Conductor–Discriminant Formula for Global Function Fields

Martha Rzedowski–Calderón

Departamento de Control Automático
Centro de Investigación y de Estudios Avanzados del I.P.N.
Apartado Postal 14-740, 07000 México, D. F., MÉXICO
mrzedowski@ctrl.cinvestav.mx

Gabriel Villa–Salvador

Departamento de Control Automático
Centro de Investigación y de Estudios Avanzados del I.P.N.
Apartado Postal 14-740, 07000 México, D. F., MÉXICO
gvilla@ctrl.cinvestav.mx

Abstract

For a finite abelian extension of a rational function field contained in a cyclotomic function field, the discriminant is equal to the product of the conductors of the characters in the group of Dirichlet characters associated to the field. In this paper we give an elementary proof of this result.

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1 Introduction

The conductor–discriminant formula, as its name says, is a formula that relates the conductors of the group of characters attached to the Galois group of an extension of global or local fields with the discriminant of the extension.

It seems that the first one to consider this formula was Dedekind [1], for a finite abelian extension of the field of rational numbers. At the beginning of the 1930’s, E. Artín and H. Hasse found a general conductor–discriminant
formula:
\[ d_{F/E} = \prod_{\chi} f(\chi)^{\chi(1)}, \] (1)

where \( d_{F/E} \) denotes the discriminant of \( F/E \), the product is taken over all irreducible characters \( \chi \) of \( G = \text{Gal}(F/E) \), \( F/E \) is a finite Galois extension of global fields and \( f(\chi) \) denotes the conductor of \( \chi \). Artin–Hasse result is valid even when \( F/E \) is not abelian. When \( F/E \) is abelian (1) reduces to

\[ d_{F/E} = \prod_{\chi} f(\chi). \] (2)

For a complete discussion of (1) see [3, Ch. VI] where a complete proof is given based on Artin representations. See also [2, Teil II, §7].

There are several proofs of (2), the simplest one uses analytic techniques, namely, the functional equation of Dirichlet–Artin \( L \)–functions. For a proof see [6, Theorem 3.11 and Ch. 4].

In this paper we present an elementary proof of (2) in the particular case when \( E \) is a rational congruence function field and \( F \) is a subfield of a cyclotomic function field.

The results and notations that we use for cyclotomic function fields can be found in [4, Ch. 12].

Let \( R_T = \mathbb{F}_q[T] \) be the ring of polynomials in one variable over the field of \( q \) elements \( \mathbb{F}_q \). Let \( k = \mathbb{F}_q(T) \) be the field of quotients of \( R_T \) and consider a monic polynomial \( M \in R_T \). Let \( \Lambda_M \) be the Carlitz–Hayes module of \( M \). Then \( k(\Lambda_M) \) is the cyclotomic function field determined by \( M \). Let \( \mathcal{O}_M \) be the integral closure of \( R_T \) in \( k(\Lambda_M) \). Then \( \mathcal{O}_M = R_T[\lambda_M] \), where \( \lambda_M \) is a generator of the \( R_T \)–module \( \Lambda_M \).

Let \( d_{F/E} \) denote the discriminant of the global function field extension \( F/E \), that is,

\[ d_{F/E} = N_{F/E} D_{F/E} \]

where \( D_{F/E} \) denotes the different of the Dedekind domain extension \( \mathcal{O}_F/\mathcal{O}_E \) and \( N_{F/E} \) denotes the norm map. Here, for any finite function field extension \( K/k \), \( \mathcal{O}_K \) denotes the integral closure of \( R_T \) in \( K \).

