The Skew-Symmetric Ortho-Symmetric Solutions of the Matrix Equations \( A^*XA = D \)

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Abstract

In this paper, the following problems are discussed.

Problem 1. Given matrices \( A \in \mathbb{C}^{n \times m} \) and \( D \in \mathbb{C}^{m \times m} \), find \( X \in SSC^n \) such that \( A^*XA = D \), where \( SSC^n = \{ X \in SSC^n / PX \in SC_{n \times n} \text{ for given } P \in OC_{n \times n} \text{ satisfying } P^* = P \} \).

Problem 2. Given a matrix \( \bar{X} \in \mathbb{C}^{n \times n} \), find \( \hat{X} \in SE \) such that

\[ \| \bar{X} - \hat{X} \| = \inf_{X \in SE} \| \bar{X} - X \|, \]

where \( \| . \| \) is the Frobenius norm, and \( SE \) is the solution set of problem 1.

Expressions for the general solution of problem 1 are derived. Necessary and sufficient conditions for the solvability of Problem 1 are determined. For problem 2, an expression for the solution is given.

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1 Introduction

Let \( C^{n \times m} \) denote the set of all \( n \times m \) complex matrices, and let \( OC^{n \times n}, SC^{n \times n}, SSC^{n \times n} \) denote the set of all \( n \times n \) orthogonal matrices, the set of all \( n \times n \) complex symmetric matrices, the set of all \( n \times n \) complex skew-symmetric matrices, respectively. The symbol \( I_k \) will stand for the identity matrix of order \( K \), \( A^\dagger \) for the Moore-penrose generalized inverse of a matrix \( A \), and \( \text{rk}(A) \) for the rank of matrix \( A \). For matrices \( A, B \in C^{n \times m} \), the expression \( A \ast B \) will be the Hadamard product of \( A \) and \( B \); also \( \| . \| \) will denote the Frobenius norm. Defining the inner product \((A, B) = \text{tr}(B^\ast A)\) for matrices \( A, B \in C^{n \times m} \), \( C^{n \times m} \) becomes a Hilbert space. The norm of a matrix generated by this inner product is the Frobenius norm. If \( A = (a_{ij}) \in C^{n \times n} \), let \( L_A = (l_{ij}) \in C^{n \times n} \) be defined as follows: \( l_{ij} = a_{ij} \) whenever \( i > j \) and \( l_{ij} = 0 \) otherwise \((i,j = 1,2,...,n)\). Let \( e_i \) be the \( i \)-th column of the identity matrix \( I_n(i = 1, 2, ..., n) \) and set \( S_n = (e_n, e_{n-1}, ..., e_1) \). It is easy to see that

\[
S_n^* = S_n, \quad S_n^* S_n = I_n.
\]

An inverse problem [2]-[6] arising in the structural modification of the dynamic behaviour of a structure calls for the solution of the matrix equation

\[
A^\ast X A = D, \tag{1.1}
\]

where \( A \in C^{n \times m}, D \in C^{m \times m} \), and the unknown \( X \) is required to be complex and symmetric, and positive semidefinite or possibly definite. No assumption is made about the relative sizes of \( m \) and \( n \), and it is assumed through that \( A \neq 0 \) and \( D \neq 0 \). Equation (1.1) is a special case of the matrix equation

\[
AXB = C. \tag{1.2}
\]

Consistency conditions for equation (1.2) were given by Penrose[7] (see also [1]). When the equation is consistent, a solution can be obtained using generalized inverses. Khatri and Mitra [8] gave necessary and sufficient conditions for the existence of symmetric and positive semidefinite solutions as well as explicit formulae using generalized inverses. In [9],[10] solvability conditions for symmetric and positive definite solutions and general solutions of Equation (1.2) were obtained through the use of generalized singular value decomposition [11]-[13].

For important results on the inverse problem \( A^\ast X A = D \) associated with several kinds of different sets \( S \), for instance, symmetric matrices, symmetric nonnegative definite matrices, bisymmetric (same as persymmetric) matrices, bisymmetric nonnegative definite matrices and so on, we refer the reader to [14]-[17].

For the case the unknown \( A \) is skew-symmetric ortho-symmetric,[18] has discussed the
inverse problem $AX = B$. However, for this case, the inverse problem $A^*XA = D$ has not been dealt with yet. This problem will be considered here.

**Definition 1.1** A matrix $P \in \mathbb{C}^{n \times n}$ is said to be a symmetric orthogonal matrix if $P^* = P$, $P^*P = I_n$.

In this paper, without special statement, we assume that $P$ is a given symmetric orthogonal matrix.

**Definition 1.2** A Matrix $X \in \mathbb{C}^{n \times n}$ is said to be a skew-symmetric ortho-symmetric matrix if $X^* = -X$, $(PX)^* = PX$. We denote the set of all $n \times n$ skew-symmetric ortho-symmetric matrices by $SSC^n_p$.

