Finite $p$-Groups which Have a Maximal Subgroup is Full-Normal ($p > 2$)

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Abstract

Let $G$ be a finite $p$-group, $M$ is a subgroup of $G$. $M$ is called full-normal if for any subgroup $K$ of $M$, we have $K \trianglelefteq G$. In this paper, we determine the structure of finite $p$-groups which have a maximal subgroup is full-normal ($p > 2$).

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1 Introduction

As is well known, Dedekind group is very important in finite group theory. There are already a great many research papers related to it, see[1, 2, 3, 5, 6]. This paper is devoted to the structure of a group similar to Dedekind group. Firstly, we introduce a new subgroup concept: Let $G$ be a finite $p$-group, $M$ is a subgroup of $G$. $M$ is called full-normal if for any subgroup $K$ of $M$, we have $K \trianglelefteq G$. In this paper, we consider the structure of $G$ that $M$ is a maximal subgroup ($p > 2$).

The next section provides some preliminaries.

2 Preliminaries

Definition 2.1 Let $G$ be a finite $p$-group. $G$ is said to be Dedekind if all its subgroups are normal.

Definition 2.2 A group $G$ is said to be homocyclic if it is a direct product of isomorphic cyclic subgroups.
Definition 2.3 Let $G$ be a finite $p$-group. We define $\Omega_s(G) = \langle a \in G \mid a^p^s = 1 \rangle$, $\Omega_0(G) = \langle a^p^s \mid a \in G \rangle$.
Furthermore, the Frattini subgroup $\Phi(G)$ of $G$ is equal to $G'\Omega_1(G)$.

Theorem 2.4 ([4]) Let $G$ be a finite Dedekind $p$-group.

1. If $p > 2$, then $G$ is abelian.
2. If $p = 2$, then $G$ is abelian or $G \cong Q_8 \times C_n^2$, which $n$ is a positive integer.

Theorem 2.5 ([7]) Let $G$ be a finite $p$-group. If $c(G) < p$, then $G$ is regular.

Theorem 2.6 ([7]) Let $G$ be a regular $p$-group. Then $G$ is $p$-abelian if and only if $\Omega_1(G') = 1$.

Lemma 2.7 ([7]) Let $G$ be a finite $p$-group. If $G$ is $p$-abelian, then $\Omega_1(G) \leq Z(G)$.

Theorem 2.8 ([7]) Let $G$ be a finite group and $G = \langle M \rangle$, then

1. $G_n = \langle [x_1, \ldots, x_n]_g \mid x_i \in M, g \in G \rangle$;
2. $G_n = \langle [x_1, \ldots, x_n], G_{n+1} \mid x_i \in M \rangle$;
Furthermore, if $G = \langle a, b \rangle$, then
3. $G_2 = G' = \langle [a, b]_g \mid g \in G \rangle$;
4. $G_2 = G' = \langle [a, b], G_3 \rangle$, so $G'/G_3$ is cyclic.

Corollary 2.9 ([8]) Let $A$ be a finite abelian $p$-group and $G$ is noncyclic. $a$ is a maximal order element of $A$, then exists $B \leq A$ such that $A = \langle a \rangle \times B$.

Lemma 2.10 ([7]) Let $G$ be a finite $p$-group, $c(G) = 2$. Then for any $x, y, z \in G$, we have

1. $[xy, z] = [x, z][y, z], [x, yz] = [x, y][x, z]$.
2. $[x^n, y] = [x, y^n]$.
3. $(xy)^n = x^n y^n [y, x]^{(n)}$.

Lemma 2.11 ([7]) Let $G$ be a metabelian group and $a, b$ the elements of $G$. Then for any positive integer $m \geq 2$,

$$(ab^{-1})^m = a^m \prod_{i+j \leq m} [ia, jb](i+j)^m b^{-m}.$$
3 Main Results

Let $G$ be a finite $p$-group $(p > 2)$, $M$ is a maximal subgroup of $G$ and $M$ is full-normal, then $M$ is Dedekind. Hence, by Theorem 2.4, $M$ is abelian.

Firstly, we have the following lemma:

**Lemma 3.1** Let $G$ be a finite $p$-group, $M \subseteq G$. If $G/M \cong C_{p^e}$, then exists $a \in G \setminus M$ such that $G = M\langle a \rangle$. Furthermore, $M$ is full-normal if and only if $M$ is abelian and exists some integer $k$ such that for any $x \in M$, we have $x^a = x^k$.

Proof. Note that $G/M \cong C_{p^e}$, so we assume that $G/M = \langle \bar{a} \rangle$. Hence $G = M\langle a \rangle$, which $a \in G \setminus M$ and $o(a) = p^e$.

