of degree 10.

Lemma 4.2.16: PSU(3, 5) $\not\subset$ G.
Proof:
Since the irreducible 2-modular characters for PSU(3, 5) by GAP are:
[[1, 1], [20, 1], [28, 3], [104, 1], [144, 2]]
(gap> CharacterDegrees(CharacterTable("U3(5)")mod 2);)
And non of these of degree 10.

Lemma 4.2.17: PSU(4, 3) $\not\subset$ G.
Proof:
Since the irreducible 2-modular characters for PSU(4, 3) by GAP are:
[[1, 1], [20, 1], [34, 2], [70, 4], [120, 1], [640, 2],[896, 1]]
(gap> CharacterDegrees(CharacterTable("U4(3)")mod 2);)

And non of these of degree 10.

Lemma 4.2.18: PSp(6, 2) $\not\subset$ G.
Proof:
Since the irreducible 2-modular characters for PSp(6, 2) by GAP are:
[[1, 1], [6, 1], [8, 1], [14, 1], [48, 1], [64, 1], [112, 1], [512, 1]]
(gap> CharacterDegrees(CharacterTable("S6(2)")mod 2);)
And non of these of degree 10.

Lemma 4.2.19: PΩ(7, 3) $\not\subset$ G.
Proof:
Since PΩ(2n+1, q), $n \geq 3$, has no projective representation in G of degree less than $q^{n-1}(q^{n-1} - 1)$, {see [13] and [14]}, thus for q = 3 and n = 3, PΩ(7, 3) has a projective representation of degree which is greater than 10.

Lemma 4.2.20: PSU(6, 2) $\not\subset$ G.
Proof:
Since the irreducible 2-modular characters for PSU(6, 2) by GAP are:
[[1, 1], [20, 1], [34, 1], [70, 2], [154, 1], [400, 1], [896, 2], [1960, 1], [3114, 1], [32768, 1]]
(gap> CharacterDegrees(CharacterTable("U6(2)")mod 2);)
And non of these of degree 10.

Now, we will determine the maximal primitive group of C_9:

Theorem 4.2: If H is a maximal primitive subgroup of G which has the property that a minimal normal subgroup M of H is not abelian group, then H is isomorphic to one of the following subgroups of G:

(i) $PSO^-(10, 2)$;
(ii) $PSO^+(10, 2)$;
(iii) SGU(5, 2);
(iv) $PSGO^+(10, 2)$.

Proof:
We will prove this theorem by finding the normalizers N of the groups of Corollary 4.1 and determine which of them are maximal:
From [11], the normalizer of SU(n, k) in SL(n, k) is SGU(n, k) = GU(n, k) $\cap$ SL(n, k).
From [10], the normalizer of SO(n, k) in SL(n, k) is SGO(n, k) = GO(n, k) $\cap$ SL(n, k). Thus,

- If Y = $PSO^-(10, 2)$, then N = $PSGO^-(10, 2)$, which prove the point (11) of theorem 1.1.
- If Y = $PSO^+(10, 2)$, then N = $PSGO^+(10, 2)$, which prove the point (10) of theorem 1.1.
- If Y = PSU(5, 2), then N = SGU(5, 2), which prove the point (12) of theorem 1.1.

- If $Y = P\Omega^{-}(6, 2)$, then $N = PSGO^{-}(6, 2)$, which is not a maximal in G since $PSGO^{-}(6, 2)$ is a subgroup of $PSGO^{-}(10, 2)$.
- If $Y = P\Omega^{+}(8, 2)$, then $N = PSGO^{+}(8, 2)$, which is not a maximal in G since $PSGO^{+}(8, 2)$ is a subgroup of $PSGO^{+}(10, 2)$.
- If $Y = P\Omega^{+}(10, 2)$, then $N = PSGO^{+}(10, 2)$, which prove the point (10) of theorem 1.1.

This completes the proof of theorem 1.1.

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