The problem studied in this paper can now be described as follows.

**Problem 1.** Given matrices $A \in \mathbb{C}^{n \times m}$ and $D \in \mathbb{C}^{m \times m}$, find a skew-symmetric ortho-symmetric matrix $X$ such that

$$A^*XA = D.$$ 

In this paper, we discuss the solvability of this problem and an expression for its solution is presented.

The Optimal approximation problem of a matrix with the above-given matrix restriction comes up in the processes of test or recovery of a linear system due to incomplete data or revising given data. A preliminary estimate $\hat{X}$ of the unknown matrix $X$ can be obtained by the experimental observation values and the information of statistical distribution. The optimal estimate of $X$ is a matrix $\hat{X}$ that satisfies the given matrix restriction for $X$ and is the best approximation of $\hat{X}$, see [19]-[21].

In this paper, we will also considered the so-called optimal approximation problem associated with $A^*XA = D$. It reads as follows.

**Problem 2.** Given matrix $\tilde{X} \in \mathbb{C}^{n \times n}$, find $\hat{X} \in S_E$ such that

$$\|\tilde{X} - \hat{X}\| = \inf_{X \in S_E} \|\tilde{X} - X\|,$$

where $S_E$ is the solution set of Problem 1.

We point out that if Problem 1 is solvable, then Problem 2 has a unique solution, and in this case an expression for the solution can be derived.

The paper is organized as follows. In section 2, we obtain the general form of $S_E$ and the sufficient and necessary conditions under which problem 1 is solvable mainly by using the structure of $SSC^n_p$ and orthogonal projection matrices. In section 3, the expression for the solution of the matrix nearness problem 2 will be determined.
2 The expression of the general solution of problem 1

In this section we first discuss some structure properties of symmetric orthogonal matrices. Then given such a matrix \( P \), we consider structural properties of the subset \( SSC^n_P \) of \( C^{n \times n} \). Finally we present necessary and sufficient conditions for the existence of and the expressions for the skew-symmetric ortho-symmetric (with respect to the given \( P \)) solutions of problem 1.

Lemma 2.1. Assume \( P \) is a symmetric orthogonal matrix of size \( n \), and let

\[
P_1 = \frac{1}{2}(I_n + P), \quad P_2 = \frac{1}{2}(I_n - P).
\]

Then \( P_1 \) and \( P_2 \) are orthogonal projection matrices satisfying \( P_1 + P_2 = I_n, P_1P_2 = 0 \).

Proof. Since

\[
P_1 = \frac{1}{2}(I_n + P), \quad P_2 = \frac{1}{2}(I_n - P).
\]

Then

\[
P_1 + P_2 = \frac{1}{2}(I_n + P) + \frac{1}{2}(I_n - P) = \frac{1}{2}(I_n + P + I_n - P) = \frac{1}{2}(2I_n) = I_n.
\]

\[
P_1P_2 = \frac{1}{2}(I_n + P)\frac{1}{2}(I_n - P) = \frac{1}{4}(I_n - P + P - P^2) = \frac{1}{4}(I_n - P^2) = \frac{1}{4}(I_n - P, P^*) = \frac{1}{4}(I_n - I_n) = 0.
\]

Lemma 2.2. Assume \( P_1 \) and \( P_2 \) are defined as (2.1) and \( \text{rank}(P_1) = r \). Then \( \text{rank}(P_2) = n - r \), and there exists unit column orthogonal matrices \( U_1 \in C^{n \times r} \) and \( U_2 \in C^{n \times (n-r)} \) such that \( P_1 = U_1U_1^*, P_2 = U_2U_2^* \), and \( U_1^*U_2 = 0 \) then \( P = U_1U_1^* - U_2U_2^* \).

Proof. Since \( P_1 \) and \( P_2 \) are orthogonal projection matrices satisfying \( P_1 + P_2 = I_n \) and \( P_1P_2 = 0 \), the column space \( R(P_2) \) of the matrix \( P_2 \) is the orthogonal complement of the column space \( R(P_1) \) of the matrix \( P_1 \), in other words, \( R^n = R(P_1) \oplus R(P_2) \). Hence, if \( \text{rank}(P_1) = r \), then \( \text{rank}(P_2) = n - r \). On the other hand, \( \text{rank}(P_1) = r \), \( \text{rank}(P_2) = n - r \), and \( P_1, P_2 \) are orthogonal projection matrices. Thus there exists unit column orthogonal matrices \( U_1 \in C^{n \times r} \) and \( U_2 \in C^{n \times (n-r)} \) such that \( P_1 = U_1U_1^*, P_2 = U_2U_2^* \). Using \( R^n = R(P_1) \oplus R(P_2) \), we have \( U_1^*U_2 = 0 \). Substituting \( P_1 = U_1U_1^*, P_2 = U_2U_2^* \), into (2.1), we have \( P = U_1U_1^* - U_2U_2^* \).
Elaborating on Lemma 2.2 and its proof, we note that $U = (U_1, U_2)$ is an orthogonal matrix and that the symmetric orthogonal matrix $P$ can be expressed as

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*.$$  \hspace{1cm} (2.2)

**Lemma 2.3.** The matrix $X \in SSC_P^n$ if and only if $X$ can be expressed as

$$X = U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*,$$  \hspace{1cm} (2.3)

where $F \in C^{r \times (n-r)}$ and $U$ is the same as (2.2).