Let \( M \in R_T \setminus \{0\} \) be a monic polynomial. A Dirichlet character \( \text{mod}M \) is a group homomorphism \( \chi: (R_T/(M))^* \rightarrow \mathbb{C}^* \). Given a Dirichlet character \( \chi \text{ mod } M \), we say that \( \chi \) can be defined \( \text{mod}N \) for a monic polynomial \( N \in R_T \) such that \( N \mid M \), if there exists a group homomorphism \( \xi: (R_T/(N))^* \rightarrow \mathbb{C}^* \) such that \( \xi \circ \varphi_{M,N} = \chi \), where \( \varphi_{M,N}: \xi: (R_T/(M))^* \rightarrow \xi: (R_T/(N))^* \) is the
natural map $\alpha \mod M \mapsto \alpha \mod N$.

\[
\begin{array}{c}
\xymatrix{(R_T/(M))^* \ar[r]^\chi \ar[d]_{\varphi_{M,N}} & \mathbb{C}^* \ar[l]_{\xi} \\
(R_T/(N))^*}
\end{array}
\]

The conductor of a Dirichlet character $\chi$ is defined as $\mathfrak{f}_\chi$, where $\mathfrak{f}_\chi \in R_T$ is the monic polynomial dividing $M$ of minimal degree such that $\chi$ can be defined mod $\mathfrak{f}_\chi$.

We consider a field $k \subseteq K \subseteq k(\Lambda_M)$. Let $X_K$ be the group of Dirichlet characters associated to $K$, that is, $K = k(\Lambda_M)^J$, where $J = \cap_{\chi \in X_K} \ker \chi$.

The conductor–discriminant formula establishes that

\[
\mathfrak{d}_{K/k} = \prod_{\chi \in X_K} \mathfrak{f}_\chi
\]

where $\mathfrak{f}_\chi$ denotes the conductor of the character $\chi$. In this paper we present an elementary proof of (3) using the theory of cyclotomic function fields.

For an irreducible monic polynomial $P \in R_T$, let $\mathfrak{d}_{K/k}(P) = P^s$, where $s$ is the exact power of $P$ dividing $\mathfrak{d}_{K/k}$. When $M = P^n$ we choose generators $\lambda_{P^n}$ of $\Lambda_{P^n}$ for $0 \leq i \leq n$ such that $\lambda_{P^n}^{P^{s_i}} = \lambda_{P^{n-i}}$.

For any $M \in R_T$ we have that $\lambda_{M}^{A}$ generates $\Lambda_{M}$ as $R_T$–module if and only if $A$ and $M$ are relatively prime. In $k(\Lambda_{P^n})/k$ the only ramified finite prime is $P$ and it is fully ramified.

In general we have $G_M := \text{Gal}(k(\Lambda_M)/k) \cong (R_T/(M))^*$, the group of units of the ring $R_T/(M)$. Therefore, the Dirichlet characters mod$M$ can be considered as the characters of the Galois group $\text{Gal}(k(\Lambda_M)/k)$. We denote by $\Phi(M)$ the cardinality of $(R_T/(M))^*$. We have that if $P$ is an irreducible polynomial of degree $d$, then $\Phi(P^n) = q^{(n-1)d}(q^d-1)$ for $n \geq 1$ and $\Phi(P^0) = 1$. Let $p_N$ be the unique prime divisor in $O_{P^n}$ above $P$. Then $p_N = (\lambda_{P^n}) = \lambda_{P^n}O_{P^n}$ and if $v_{p_N}$ denotes the valuation corresponding to $p_N$, then $v_{p_N}(P^n) = \Phi(P^n)$. In fact $(P) = P\mathcal{O}_{P^n} = (\lambda_{P^n})^{\Phi(P^n)}$. We have that $v_{p_N}(\lambda_{P^n}) = \frac{\Phi(P^n)}{\Phi(P^0)} = q^{(n-i)d}$ for $1 \leq i \leq n$.

For any Dirichlet character $\chi$ defined modulo $M$, $\chi : (R_T/(M))^* \rightarrow \mathbb{C}^*$, the canonical decomposition

\[
(R_T/(M))^* \cong (R_T/(P_{1}^{\alpha_1}))^* \times \cdots \times (R_T/(P_{r}^{\alpha_r}))^*,
\]

where $M = P_1^{\alpha_1} \cdots P_r^{\alpha_r}$ is the decomposition of $M$ as product of powers of different irreducible polynomials $P_1, \ldots, P_r$, gives a decomposition of $\chi$:

\[
\chi = \chi_{P_{1}} \cdots \chi_{P_{r}}, \quad \text{where} \quad \chi_{P_{i}} : (R_T/(P_{i}^{\alpha_i}))^* \rightarrow \mathbb{C}^*.
\]
For any group $G$, $\hat{G}$ denotes the group of characters of $G$. For any group $X$ of Dirichlet characters we denote $X_P = \{ \chi_P \mid \chi \in X \}$. Note that if $X \subseteq \hat{G}_M$ and $P \nmid M$ then $X_P = \{1\}$. If $\sigma, \theta$ are two Dirichlet characters such that $\mathfrak{f}_\sigma$ and $\mathfrak{f}_\theta$ are relatively prime, then $\mathfrak{f}_{\sigma \theta} = \mathfrak{f}_\sigma \mathfrak{f}_\theta$.

