On \(n\)-Strongly Gorenstein Rings

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Abstract

This paper introduces and studies a particular subclass of the class of commutative rings with finite Gorenstein global dimension.

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1 Introduction

Throughout the paper, all rings are commutative with identity, and all modules are unitary.

Let \(R\) be a ring, and let \(M\) be an \(R\)-module. As usual, we use \(\text{pd}_R(M)\), \(\text{id}_R(M)\), and \(\text{fd}_R(M)\) to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of \(M\).

For a two-sided Noetherian ring \(R\), Auslander and Bridger [1] introduced the \(G\)-dimension, \(\text{Gdim}_R(M)\), for every finitely generated \(R\)-module \(M\). They showed that \(\text{Gdim}_R(M) \leq \text{pd}_R(M)\) for all finitely generated \(R\)-modules \(M\), and equality holds if \(\text{pd}_R(M)\) is finite.

Several decades later, Enochs and Jenda [8, 9] introduced the notion of Gorenstein projective dimension (\(G\)-projective dimension for short), as an extension of \(G\)-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (\(G\)-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [11] introduced the Gorenstein flat dimension. Some references are [3, 6, 7, 8, 9, 11, 12].
Recall that an $R$-module $M$ is called Gorenstein projective, if there exists an exact sequence of projective $R$-modules:

$$P : \ldots \to P_1 \to P_0 \to P^0 \to P^1 \to \ldots$$

such that $M \cong \text{Im}(P_0 \to P^0)$ and such that the functor $\text{Hom}_R(\cdot, Q)$ leaves $P$ exact whenever $Q$ is a projective $R$-module. The complex $P$ is called a complete projective resolution. The Gorenstein injective $R$-modules are defined dually.

The Gorenstein projective and injective dimensions are defined in terms of resolutions and denoted by $\text{Gpd}(\cdot)$ and $\text{Gid}(\cdot)$, respectively ([6, 10, 12]).

In [3], the authors proved, for any associative ring $R$, the equality

$$\sup \{ \text{Gpd}_R(M) \mid M \text{ is a (left) } R\text{-module} \} = \sup \{ \text{Gid}_R(M) \mid M \text{ is a (left) } R\text{-module} \}.$$  

They called the common value of the above quantities the left Gorenstein global dimension of $R$ and denoted it by $l.Ggldim(R)$. Since in this paper all rings are commutative, we drop the letter $l$.

Recently, in [15], particular modules of finite Gorenstein projective, injective, and flat dimensions are defined as follows:

**Definitions 1.1.** Let $n$ be a positive integer.

1. An $R$-module $M$ is said to be strongly $n$-Gorenstein projective, if there exists a short exact sequence of $R$-modules $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$ where $\text{pd}_R(P) \leq n$ and $\text{Ext}^{n+1}_R(M, Q) = 0$ whenever $Q$ is projective.

2. An $R$-module $M$ is said to be strongly $n$-Gorenstein injective, if there exists a short exact sequence of $R$-modules $0 \longrightarrow M \longrightarrow I \longrightarrow M \longrightarrow 0$ where $\text{id}_R(I) \leq n$ and $\text{Ext}^{n+1}_R(E, M) = 0$ whenever $E$ is injective.

Clearly, strongly 0-Gorenstein projective and injective are just the strongly Gorenstein projective, injective, and flat modules, respectively ([3, Propositions 2.9 and 3.6]).

In this paper, we investigate these modules to characterize a new class of rings with finite Gorenstein global dimension, which we call $n$-strongly Gorenstein rings.

## 2 $n$-Strongly Gorenstein rings

In [15], the authors proved the following proposition:
Proposition 2.1 (Proposition 2.16, [15]). Let \( R \) be a ring. The following statements are equivalent:

1. Every module is strongly \( n \)-Gorenstein projective.
2. Every module is strongly \( n \)-Gorenstein injective.

Thus, we give the following definition:

Definition 2.2. Let \( n \) be a positive integer. A ring \( R \) is called \( n \)-strongly Gorenstein (\( n \)-SG ring for short), if \( R \) satisfies one of the equivalent conditions of Proposition 2.1.

