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On a Lie Ring of Generalized Derivations of Non-Commutative Rings

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Abstract

Let R be a 2-torsion free non-commutative ring with centre $Z(R) \neq \{0\}$. Let F be a nonzero central generalized derivation, with associated central derivation d such that $d(Z(R)) \neq \{0\}$, of R. If for some nonzero $c \in Z(R)$, R is cF-prime as well as cd-prime, then $Z(R)F$ is a prime Lie ring.

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1 Introduction

Let R be an associative ring with centre $Z(R) \neq \{0\}$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Derivations have been generalized as generalized derivations by Brešar [2] and Hvala [4] and since then have been extensively studied by various researchers (see [1] and references therein). An additive mapping $F : R \to R$ is called a generalized derivation, with associated derivation d, if there is a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Obviously every derivation d is a generalized derivation (in case $F = d$). A mapping $f : R \to R$ is called central if $f(x) \in Z(R)$ for all $x \in R$. Every mapping $f : R \to R$, where R is commutative, is central. It is known that all commuting derivations on semiprime rings are central. By $[x, y]$ we mean $xy - yx$, $x, y \in R$.

Lie rings of derivations of prime rings and commutative rings with identity have been studied by various researchers (see [3,5-7] and references therein). In [3], Chebotar and Lee proved the following result:

Theorem A. Let R be a 2-torsion free commutative ring with identity and δ a nonzero derivation of R such that R is δ-prime. Then $R\delta$ is a prime Lie ring.

The motivation behind this paper is to prove an analog of Theorem A for generalized derivations on non-commutative rings without identity and to initiate a study of Lie rings of generalized derivations on non-commutative rings without identity.

In this paper we show that:
Let R be a 2-torsion free non-commutative ring with centre $Z(R) \neq \{0\}$. Let F be a nonzero central generalized derivation, with associated central derivation d such that $d(Z(R)) \neq \{0\}$, of R. If for some nonzero $c \in Z(R)$, R is cF-prime as well as cd-prime, then $Z(R)F$ is a prime Lie ring.

Let $f : R \to R$ be a mapping. A nonempty subset S of R is called f-invariant if $f(S) \subseteq S$, and an ideal of R is called an f-ideal if it is f-invariant. R is called f-prime if $AB \neq 0$ for nonzero f-ideals A and B of R.

Let S be a subset of R. The set $ann(S) = \{a : a \in R \ and \ aS = Sa = 0\}$ is called annihilator of S in R. It is known that if A is an ideal of R, then $ann(A)$ is an ideal of R. We shall need the following lemmas in the sequel.

Lemma 1.1 *Let $F : R \to R$ be a central generalized derivation, with associated central derivation d. Let A be an F-invariant ideal of R. Then $ann(A)$ is F-invariant as well as d-invariant.*

Proof. Let $x \in ann(A)$. Then $xA = 0 = Ax$. Thus $F(xA) = F(0) = F(Ax)$ and $d(xA) = d(0) = d(Ax)$. Since $0 = F(0) = F(Ax) = F(A)x + Ad(x)$ and $F(A) \subseteq A$, therefore the last equation implies $Ad(x) = 0$, which along with the hypothesis that d is central implies

$$0 = Ad(x) = d(x)A. \tag{1}$$

Further $0 = d(0) = d(xA) = d(x)A + xd(A)$ along with (1) implies $xd(A) = 0$. Considering $F(xA) = 0$, we get $F(x)A + xd(A) = 0$, which along with $xd(A) = 0$ gives $F(x)A = 0$. Since F is central, therefore

$$F(x)A = AF(x) = 0. \tag{2}$$

From (1) and (2) we get that $ann(A)$ is invariant under F as well as its associated derivation d.

Lemma 1.2 *Let R be a ring and $F : R \to R$ a nonzero central generalized derivation, with associated nonzero derivation d, such that R is F-prime. Then*
(a) If S is a nonzero F-invariant additive subgroup of R such that $RS = SR$, then $ann(S) = 0$.
(b) If $r \in R$, and $rF = 0$, then $r = 0$.