We have that if $E \subseteq F \subseteq L$ is a tower of global function fields, then $\mathcal{O}_{L/E} = \mathcal{O}_{L/F} \cap \mathcal{O}_{F/E}$, where $\cap$ is the conorm map. For any algebraic extension $F/E$ and $\alpha \in F$, $\text{Irr}(u, \alpha, E) \in E[u]$ denotes the irreducible polynomial of $\alpha$ over $E$. Finally, for any finite Galois function field extension $K/k$ and for an irreducible polynomial $P \in R_T$, let $\text{con}_{k/K} P = P\mathcal{O}_K = (p_1 \cdots p_h)^e$. Then $efh = [K : k]$ where $f = [\mathcal{O}_K/p_i : R_T/(P)]$, $1 \leq i \leq h$.

2 Case $K \subseteq k(\Lambda P^n)$

In this section we consider a field $K$ such that $k \subseteq K \subseteq k(\Lambda P^n)$, where $P \in R_T$ is a monic irreducible polynomial of degree $d$. Let $X_K$ be the group of Dirichlet characters associated to $K$.

**Theorem 2.1** Under the above conditions, we have $\mathfrak{d}_{K/k} = \prod_{\chi \in X_K} \mathfrak{f}_\chi$.

**Proof:** Let $K_i := K \cap k(\Lambda P^n)$, $i = 0, 1, 2, \ldots, n$. Then $K_n = K$ and $K_0 = k$. We have that a character $\chi$ has conductor $P^j$ if and only if $\chi$ is a character associated to the field $k(\Lambda P^j)$ but not to $k(\Lambda P^{j-1})$. It follows that $X_K$ contains precisely $[K_j : k] - [K_{j-1} : k]$ characters of conductor $P^j$, $1 \leq j \leq n$. Thus $\prod_{\chi \in X_K} \mathfrak{f}_\chi = P^\alpha$ where

$$\alpha = \sum_{j=1}^n j([K_j : k] - [K_{j-1} : k]) = n[K_n : k] - \sum_{j=0}^{n-1} [K_j : k].$$

Therefore

$$\prod_{\chi \in X_K} \mathfrak{f}_\chi = P^\alpha \quad \text{with} \quad \alpha = n[K : k] - \sum_{j=0}^{n-1} [K_j : k]. \quad (4)$$

Now, if we prove that $\mathcal{O}_{K/k} = p_K^\mathfrak{g}_K$, where $p_K := p_n \cap \mathcal{O}_K$, since the relative degree of $p_K$ over $P$ is one, it will follow that $\mathfrak{d}_{K/k} = N_{K/k} \mathcal{O}_{K/k} = P^\alpha$.

Let $\mathcal{D}_{K/k} = p_K^\mathfrak{g}_K$. We have $\mathcal{O}_{P^n} = R_T[\lambda_{P^n}]$ and $\mathcal{O}_{P^n} = \mathcal{O}_K[\lambda_{P^n}]$.

