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Boolean-Like Rings and Related Rings

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Abstract

A *Boolean-Like* ring is a ring R such that $x^2y - xy^2$ is nilpotent for certain elements x, y of R. A *strongly Boolean-like ring* is a ring in which $x^2y = xy^2$ for some elements x, y in R. The commutaivity behavior of such rings is considered, Also, certain conditions which imply that these rings have a nil commutator ideal are established.

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Throughout, R is a ring not necessarily with identity, N, J, and C denote the set of nilpotents, the Jacobson radical, and center of R, respectively. For x, y in R, [x,y] denotes the commutator xy-yx. Z will denote the ring of integers.

Definition 1. A ring R is called a *Boolean-Like* if

(1) $x^2y - xy^2 \in N$ for all x, y, in $R \setminus (N \cup C)$.

R is called *strongly Boolean-Like ring* if

(2) $x^2 y = xy^2$ For all x, y, in $R \setminus (N \cup C)$.

Although all commutative rings are both Boolean-Like and strongly Boolean-Like, the converse is not true. Indeed, the ring

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} : 0,1 \in GF(2) \right\}$$

is both Boolean-Like and strongly Boolean-Like but it is not commutative. In Theorem 3 we characterize all *commutative* Boolean-Like and strongly Boolean-Like rings.

The following lemmas will be needed in the proofs of the main theorems.

Lemma 1 ([2]). *Suppose* R *is a ring such that*
(3) $x - x^2 \in C$ *for all x in R.*
Then R is commutative.

Lemma 2 ([1]). *Suppose* R *is a ring in which every element x is central or potent in the sense that* $x^k = x$ *for some* integer k>1. *Then* R *is commutative*.

We are now in a position to prove our main theorems.

Theorem 1. *A Boolean-Like ring* R *with central nilpotents is commutative*.

Proof. Since N $\subseteq$C, (1) implies that

(4) $x^2y - xy^2 \in N$ for all x, y $\in$ $R \setminus C$.

We claim that

(5) $x - x^2 \in C$ for all x $\in$ R.

Suppose not. Let x $\in R$ be such that $x - x^2 \notin C$. Then, x $\notin$ C and $x - x^2 \notin$ C, and hence by (4)

$$x^2(x - x^2) - x(x - x^2)^2 \in N,$$

which implies $x^4 - x^5 \in N$. Hence, for some $f(\lambda) \in Z[\lambda]$,

$$(x - x^2)^4 = (x - x^2)x^3 f(x) = (x^4 - x^5) f(x) \in N ,$$

and thus $x - x^2 \in N \subseteq C$. Therefore, $x - x^2 \in C$, contradiction. This contradiction proves (5), and hence R is commutative, by Lemma 1.

Corollary 1. *A reduced Boolean-Like ring is commutative.*

Corollary 2. *A Boolean ring is commutative.*

Theorem 2. *Suppose R is a Boolean-Like ring. Then,*

(i) *For all* $x \in R \setminus C$, $x - x^2 \in N$.
(ii) *For all* $x \in R \setminus C$, $x^q = x^q e$ *for some* $q \geq 1$ *and some* $e \in xZ[x]$ *for which* $e^2 = e$.
(iii) *Every subring and every homomorphic image of a Boolean-Like ring is Boolean-Like.*

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Proof (*i*). Suppose not. Let

(6) $x \notin C$, $x - x^2 \notin N$.

Then, $x \notin N$, $x^2 \notin N$.

Case 1. $x^2 \notin C$. Then, $x \notin (NUC)$, $x^2 \notin (NUC)$, and hence by (1), with $y = x^2$ we get

(7) $x^4 - x^5 \in N$,

which, as we saw in the proof of Theorem 1, implies that $x - x^2 \in N$, contradicting (6).

Case 2. $x^2 \in C$. Then $x - x^2 \notin C$ (since $x \notin C$), and $x - x^2 \notin N$ (see (6)). The net result is: $x - x^2 \notin (N \cup C)$ and $x \notin (N \cup C)$ either. So, by setting $y = x - x^2$ in (1), we see that (7) holds, and thus, (as shown above), $x - x^2 \in N$, contradicting (6) again. This contradiction proves (i).