The 0-SG rings and 1-SG rings are already studied in [5, 14] and they are called strongly Gorenstein semi-simple rings and strongly Gorenstein hereditary rings, respectively. Clearly, by definition, every \( n \)-SG ring is \( m \)-SG whenever \( n \leq m \).

Our first result gives a characterization of strongly \( n \)-Gorenstein rings.

Proposition 2.3. For a ring \( R \) and a positive integer \( n \), the following statements are equivalent:

1. \( R \) is an \( n \)-SG ring.
2. \( \text{Ggldim}(R) \leq n \) and for every \( R \)-module \( M \) there exists a short exact sequence of \( R \)-modules

\[
0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0
\]

where \( \text{pd}_R(P) < \infty \).
3. \( \text{Ggldim}(R) < \infty \) and for every \( R \)-module \( M \) there exists a short exact sequence of \( R \)-modules

\[
0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0
\]

where \( \text{pd}_R(P) \leq n \).

Proof. (1 \( \Rightarrow \) 2) Clear since for every \( n \)-SG ring \( R \) we have \( \text{Ggldim}(R) \leq n \) (by [15, Proposition 2.2(1)]).
(2 \( \Rightarrow \) 3) Follows directly from [3, Corollary 2.7].
(3 \( \Rightarrow \) 1) Follows from [15, Proposition 2.10].

The next result studies the direct product of \( n \)-SG rings.

Theorem 2.4. Let \( \{R_i\}_{i=1}^m \) be a family of rings and set \( R := \prod_{i=1}^m R_i \). Then, \( R \) is an \( n \)-SG ring if and only if \( R_i \) is an \( n \)-SG ring for each \( i = 1, \ldots, m \).
Proof. By induction on $m$ it suffices to prove the assertion for $m = 2$. First suppose that $R_1 \times R_2$ is an $n$-SG ring. We claim that $R_1$ is an $n$-SG ring. Let $M$ be an arbitrary $R_1$ module. $M \times 0$ can be viewed as an $R_1 \times R_2$-module. For such module and since $R_1 \times R_2$ is an $n$-SG ring, there is an exact sequence $0 \rightarrow M \times 0 \rightarrow P \rightarrow M \times 0 \rightarrow 0$ where $\text{pd}_{R_1 \times R_2}(P) \leq n$. Thus, since $R_1$ is a projective $R_1 \times R_2$ module, by applying $- \otimes_{R_1 \times R_2} R_1$ to the sequence above, we find the short exact sequence of $R$-modules: $0 \rightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \rightarrow P \otimes_{R_1 \times R_2} R_1 \rightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \rightarrow 0$. Clearly $\text{pd}_{R_1}(P \otimes_{R_1 \times R_2} R_1) \leq \text{pd}_{R_1 \times R_2}(P) \leq n$. Moreover, we have the isomorphism of $R$-modules:

$$M \times 0 \otimes_{R_1 \times R_2} R_1 \cong M \times 0 \otimes_{R_1 \times R_2} (R_1 \times R_2)/(0 \times R_2) \cong M.$$ 

Thus, we obtain an exact sequence of $R$-module with the form: $0 \rightarrow M \rightarrow P \otimes_{R_1 \times R_2} R_1 \rightarrow M \rightarrow 0$. On the other hand, by [4, Theorem 3.1], we have $\text{Ggldim}(R_1) \leq \text{Ggldim}(R_1 \times R_2) \leq n$. Thus, using Proposition 2.3, $R_1$ is an $n$-SG ring, as desired. By the same argument, $R_2$ is also an $n$-SG ring.