First, we consider the case $K = k(\Lambda P^n)$. In this case we have that $K_i = k(\Lambda P^i)$. Let $G := G_{P^n} \cong (R_T/(P^n))^*$. For $i \geq -1$ let $G_i$ be the $i$-th ramification group of $G$. That is $G_i = \{ \sigma \in G \mid v_{p_n}(\sigma(\lambda_{P^n}) - \lambda_{P^n}) \geq i + 1 \}$ where $A \in R_T$ is a polynomial relatively prime to $P$, $\sigma(\lambda_{P^n}) := \lambda_{P^n}^A$ and $v_{p_n}$ is the valuation associated to $p_n$. We have for $A \not\equiv 1 \mod P^n$, say $A = 1 + P^{\alpha_A} S_A$ with $S_A \in R_T$ relatively prime to $P$ and $0 \leq \alpha_A = v_P(A - 1) \leq n - 1$:

$$i_{\sigma_A} := v_{p_n}(\sigma_A(\lambda_{P^n}) - \lambda_{P^n})) = v_{p_n}(\lambda_{P^n}^{A-1}) = v_{p_n}(\lambda_{P^n}^{P^{\alpha_A} S_A}) = q^{\alpha_A d}. \quad (5)$$
From (5) it follows that the ramification groups satisfy

\[ G_{-1} = G_0 = G, \]
\[ G_i \cong D_{p_n, p}, \quad 1 \leq i \leq q^d - 1, \]
\[ G_i \cong D_{p^n, p^2}, \quad q^d \leq i \leq q^{2d} - 1, \]
\[ \cdots \]
\[ G_i \cong D_{p_{n-1}, p}, \quad q^{(n-2)d} \leq i \leq q^{(n-1)d} - 1, \]
\[ G_i = \{1\} \quad \text{for} \quad i \geq q^{(n-1)d}, \tag{6} \]

where

\[ D_{p_n, p} := \{ A \mod P_n | A \in RT \text{ relatively prime to } P \text{ and } A \equiv 1 \mod P^j \} \]
\[ \cong \text{Gal}(k(\Lambda_{p_n})/k(\Lambda_{p_j})), \quad 1 \leq j \leq n - 1. \]

It follows by Hilbert’s formula ([3, Ch. IV, Proposition 4]) that \( \mathfrak{D}_{k(\Lambda_{p_n})/k} = p^n_0 \) where

\[ \beta = \sum_{j=0}^{\infty} (|G_j| - 1) = (|G| - 1) + \sum_{j=1}^{n-1} (q^{jd} - q^{(j-1)d})(|D_{p_n, p_j}| - 1) \]
\[ = [k(\Lambda_{p_n}) : k] - 1 + \sum_{j=1}^{n-1} [k(\Lambda_{p_j}) : k][k(\Lambda_{p_n})] - 1) \]
\[ = n[k(\Lambda_{p_n}) : k] - \sum_{j=0}^{n-1} [k(\Lambda_{p_j}) : k] = \alpha. \tag{7} \]

This proves the theorem for the case \( K = k(\Lambda_{p_n}). \)

We observe that the different \( \mathfrak{D}_{k(\Lambda_{p_n})/k} \) is well known. In fact it appears in the original paper of Hayes, see [4, Proposition 12.7.1]. We include the previous computation since we are going to use ramification groups also for any subfield \( k \subseteq K \subseteq k(\Lambda_{p_n}). \)

Now let \( K \) be such a field. We have ([3, Ch. IV, Proposition 2]) that for any subgroup \( H < G \) and every \( \sigma \in H, \ i_H(\sigma) = i_G(\sigma) \) and \( H_i = G_i \cap H. \) Let \( H := \text{Gal}(k(\Lambda_{p_n}/K)) \) and \( \mathfrak{D}_{k(\Lambda_{p_n})/K} = p_n^5. \) We have \( G_i \cong D_{p_n, p_{r_i}} \) for some \( r_i. \) Then \( H_i = G_i \cap H = \text{Gal}(k(\Lambda_{p_n}/Kk(\Lambda_{p_{r_i}}))). \) Therefore

\[ \delta = \sum_{\sigma \in H \setminus \{1\}} i_H(\sigma) = \sum_{\sigma \in H \setminus \{1\}} i_G(\sigma) = \sum_{j=0}^{\infty} (|H_j| - 1) = \sum_{j=0}^{\infty} (|G_j \cap H| - 1) \]
\[ = ([k(\Lambda_{p_n}) : K] - 1) + \sum_{j=1}^{n-1} [k(\Lambda_{p_j}) : k][k(\Lambda_{p_n})/Kk(\Lambda_{p_j})] - 1). \tag{8} \]
Since $\mathcal{O}_{k(\Lambda P_n)/k} = \mathcal{O}_{k(\Lambda P_n)/K \cap K/k(\Lambda P_n)} \mathcal{O}_{K/k}$ and $p_K$ is totally ramified in $k(\Lambda P_n)/K$, we obtain from (7) and (8) that