$\implies$: Since $M$ is full-normal, $M$ is Dedekind group. Note that $p > 2$, by Theorem 2.4, we have $M$ is abelian. It follows that we may assume that \{x_1, x_2, \ldots, x_r\} is a basis of $M$. Since $M$ is full-normal, $\langle x_1 x_2 \cdots x_r \rangle \subseteq G$. Hence there exists some integer $k$ such that $(x_1 x_2 \cdots x_r)^a = (x_1 x_2 \cdots x_r)^k$. So $x_1^k x_2^k \cdots x_r^k = x_1^k x_2^k \cdots x_r^k$. Thus $x_i^k = x_i^k$, $i = 1, 2, \ldots, r$.

Next, for any $x \in M$, without loss of generality, we may assume that $x = x_1 x_2 \cdots x_r$. Obviously, $x^a = (x_1^a x_2^a \cdots x_r^a) = (x_1^a x_2^a \cdots x_r^a)^a = x_1 x_2 \cdots x_r = x_1 x_2 \cdots x_r = \cdots x_1 x_2 \cdots x_r = x^k$.

$\impliedby$: Note that $G = M\langle a \rangle$, for any $g \in G$, we may assume that $g = m a^j$ where $m \in M$. Let $H \leq M$. For any $h \in H$, note that $M$ is abelian, we have $h^g = h^m a^j = h^{a^j} = h^{k^j} \in H$. It follows that $H^g \subseteq H$. So $H \leq G$. i.e., $M$ is full-normal.

**Lemma 3.2** Let $G$ be a finite nonabelian $p$-group, $M \triangleleft G$ and $\exp M = p^e, p > 2$. If $M$ is full-normal, then we may assume that $k = 1 + p^{e-1}$ in Lemma 3.1.

Proof. Obviously, for minimal nonnegative simplified residue system module $p^e$, there exists $l$ such that $k \equiv l \pmod{p^e}$. Since $\exp M = p^e$, we have $x^k = x^l$ for any $x \in M$. Hence by Lemma 3.1 we have $x^a = x^l$. Thus $x^{ap} = x^{lp}$. Since $G/M \cong C_{p^e}$, we may let $G = M\langle a \rangle$ where $a^p \in M$. Note that $M$ is abelian and $a^p \in M$, we have $x = x^{a^p} = x^{lp}$. So $lp \equiv 1 \pmod{p^e}$.

We claim that $e \geq 2$. If not, we have $e = 1$. Hence $lp \equiv 1 \pmod{p}$, $a^p \equiv 1 \pmod{p}$. Note that $|Z_p^*| = p - 1$, we have $lp^{-1} \equiv 1 \pmod{p}$. So $l = 1$. i.e., for any $x \in M$, $x^{a^p} = x^{a^i} x^{a^j}$. It follows that $G$ is abelian, a contradiction. Since $l \in Z_{\varphi(p^e)}$ and $lp \equiv 1 \pmod{p^e}$, $lp^{-1} = 1$. Furthermore, since $(l + p^{e-1})$ is a unique subgroup of order $p$, we have $\bar{l} = \bar{l} + p^{e-1}$ where $(m, p) = 1$. Note that $l$ belongs to minimal nonnegative simplified residue system module $p^e$, we have $l = 1 + mp^{e-1}$. Since $(m, p) = 1$, we may find integers $i$ and $j$ such that $mi + jp = 1$. Letting $a' = a^i$, then $G = M\langle a' \rangle$ and $x^{a'} = x^{(1+mp^{e-1})i} = x^{1+mp^{e-1}} = x^{1+lp^{e-1}}$. 


Theorem 3.3 Let $G$ be a finite nonabelian $p$-group, $p > 2$, $M\triangleleft G$, exp $M = p^e$. If $M$ is full-normal, then:

1. $M$ is abelian, exp $M \geq p^2$ and $G$ is metabelian;
2. There exists $a \in G\setminus M$ such that $G = M \langle a \rangle$, and for any $x \in M$, we have $x^a = x^{1+p^{e-1}}$ and $a^p \in \Omega_{e-1}(M)$.

Proof. (1) By Lemma 3.1 and Lemma 3.2, $M$ is abelian and $e \geq 2$. Hence exp $M \geq p^2$. Next, since $M$ is abelian and $G' \leq \Phi(G) \leq M$, $G'$ is abelian. Thus $G$ is metabelian.

(2) By Lemma 3.1 and Lemma 3.2, there exists $a \in G\setminus M$ such that $G = M \langle a \rangle$ and for any $x \in M$, we have $x^a = x^{1+p^{e-1}}$. Since $a^p \in M$, $(a^p)^a = a^{p(1+p^{e-1})}$. Note that $(a^p)^a = a^p$, we have $a^p = a^{p(1+p^{e-1})}$. So $a^{p^e} = 1$. Thus $a^p \in \Omega_{e-1}(M)$.