Now, suppose that $R_1$ and $R_2$ are $n$-SG rings and we claim that $R_1 \times R_2$ is an $n$-SG ring. Let $M$ be an arbitrary $R_1 \times R_2$-module. We have

$$M \cong M \otimes_{R_1 \times R_2} (R_1 \times R_2) \cong M \otimes_{R_1 \times R_2} ((R_1 \times 0) \oplus (R_2 \times 0)) \cong M_1 \times M_2$$

where $M_i = M \otimes_{R_1 \times R_2} R_i$ for $i = 1, 2$. For each $i = 1, 2$, there is an exact sequence $0 \rightarrow M_i \rightarrow P_i \rightarrow M_i \rightarrow 0$ where $\text{pd}_{R_i}(P_i) \leq n$ since $R_i$ is an $n$-SG ring. Thus, we have the exact sequence of $R_1 \times R_2$-modules:

$$0 \rightarrow M_1 \times M_2 \rightarrow P_1 \times P_2 \rightarrow M_1 \times M_2 \rightarrow 0.$$ 

On the other hand, $\text{pd}_{R_1 \times R_2}(P_1 \times P_2) = \sup \{\text{pd}_{R_i}(P_i)\}_{1,2} \leq n$ (by [13, Lemma 2.5 (2)]). Moreover, by [4, Theorem 3.1], $\text{Ggldim}(R_1 \times R_2) = \sup \{\text{Ggldim}(R_i)\}_{1,2} \leq n$. Thus, from Proposition 2.3, $R_1 \times R_2$ is an $n$-SG ring, as desired.

Let $T := R[X_1, X_2, \ldots, X_n]$ be the polynomial ring in $n$ indeterminates over $R$. If we suppose that $T$ is an $m$-SG ring, it is easy, by [4, Theorem 2.1], to see that $n \leq m$.

**Theorem 2.5.** If $R[X_1, X_2, \ldots, X_n]$ is an $m$-SG ring then $R$ is an $(m - n)$-SG ring.

Proof. By induction on $n$ is suffices to prove the result for $n = 1$. So, suppose that $R[X]$ is an $m$-SG ring. Let $M$ be an arbitrary $R$-module. For the $R[X]$-module $M[X] := M \otimes_R R[X]$ there is an exact sequence of $R[X]$-modules $0 \rightarrow M[X] \rightarrow P \rightarrow M[X] \rightarrow 0$ where $\text{pd}_{R[X]}(P) \leq m$. Applying $- \otimes_{R[X]} R$ to the short exact sequence above and seeing that $M \cong_R M[X] \otimes_{R[X]} R$, we
obtain a short exact sequence of $R$-modules with the form $0 \rightarrow M \rightarrow P \otimes_{R[X]} R \rightarrow M \rightarrow 0$ (see that $R$ is a projective $R[X]$-module). Moreover, $\text{pd}_{R[X]}(P) \leq \text{pd}_{R}(P) < \infty$. On the other hand, by [4, Theorem 2.1], $\text{Ggldim}(R) = \text{Ggldim}(R[X]) - 1 \leq m - 1$. Hence, by Proposition 2.3, $R$ is an $(m - 1)$-SG ring.

Trivial examples of $n$-SG-ring are the rings with global dimension $\leq n$. The following example gives a new family of $n$-SG rings with infinite weak global dimension.

**Example 2.6.** Consider the non semi-simple quasi-Frobenius rings $R_1 := K[X]/(X^2)$ and $R_2 := K[X]/(X^3)$ where $K$ is a field, and let $S$ be a non Noetherian ring such that $\text{gldim}(S) = n$. Then,

1. $\text{Ggldim}(R_1) = \text{Ggldim}(R_2) = 0$ and $R_1$ is 0-SG ring but $R_2$ is not.
2. $R_1 \times S$ is a non Noetherian $n$-SG ring with infinite weak global dimension.
3. $\text{Ggldim}(R_2 \times S) = n$ but $R_2 \times S$ is not an $n$-SG ring.

**Proof.** From [5, Corollary 3.9] and [3, Proposition 2.6], $\text{Ggldim}(R_1) = \text{Ggldim}(R_2) = 0$ and $R_1$ is 0-SG ring but $R_2$ is not. So, (1) is clear. Moreover $R_1$ and $R_2$ have infinite weak global dimensions. By [4, Theorems 2.1] and Theorem 2.4, it is easy to see that, $\text{Ggldim}(R_2 \times S) = n$ and that $R_2 \times S$ is not an $n$-SG ring.

**References**


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