$$\gamma = \frac{\beta - \delta}{[k(\Lambda P_n) : K]}$$

$$= \frac{1}{[k(\Lambda P_n) : K]} \left( ([k(\Lambda P_n) : k] - [k(\Lambda P_n) : K])
+ \sum_{j=1}^{n-1} [k(\Lambda P_j) : k]([k(\Lambda P_n) : k(\Lambda P_j)] - [k(\Lambda P_n) : Kk(\Lambda P_j)]) \right)$$

$$= ([K : k] - 1) + \frac{1}{[k(\Lambda P_n) : K]} \left( \sum_{j=1}^{n-1} ([k(\Lambda P_n) : k] - [k(\Lambda P_n) : Kk(\Lambda P_j)][k(\Lambda P_j) : k]) \right).$$

Therefore

$$\gamma = [K : k] - 1 + \sum_{j=1}^{n-1} \left( [K : k] - \frac{[k(\Lambda P_j) : k]}{Kk(\Lambda P_j) : K} \right)$$

$$= n[K : k] - \sum_{j=0}^{n-1} \frac{[k(\Lambda P_j) : k]}{Kk(\Lambda P_j) : K}. \quad (9)$$

We have

$$k(\Lambda P_j) \quad Kk(\Lambda P_j) \quad [Kk(\Lambda P_j) : K] = [k(\Lambda P_j) : K_j].$$

$$K_j = K \cap k(\Lambda P_j) \quad \quad \quad \quad \quad K$$

Thus

$$\frac{[k(\Lambda P_j) : k]}{[Kk(\Lambda P_j) : K]} = \frac{[k(\Lambda P_j) : k]}{[k(\Lambda P_j) : K_j]} = [K_j : k]. \quad (10)$$

It follows from (4), (9) and (10) that

$$\gamma = n[K : k] - \sum_{j=0}^{n-1} [K_j : k] = \alpha.$$  

This proves the theorem.
3 General case

The proof in the general case is similar to that of the number field case [5]. We present it for the sake of completeness. First we reproduce Theorem 12.6.36 of [4] since we use it for the proof of our main result.

**Theorem 3.1** Let $X$ be a finite group of Dirichlet characters and $K_X$ the field associated to $X$. Let $P \in R_T$ be a monic irreducible polynomial and $p$ a prime divisor in $K_X$ above $P$. Then the ramification index of $p$ over $P$ is $e = e_P(K_X/k) = e(p|P) = |X_P|$.

**Proof:** Let $M$ be the least common multiple of $\{\mathfrak{f}_\chi \mid \chi \in X\}$. Then $K_X \subseteq k(\Lambda_M)$. Let $M = P^aA$ with $A$ relatively prime to $P$. Set $L := K_X(\Lambda_A) = K_Xk(\Lambda_A)$. Then we have the following diagram, where the ramification refers to $P$ or prime divisors above it.

Let $G_A = \text{Gal}(k(\Lambda_A)/k)$. Then $L = K_Xk(\Lambda_A) = K_XK_{\hat{G}_A} = K_{\langle X, \hat{G}_A \rangle}$. That is, $L$ is the field associated to the group $\langle X, \hat{G}_A \rangle$. Equivalently, the group of Dirichlet characters associated to $L$ is generated by $X$ and those characters of $G_M$ whose conductor is relatively prime to $P$. It follows that $\langle X, \hat{G}_A \rangle \cong X_P \times \hat{G}_A$. We have $K_{X_P} \subseteq k(\Lambda_{P^a})$ and $L = K_{X_P}k(\Lambda_A)$ with $K_{X_P} \cap k(\Lambda_A) = k$.

Since $P$ is unramified in $k(\Lambda_A)/k$, the ramification index of $P$ in $K_X/k$ is the same as the one in $L/k$. We also have that $L/K_{X_P}$ is unramified at the prime divisors above $P$ and that $P$ is totally ramified in $K_{X_P}/k$. It follows that $e = [K_{X_P} : k] = |X_P|$.