(ii) By part (i), if $x \notin C$, then $x - x^2 \in N$, and hence $(x - x^2)^q = 0$ for some $q \geq 1$. Thus, $x^q = x^q (x\, g(x))$ for some $g(\lambda) \in Z[\lambda]$. Let $e = (x\, g(x))^q$. It is readily verified that $x^q = x^q e$ and $e^2 = e \in xZ[x]$. This proves *(ii).*

(iii) This follows at once from the definition of a Boolean-Like ring.

Theorem 3. *A Boolean-Like ring is commutative if and only if the idempotents of* R *are central and* N∩J *is commutative. (This also holds for strongly Boolean-Like rings*).

Proof. Clearly a commutative Boolean-Like ring satisfies the above stated conditions on the idempotents and on N∩J. To prove the converse, suppose that R is a Boolean-Like ring for which

(8) $E \subseteq C$, N∩J is commutative, (E is the set of idempotents).

We claim that

(9) $N \subseteq J$.

To prove this, let $a \in N$, $x \in R$. By Theorem 2 (i), $ax \in C$ or $ax - (ax)^2 \in N$. If $ax \in C$,
then $ax \in N$. On the other hand, if $ax \notin C$, then by Theorem 2 (ii), $(ax)^q = (ax)^q e$ for some idempotent $e \in axZ[ax]$. Hence,

(10) $e = e\,e = e\;a\,r = a\,e\,r$, for some $r \in R$, (see (8)).

Re-iterating, we see that (10) implies that

(11) $e = a\,e\,r = a^2 e\,r^2 = \ldots = a^k e\,r^k$ for all positive integers k.

Since $a \in N$, let $a^k = 0$. Then, by (11), $e = 0$, and hence $(ax)^q = (ax)^q e = 0$, which implies that $ax \in N$. Hence, in any case, $ax \in N$, and thus ax is right quasi regular for all $x \in R$. So $a \in J$, proving (9). Since $N \subseteq J$, $N \cap J = N$ is commutative (by (8)). The net result is:

(12) N is commutative.

As is well known,

(13) $R \cong a$ subdirect sum of subdirectly irreducible rings R_i, $(i \in \Gamma)$.

Let $\sigma : R \to R_i$ be the natural homomorphism of R onto R_i, and let $\sigma : x \to x_i$. Our next objective is to prove that

(14) The set N_i of nilpotents of $R_i \subseteq \sigma(N) \cup C_i$,

where C_i denotes the center of R_i. To prove this, let

$d_i \in N_i$, $d_i \notin C_i$, and let $\sigma(d) = d_i$, $d \in R$. Then $d \notin C$, and hence by Theorem2 (i), $d-d^2 \in N$. Suppose $d_i^k = 0$. Then,

$$d-d^{k+1} = (d-d^2) + d(d-d^2) + \ldots + d^{k-1}(d-d^2) \in N \text{ (since } d-d^2 \in N),$$

which implies that

$$\sigma(d-d^{k+1}) \in \sigma(N).$$

Hence, $d_i-d_i^{k+1} \in \sigma(N)$; that is, $d_i \in \sigma(N)$, which proves (14).

Our next goal is to prove that,

(15) Every element of R_i is nilpotent or a unit or central.

To prove this, let $x_i \in R_i \setminus C_i$, and let $\sigma : x \to x_i$, $x \in R$. Then $x \notin C$, and hence by Theorem 2 (ii), $x^q = x^q e$ for some $q \geq 1$, $e \in x Z[x]$, $e^2 = e$. Therefore, since $e \in C$ (by (8)), $e_i = \sigma(e) \in C_i$ and, of course, $e_i^2 = e_i$. The net result is:

$$x_i^q = x_i^q e_i, \quad e_i \text{ is a } \textit{central} \text{ idempotent in } R_i .$$

Moreover, since R_i is subdirectly irreducible, $e_i = 0$ or $e_i = 1$.
If $e_i = 0$, then x_i is nilpotent, while if $e_i = 1$, then x_i is a unit (since $e_i \in x_i Z[x_i]$), which proves (15).

Next, we prove that

(16) $N_i \subseteq C_i$.

Suppose not. Then there exists $a_i \in N_i$, $a_i \notin C_i$, and thus for some $b_i \in R_i$, $[a_i, b_i] \neq 0$. The net result is:

(17) $[a_i, b_i] \neq 0$, $a_i \in N_i$, $b_i \in R_i$.