Corollary 3.4 Assume that $G$, $M$ such as Theorem 3.3 supposed. If $M = A \times B$, $A$ is homcyclic, exp $A = p^e$ and exp $B \leq p^{e-1}$, then exists $a \in G\setminus M$ such that $G = M \langle a \rangle$, and satisfies for any $x \in A$, $y \in B$, we have $x^a = x^{1+p^{e-1}}$, $y^a = y$ and $a^p = 1$ or $a^p \in B \setminus \mathcal{U}_1(B)$.

Proof. By Theorem 3.3, there exists $a \in G\setminus M$ such that $G = M \langle a \rangle$ and $a^p \in \Omega_{e-1}(M)$.

(1) Claim that $a^p \in B$

Note that exp $A = p^e$ and exp $B \leq p^{e-1}$. So, by Theorem 3.3, we have $x^a = x^{1+p^{e-1}}$, $y^a = y^{1+p^{e-1}} = y$ for any $x \in A$, $y \in B$. Since $M$ is abelian, exp $A = p^e$ and exp $B \leq p^{e-1}$, we have $\Omega_{e-1}(M) = \mathcal{U}_1(A) \times B$. Note that $a^p \in \Omega_{e-1}(M)$, we may assume that $a^p = x_1^p y_1$ where $x_1 \in A$, $y_1 \in B$. Since $G$ is metabelian, by Lemma 2.11, we have $(x_1^{-1})^p = a^p x_1^{-p} [a,x_1]^{(\frac{1}{p})} = a^p x_1^{-p}$. Replacing $a$ by $ax_1^{-1}$, we have $a^p = y_1 \in B$ and $G = M \langle a \rangle$. Furthermore, for any $x \in A$, $y \in B$, we have $x^a = x^{1+p^{e-1}}$, $y^a = y$.

(2) $a^p = 1$ or $a^p \in B \setminus \mathcal{U}_1(B)$;

If $a^p \notin B \setminus \mathcal{U}_1(B)$, then $a^p \in \mathcal{U}_1(B)$. Hence exists $y_1 \in B$ such that $a^p = y_1^p$. Since $[a,y_1] = 1$, $(ay_1^{-1})^p = 1$. Replacing $a$ by $ay_1^{-1}$, we have $a^p = 1$, $G = M \langle a \rangle$ and for any $x \in A$, $y \in B$, we have $x^a = x^{1+p^{e-1}}$, $y^a = y$.

Corollary 3.5 Assume that $G$, $M$, $A$, $B$ such as Corollary 3.4 supposed. If $M$ is full-normal, then $c(G) = 2$, $G$ is regular, $p$-abelian and $\mathcal{U}_1(G) \leq Z(G)$.

Proof. Since $A$ is homcyclic, without loss of generality, we may assume that $A = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle$, $o(x_i) = p^e$, $e \geq 2$.

Since for any $y \in B$, we have $[x_i, y] = [y,a] = 1$. So $G' = \langle [x_i,a]^g \mid \forall g \in G \rangle$. Note that $G = M \langle a \rangle$, thus for any $g \in G$, let $g = ma^i$ which $m \in M$. By computation, we have $[x_i,a]^g = (x_i^{p^{e-1}})^g = (x_i^{p^{e-1}})^ma^i = x_i^{p^{e-1}} = [x_i,a]$. Hence $G' = \langle x_1^{p^{e-1}}, x_2^{p^{e-1}}, \cdots, x_k^{p^{e-1}} \rangle$. Since $\langle x_i^{p^{e-1}} \rangle \leq G$ and $|\langle x_i^{p^{e-1}} \rangle| = p$,
\( \langle x_i^{p^{e-1}} \rangle \leq Z(G) \). So \( G' \leq Z(G) \). Hence \( c(G) = 2 < p \). Thus \( G \) is regular by Theorem 2.5.

Since \( \exp G' = p \), \( \mathcal{U}_1(G') = 1 \). Hence by Theorem 2.6, \( G \) is \( p \)-abelian. Furthermore, by Lemma 2.7, we have \( \mathcal{U}_1(G) \leq Z(G) \).

**Corollary 3.6** Let \( G \) be a finite nonabelian \( p \)-group, \( p > 2 \) and have a maximal subgroup is full-normal. Then the number of full-normal maximal subgroup of \( G \) is 1 or \( p \).

Proof. Assume that \( G, M, A, B \) such as Corollary 3.4 supposed. Then we distinguish the following two cases.

(1) \( d(A) > 1 \)

Without loss of generality, we may let \( A = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle \), \( B = \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_l \rangle \), \( k \geq 2, l \geq 0 \).