We are ready to prove our main result.

**Theorem 3.2** (conductor–discriminant formula) Let $X$ be a group of characters of Dirichlet and let $K_X$ be the field associated to $X$. Then $\mathfrak{d}_{K_X/k} = \prod_{\chi \in X} \mathfrak{f}_\chi$. 
Theorem 3.1 we have
\[ \mathfrak{d}_{L/k} = \mathfrak{d}_{L/K_X} \cap_{\mathfrak{d}_{K_X/L}} \mathfrak{d}_{K_X/k} = \mathfrak{d}_{L/K_X^P} \cap_{\mathfrak{d}_{K_X^P/L}} \mathfrak{d}_{K_X^P/k}, \]
\[ \mathfrak{d}_{L/k}(P) = (N_{L/k}(\mathfrak{d}_{L/K_X^P}))(P) \cdot \mathfrak{d}_{K_X^P/k}(P). \]

Since \( P \) is unramified in \( L/K_X \) and in \( L/K_X^P \) we have
\[ (N_{L/k}(\mathfrak{d}_{L/K_X^P}))(P) = (N_{L/k}(\mathfrak{d}_{L/K_X^P}))(P) = 1. \]

Therefore
\[ \mathfrak{d}_{K_X^P/k}(P) = \left( \mathfrak{d}_{K_X^P/k}(P) \right)^{[L:K_X^P]/[L:K_X]}. \]

We have \( [L : K_X^P] = [k(\Lambda_A) : k] = \Phi(A) \) and \( [L : K_X] = [Y : X] \) where \( Y \) is the group of Dirichlet characters associated to \( L \), that is, \( Y = X_P \times \hat{G}_A \). It follows that
\[ [L : K_X] = \frac{|Y|}{|X|} = \frac{|X_P||\hat{G}_A|}{|X|} = \frac{|X_P|\Phi(A)}{|X|} \]
and
\[ \frac{[L : K_X^P]}{[L : K_X]} = \frac{\Phi(A)}{|X_P|\Phi(A)|X|} = \frac{|X|}{|X_P|} = \frac{|K_X : k|}{e} = \frac{eh}{e} = fh. \]

Since \( K_X^P \subseteq k(\Lambda_{A^e}) \), from Theorem 2.1 we obtain that \( \mathfrak{d}_{K_X^P/k}(P) = \mathfrak{d}_{K_X^P/k} = \prod_{\varphi \in X_P} \mathfrak{F}_\varphi \). Hence
\[ \mathfrak{d}_{K_X^P/k}(P) = \left( \prod_{\varphi \in X_P} \mathfrak{F}_\varphi \right)^{([L:K_X^P]/[L:K_X])} = \left( \prod_{\varphi \in X_P} \mathfrak{F}_\varphi \right)^{fh} = \prod_{\varphi \in X_P} \mathfrak{F}_\varphi^{fh}. \]

From the natural epimorphism \( \pi : X \to X_P, \chi \mapsto \chi_P \) we obtain that \( |\ker \pi| = \frac{|X|}{|X_P|} = fh \). Therefore, for each \( \varphi \in X_P \), \( |\pi^{-1}(\varphi)| = fh \). It follows that for each \( \varphi \in X_P \) there exist precisely \( fh \) elements \( \chi \in X \) such that \( \pi(\chi) = \chi_P = \varphi \). Thus
\[ \mathfrak{d}_{K_X^P/k}(P) = \prod_{\varphi \in X_P} \mathfrak{F}_\varphi^{fh} = \prod_{\chi \in X} \mathfrak{F}_\chi. \]

Finally, we have that \( \mathfrak{F}_\chi = \prod_{\varphi \in X_P} \mathfrak{F}_{\chi_P} \) for \( \chi \in X \) and \( \mathfrak{d}_{K_X^P/k} = \prod_{\varphi \in X_P} \mathfrak{F}_\varphi^{fh} \).

Therefore \( \mathfrak{d}_{K_X^P/k} = \prod_{\chi \in X} \mathfrak{F}_\chi. \)

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References


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