Now, by (14), $N_i \subseteq \sigma(N) \cup C_i$, and furthermore, N is commutative, by (12). So N_i is commutative. Combining this fact with (17), we see that $b_i \notin N_i$ and, of course, $b_i \notin C_i$. Hence, by (15), it follows that b_i is a unit in R_i. Moreover, by Theorem 2 (i), $b_i - b_i^2 \in N_i$, and hence $b_i^{-1}(b_i - b_i^2) \in N_i$; that is, $1- b_i \in N_i$. Since $a_i \in N_i$ and N_i is commutative, $[a_i, 1-b_i] = 0$, which implies that $[a_i, b_i] = 0$, a contradiction (see (17)). This contradiction proves (16).

Note, too, that R_i, as a homomorphic image of R, satisfies (in particular) conclusion (i) of Theorem 2; that is,

(18) $x_i \in R_i \setminus C_i$ implies that $x_i - x_i^2 \in N_i$.

Combining (16), (18), and Lemma 1, we conclude that R_i is commutative, and hence the ground ring R is commutative (see (13)). This proves the theorem.

Corollary 3. *A Boolean-Like ring with central idempotents and commuting nilpotents is commutative.*

A concept related to commutativity is that of a ring having a nil commutator ideal. In this connection, we have the following:

Theorem 4. *The commutator ideal of a Boolean-Like ring R is nil.*

Proof. First, we show that

(19) The commutator ideal C (R) is contained in J.

To prove this, recall that

(20) $R/J \cong$ *a* subdirect sum of primitive rings R_i ($i \in \Gamma$).

By Theorem 2 (iii), R_i is a Boolean-Like ring . If the primitive ring R_i is a division ring, then for any x_i which is *not* central, we have, by (1),

$$x_i^2 (x_i + 1) - x_i (x_i + 1)^2 = 0,$$

and hence $x_i = 0$ or $x_i = -1$, a contradiction. This contradiction proves that R_i is indeed commutative in this case. Next, suppose that R_i is a primitive ring which is not a division ring. Since (1) is inherited by all subrings and all homomorphic images of R_i, it follows, by Jacobson's density theorem [3; p. 33], that there exists a division ring D and an integer $k > 1$ such that the complete matrix ring D_k satisfies (1). This, however, is false, as can be seen by taking

$$x = E_{11},\ y = E_{12} + E_{21}, \quad (x, y \text{ in } D_k).$$

(To verify this, note that $(x^2y - x y^2)^3 = x^2y - x y^2 \notin N$.)

This contradiction proves that R_i must be a division ring, and hence (as shown above), R_i is commutative. So R / J is commutative, which proves (19).

Our next goal is to show that

(21) $J \subseteq N \cup C$.

Let $j \in J$, $j \notin C$. Then, by Theorem 2 (ii),

$j^q = j^q e$ for some $q \geq 1$ and some $e \in j Z[j]$, $e^2 = e$,

which implies that e = 0. Thus,

$j^q = j^q e = 0$, and hence $j \in N$, proving (21).

Combining (19) and (21), we see that

(22) $C(R) \subseteq J \subseteq N \cup C$.

Next, we prove that

(23) $N \subseteq J$.

Let $a \in N$, $x \in R$, and suppose $a^n = 0$. Let $\bar{x} = x + J \in R / J$.
Since, by (19), R / J is commutative, we have

$(\bar{a}\ \bar{x})^n = (\bar{a})^n (\bar{x})^n = \bar{0}$, and hence

$(a x)^n \in J \subseteq N \cup C$ (by (21)).

Therefore, $(a x)^n \in N$ (which implies that $ax \in N$), or $(a x)^n \in C$. If $(a x)^n \in C$, then

$((a x)^n)^n = (a x)^n (a x)^n \dots (a x)^n = a^n r$, for some $r \in R$,

and hence $((a x)^n)^n = 0$ (since $a^n = 0$). Again, $ax \in N$. It follows that $a x \in N$ for all x in R, and thus $a x$ *is* right quasi regular for all x in R. Hence, $a \in J$, proving (23).
Combining (23) and (21), we get

(24) $N \subseteq J \subseteq N \cup C$.

Next, we prove that

(25) N is an ideal.

Let $a \in$N, x $\in$R. By (24), $a \in$J, x $\in$R, and hence $ax \in$J $\subseteq$ N $\cup$ C (by (24)). So $ax \in$N or $ax \in$C. Of course, if $ax \in$C, then $ax \in$N, and hence $ax \in$N in any case. Similarly, $xa \in$N.