Proof. (a) Let $I = S + SR$. Then it easily follows that I is a nonzero ideal of R and $ann(I) = ann(S)$. Further $F(I) = F(S + SR) = F(S) + F(SR) = F(S) + F(S)R + Sd(R) \subseteq F(S) + SR + SR \subseteq S + SR = I$. Hence I is F-invariant. By Lemma 1.1, $ann(I)$ is F-invariant. Since I is nonzero and $Iann(I) = 0$, therefore F-primeness of R implies $ann(I) = 0$.
(b) Obviously $F(F(R)) \subseteq F(R)$ and $F(R)$ is a nonzero additive subgroup of R. Thus $F(R)$ is an additive subgroup of R invariant under F. Since F is central, so $F(R)R = RF(R)$. Let $rF = 0$. Then $(rF)(x) = rF(x) = 0$, which implies $F(x)r = 0$ for all $r, x \in R$, because F is central. So $r \in ann(F(R))$. Thus part(a) of the lemma implies $r = 0$.

Remark 1.3 *Since every derivation d on a ring R is a generalized derivation with associated derivation d, so the following corollary follows from Lemma 1.2.*

Corollary 1.4 *Let R be a ring and $d : R \to R$ a nonzero central derivation such that R is d-prime. Then*
(a) If S is a nonzero d-invariant additive subgroup of R such that $RS = SR$, then $ann(S) = 0$.
(b) If $r \in R$ and $rd = 0$, then $r = 0$.

Let $GDer(R) = \{F : F : R \to R \text{ is a generalized derivation}\}$. Then $GDer(R)$ is a Lie ring under the product $F_1 \circ F_2 = [F_1, F_2]$, $F_1, F_2 \in GDer(R)$. If F_1 and F_2 are generalized derivations with associated derivations d_1, d_2 respectively, then it easily follows that $[F_1, F_2]$ is a generalized derivation with associated derivation $[d_1, d_2]$. Further $GDer(R)$ is a left $Z(R)$-module: For $r \in Z(R)$, $F \in GDer(R)$, rF is defined as $(rF)(x) = rF(x)$ for $x \in R$. It easily follows that rF, $r \in Z(R)$, is a generalized derivation with associated derivation rd.

2 The Results

Let R be a ring and $F : R \to R$ a nonzero generalized derivation. Let $r \in Z(R)$. Then $rF : R \to R$ is also a generalized derivation, with associated derivation rd. Let $Z(R)F = \{rF : r \in Z(R)\} \subseteq GDer(R)$. It is easy to verify that Lie multiplication of $GDer(R)$ restricted to $Z(R)F$ is given by $[rF, sF] = (rF(s) - sF(r))F$ for $r, s \in Z(R)$. Obviously $Z(R)F$ is a Lie subring of $GDer(R)$ and is also a left $Z(R)$-submodule of $GDer(R)$.

We recall that a Lie ring L is prime if $[A, B] \neq 0$ for any two nonzero ideals A, B of L.

In the sequel we shall identify cx by $cI(x)$, $x \in R$ and c is a fixed element of $Z(R)$. Here I is the identity mapping on R. It is easy to verify that if F is a generalized derivation of R, with associated derivation d, then $cF - hI$, $c \in Z(R)$ and $h \in R$, is a generalized derivation of R, with associated derivation cd.

Lemma 2.1 *Let R be a ring and F a nonzero generalized derivation, with associated nonzero derivation d, of R. Let c be a nonzero element of $Z(R)$ and $h \in R$. Then*
(a) R is $(cF - hI)$-prime if and only if R is cF-prime.
(b) If R is cF-prime, then R is F-prime.

Proof. (a) It is sufficient to prove that A is cF-invariant if and only if A is $(cF - hI)$-invariant. Now an ideal A of R is a cF-invariant if and only if $(cF)(A) = cF(A) \subseteq A$, which holds if and only if $cF(A) - hI(A) \subseteq A - hI(A) = A - hA \subseteq A$, or $(cF - hI)(A) \subseteq A$. Thus A is cF-invariant if and only if A is $(cF - hI)$-invariant.
(b) Let A, B be F-invariant ideals of R such that $[A, B] = 0$. Now $F(A) \subseteq A$ implies $cF(A) \subseteq cA \subseteq A$, so A is a cF-invariant ideal. Similarly B is a cF-invariant ideal. Since R is cF-prime, so either $A = 0$ or $B = 0$. Hence R is F-prime.

Since every derivation d is also a generalized derivation, with associated derivation d, therefore following corollary immediately follows from Lemma 2.1.(b).

Corollary 2.2 *Let R be a ring and d a nonzero derivation of R. If, for some nonzero $c \in Z(R)$, R is cd-prime, then R is d-prime.*

We are now ready to prove our results regarding the Lie ring $Z(R)F$.