Since \( a \notin M \), we may assume that \( M = \langle x_1 a^{i_1}, x_2 a^{i_2}, \cdots, x_k a^{i_k}, y_1 a^{j_1}, \cdots, y_l a^{j_l}, \Phi(G) \rangle \), which \( i_1, \cdots, i_k, j_1, \cdots, j_l = 0, 1, \cdots, p-1 \). Since \( M \) is full-normal, \( M \) is abelian. Hence \( [x_m a^{i_m}, x_n a^{j_n}] = 1 \), \( [x_m a^{i_m}, y_l a^{j_l}] = 1 \) which \( m, n = 1, 2, \cdots, k, t = 1, 2, \cdots, l \). Note that \( c(G) = 2 \) by Corollary 3.5, so by computation, we have \( i_1 = i_2 = \cdots = i_k = j_1 = \cdots = j_l = 0 \). Hence \( M = \langle x_1, \cdots, x_k, y_1, \cdots, y_l, \Phi(G) \rangle = \langle x_1, \cdots, x_k, y_1, \cdots, y_l, a^p \rangle \). Since \( a^p = 1 \) or \( a^p \in B \setminus \mathcal{U}_1(B) \) by Corollary 3.4, we have \( M = \langle x_1, \cdots, x_k, y_1, \cdots, y_l \rangle \). It is easy to see that \( M \) is abelian and \( M \) is full-normal. Hence \( G \) has only one maximal subgroup is full-normal.

(2) \( d(A) = 1 \)

Without loss of generality, we may let \( A = \langle x \rangle \), \( B = \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_l \rangle \), \( l \geq 0 \).

Since \( a \notin M \), we may assume that \( M = \langle x a^i, y_1 a^{j_1}, \cdots, y_l a^{j_l}, \Phi(G) \rangle \), which \( i, j_1, \cdots, j_l = 0, 1, \cdots, p-1 \). Since \( M \) is full-normal, \( M \) is abelian. By computation, we have \( j_1 = j_2 = \cdots = j_l = 0 \). It follows that \( M = \langle x a^i, y_1, \cdots, y_l, \Phi(G) \rangle = \langle x a^i, y_1, \cdots, y_l, a^p \rangle \). Note that \( a^p = 1 \) or \( a^p \in B \setminus \mathcal{U}_1(B) \) by Corollary 3.4.

If \( a^p = 1 \), then \( M = \langle x a^i, y_1, \cdots, y_l \rangle \), which \( i = 0, 1, \cdots, p-1 \). We will prove \( M \) is full-normal in the following. Note that \( G \) is \( p \)-abelian by Corollary 3.5, we have \( (x a^i)^{1+p^{e-1}} = (x a^i)(x a^{i})^{p^{e-1}} = x a^i x^{p^{e-1}} a^{i} = x^{1+p^{e-1}} a \) and \( (x a^i)^{1+ip^{e-1}} = x^{1+ip^{e-1}} a^{i} = x^{1-ip^{e-1}} a^{i} \). So \( (x a^i)^a = x^a a^i = x^{1+p^{e-1}} a \) and \( (x a^i)^x = x(a^i)^x = x(a^i) = x a^i x^{ip^{e-1}} = x^{1-ip^{e-1}} a^{i} = (x a^i)^{1-ip^{e-1}} \).

And it is easy to see that \( (x a^i)^{y_1} = x a^i \). So we claim that \( \langle x a^i \rangle \leq G \). Hence \( M \) is full-normal. It follows that the number of full-normal maximal subgroup of \( G \) is \( p \).

If \( a^p \in B \setminus \mathcal{U}_1(B) \), we may assume that \( a^p = y_1 \). Then \( M = \langle x a^i, y_1, \cdots, y_l \rangle \), which \( i = 0, 1, \cdots, p-1 \). Note that \( (x a^i)^a = x^{1+p^{e-1}} a^i \) and \( (x a^i)^a = (x a^i)^{1+p^{e-1}} = x^{1+p^{e-1}} a^{i} + ip^{e-1} \). It follows that \( a^{ip^{e-1}} = 1 \).
When $o(a) = p^e$, then $i = 0$. Hence $M = \langle x, y_1, \cdots, y_l \rangle$. It is easy to see that $M$ is abelian and $M$ is full-normal. Hence $G$ has only one maximal subgroup is full-normal.

When $o(a) < p^{e-1}$, then for any $i = 0, 1, \cdots, p - 1$, we have $a^{ip^{e-1}} = 1$. Hence $M = \langle xa^i, y_1, \cdots, y_l \rangle$ where $i = 0, 1, \cdots, p - 1$. Similarly, we can prove that $M$ is full-normal. So the number of full-normal maximal subgroup of $G$ is $p$.

References


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