Finally, suppose $a \in$N, b$\in$N. Then, by (24), $a \in$J, b$\in$J, and hence a - b $\in$J $\subseteq$ N $\cup$ C.
So a - b $\in$N, or a - b $\in$C. If a - b $\in$C, then a commutes with b and hence again a - b $\in$N, proving (25).

To complete the proof, since N is an ideal (by (25)), R/N is a Boolean-Like ring (by Theorem 2 (iii)), and hence by Theorem 2 (i) (with R/N playing the role of R),

(26) Every element of R/N is central or potent.

Therefore, by Lemma 2, R/N is commutative, which proves the theorem.

We now turn our attention to strongly Boolean-Like rings. In this connection, we have:

Theorem 5. *Every strongly Boolean-Like ring* R *with identity is commutative.*

Proof. First, we prove that

(27) the set N of nilpotents of R is commutative.

To prove this, let a, $a' \in$N and suppose $[a,a'] \neq 0$. Since $1+a \in$ R\(N $\cup$C) and $1+a' \in$ R\(N $\cup$C), it follows by (2) that

(28) $(1+a)^2(1+a') = (1+a)(1+a')^2$,

and hence $1+a = 1+a'$, which contradicts the hypothesis that $[a,a'] \neq 0$, proving (27).

Next, we prove that all idempotents are central.

To this end, suppose $e^2 = e$, x $\in$R, a = xe – exe. Suppose $a \neq 0$. Then e $\neq$ 0, and hence e $\notin$(N $\cup$C), since e$\notin$C. Also, $1+a \notin$C (since $1+a \in$C implies $a \in$C, and hence $ea = ae$, which yields the contradiction $0 = a$). Moreover, clearly $1+a \notin$N, and thus $1+a \notin$(N $\cup$C). Since e $\notin$(N $\cup$C) and $1+a \notin$(N $\cup$C), it follows by (2) that

$$(1+a)^2 e = (1+a)e^2 ,$$

which implies that $(1+a)$ e = e^2 = e, and hence $ae = 0$; that is, $a = 0$, contradiction. This contradiction proves that $a = 0$, and thus xe = exe. A similar argument shows that ex = exe, and hence

(29) All idempotents are central.

Theorem 5 now follows from (27), (29), and Corollary 3.

Our final theorem is the following:

Theorem 6. *Suppose* R *is a strongly Boolean-Like ring with central idempotents. Then*, R *is isomorphic to a subdirect sum of rings* R_i, *where* R_i *is either nil or commutative*.

Proof. Since R is also a Boolean-Like ring, it follows, by Theorem 2 (ii), that

(30) For all $x \in R \setminus C$, $x^q = x^q e$ for some $q \geq 1$, $e^2 = e \in xZ[x]$.

Write R as a subdirect sum of subdirectly irreducible rings R_i. Since R_i inherits (30) from the ground ring R, we see that

(31) For all $x_i \in R_i \setminus C_i$, $x^q_i = x^q_i e_i$, $q \geq 1$, $e_i^2 = e_i \in x_i Z[x_i]$.

Moreover, since (by hypothesis), e is central in R, e_i is *central* in the subdirectly irreducible ring R_i, which implies that $e_i = 0$ or $e_i = 1$. If R_i, does *not* have an identity then $e_i = 0$, and hence by (31),

(32) $R_i = N_i \cup C_i$, N_i is the set of nilpotents of R_i, C_i is the center of R_i.

It is easy to see that (32) implies that $R_i = N_i$ or $R_i = C_i$ (if R_i, does not have an identity). If, on the other hand, $1 \in R_i$, then R_i is a strongly Boolean-Like ring *with identity*, and hence R_i is commutative, by Theorem 5. This proves the Theorem.

We conclude with the following:

Remark 1: Theorems 3 and 5 are not true if we replace the exponent 2 in (1) or (2) by a prime p > 2. To see this, consider the ring

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} : a, b, c \in GF(4) \right\}.$$

It is readily verified that

(i) $x^7 y = xy^7$ for all $x, y \in R \setminus (N \cup C)$;

(ii) All idempotents of R are central;
(iii) N is commutative;
(iv) $1 \in R$.
But R is *not* commutative.

Remark 2: Theorem 5 is not true if $1 \notin R$, as can be seen by considering the four-element ring of matrices in the introduction.

Related work appears in [4].

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