Theorem 2.3 *Let R be a non-commutative ring of characteristic different from 2. Let $F : R \to R$ be a nonzero central generalized derivation, with associated nonzero central derivation d such that $d(Z(R)) \neq \{0\}$, of R. Let c be a nonzero fixed element of $Z(R)$ and R a cF-prime as well as cd-prime ring, then $Z(R)F$ is a prime Lie ring.*

Proof. Let $A \neq 0$ and $B \neq 0$ be ideals of the Lie ring $Z(R)F$ such that $[A, B] = 0$. Let $A' = \{a : a \in Z(R) \textit{ and } aF \in A\}$ and $B' = \{b : b \in Z(R) \textit{ and } bF \in B\}$. Since $A \neq 0, B \neq 0$, therefore A' and B' are nonzero subsets of $Z(R)$ and $RA' = A'R$, $RB' = B'R$. Let $c \neq 0$ be an element of $Z(R)$. Then $[cF, aF] = cF(aF) - aF(cF) = (cF(a) - F(c)a)F = ((cF - F(c)I)(a))F \in A$, which implies $(cF - F(c)I)(a) \in A'$. Thus A' is invariant under $(cF - F(c)I)$. Similarly B' is invariant under $(cF - F(c)I)$. Further by Lemma 2.1(a), we get that A' and B' are cF-invariant. Moreover it easily follows that A' and B' are additive subgroups of R. Since $[A, B] = 0$, therefore $[aF, bF] = 0$ for all $a \in A', b \in B'$, which implies $(aF(b) - bF(a))F = 0$. Hence $c(aF(b) - bF(a))F = 0$. That is, $(aF(b) - bF(a))cF = 0$, which by Lemma 1.2(b) and cF-primeness of R implies

$$aF(b) = bF(a) \ \ for \ \ a \in A', b \in B'. \tag{3}$$

Let $r \in Z(R)$. Then $brF \in Z(R)F$. Since A is an ideal of $Z(R)F$, therefore $[aF, brF] \in A$. That is, $(aF(br) - brF(a))F \in A$, which implies $[aF(b)r + abd(r) - brF(a)]F \in A$. By (3) and the last relation we get $(abd(r))F \in A$, which implies

$$abd(r) \in A'. \tag{4}$$

Let $k \in B'$. Then $kF \in B$. Now from (3) and (4), we get $abd(r)F(k) = kF(abd(r))$. That is,

$$abd(r)F(k) = kF(a)bd(r) + kad(b)d(r) + kabd^2(r). \tag{5}$$

Using (3) from (5) we get $kad(b)d(r) + kabd^2(r) = 0$, which implies

$$ak[d(b)d(r) + bd^2(r)] = 0. \tag{6}$$

Thus

$$[d(b)d(r) + bd^2(r)](ak) = 0 \ \ for \ \ all \ \ b \in B', \ \ k \in B', \ \ a \in A' \ \ and \ \ r \in Z(R). \tag{7}$$

Replacing r by r^2 in (7), we get $[2rd(b)d(r) + 2rbd^2(r) + 2b(d(r))^2](ak) = 0$, which along with (7) and the hypothesis that R is not of characteristic 2, implies $[b(d(r))^2]ak = 0$. That is,

$$(d(r))^2bak = 0. \tag{8}$$

Since R is cF-prime and it has already been shown that A' and B' are nonzero cF-invariant additive subgroups of R satisfying $RA' = A'R$ and $RB' = B'R$, therefore by using Lemma 1.2(a) thrice, from (8), we get $(d(r))^2 = 0$. Linearizing the last relation, we get $2d(r)d(s) = 0$ for all $r, s \in Z(R)$. Since R is not of characteristic 2, so last relation implies $d(r)d(s) = 0$ for all $r, s \in Z(R)$, which implies $c(d(r)d(s)) = 0$ for all $r, s \in Z(R)$. Thus

$$d(r)cd(s) = 0 \ \ for \ \ all \ \ r, s \in Z(R). \tag{9}$$

Obviously $Z(R)$ is nonzero additive subgroup of R, invariant under cd and $RZ(R) = Z(R)R$. Since R is cd-prime, therefore from (9) and Corollary 1.4(b) we get $d(r) = 0$ for all $r \in Z(R)$, a contradiction. Hence either $A = 0$

or $B = 0$. Thus $Z(R)F$ is a prime Lie ring.

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