**proof** Assume $X \in SSC_P^n$. By lemma 2.2 and the definition of $SSC_P^n$, We choose $P_1 = \frac{I + P}{2}, P_2 = \frac{I - P}{2}$

$$P_1XP_1 = \frac{I + P}{2} X \frac{I + P}{2} = \frac{1}{4} (X + PX + XP + PXP)$$

$$= \frac{1}{4} \left( U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* + PX + XP + U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^* \right).$$

$$= \frac{1}{4} \left( U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* + PX + XP + U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^* \right).$$

$$= \frac{1}{4} \left( U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* + PX + XP + U \begin{pmatrix} 0 & -F \\ F^* & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^* \right).$$

$$P_1XP_1 = \frac{1}{4} (XP + PX).$$

Similarly

$$P_2XP_2 = -\frac{1}{4} (XP + PX).$$

Hence,

$$X = (P_1 + P_2)X(P_1 + P_2) = P_1XP_1 + P_1XP_2 + P_2XP_1 + P_2XP_2$$
\[ X = P_1XP_2 + P_2XP_1 \quad (since P_1XP_1 + P_2XP_2 = 0). \]

\[ X = P_1XP_2 + P_2XP_1 = U_1^*XU_2U_2^* + U_2^*XU_1U_1^* \quad (since P_1 = U_1^* and P_2 = U_2^*) \]

\[ = U_1FU_2^* + U_2GU_1^* \]

Let \( F = U_1^*XU_2 \) and \( G = U_2^*XU_1 \). It is easy to verify that \( F^* = -G \).

\[ (since F = U_1^*XU_2, \quad F^* = (U_1^*XU_2)^* = U_2^*X^*U_1 = -U_2^*U_1 = -G) \]

Then we have

\[ X = U_1FU_2^* + U_2GU_1^* = U \left( \begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) U^* \]

Conversely, for any \( F \in C^{r \times (n-r)} \), Let

\[ X = U \left( \begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) U^* \]

It is easy to verify that \( X^* = -X \)

\[ X = U_1FU_2^* + U_2GU_1^* \quad X^* = (U_1FU_2^*)^* + (U_2GU_1^*)^* = U_2F^*U_1^* + U_1G^*U_2^* \]

\[ = -U_2GU_1^* - U_1FU_2^* = -(U_1FU_2^* + U_2GU_1^*) = -X. \]

Using (2.2), we have

\[ PXP = PU \left( \begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) U^* P \]

\[ = U \left( \begin{smallmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{smallmatrix} \right) U^* U \left( \begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) U^* U \left( \begin{smallmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{smallmatrix} \right) U^* \]

\[ = U \left( \begin{smallmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{smallmatrix} \right) \left( \begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) \left( \begin{smallmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{smallmatrix} \right) U^* \]

\[ = U \left( \begin{smallmatrix} 0 & F \\ -I_{n-r} & 0 \end{smallmatrix} \right) U^* = U \left( \begin{smallmatrix} 0 & -F \\ 0 & I_{n-r} \end{smallmatrix} \right) U^* = -X \]

Thus

\[ X = U \left( \begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) U^* \in SSC_n^p. \]
Lemma 2.4. Let $A \in \mathbb{C}^{n \times n}, D \in \text{SSC}^{n \times n}$ and assume $A - A^* = D.$ Then there is precisely one $G \in \text{SC}^{n \times n}$ such that $A = L_D + G,$ and $G = \frac{1}{2}(A + A^*) - \frac{1}{2}(L_D + L_D^*).$

Proof. For given $A \in \mathbb{C}^{n \times n}, D \in \text{SSC}^{n \times n}$ and $A - A^* = D.$ It is easy to verify that there exists unique

$$G = \frac{1}{2}(A + A^*) - \frac{1}{2}(L_D + L_D^*) \in \text{SC}^{n \times n},$$

and we have

$$A = \frac{1}{2}(A - A^*) + \frac{1}{2}(A + A^*) = \frac{1}{2}(L_D - L_D^*) + \frac{1}{2}(A + A^*)$$

$$A = \frac{1}{2}(A + A^*) + L_D - \frac{1}{2}(L_D + L_D^*) (\text{since } L_D = \frac{1}{2}(L_D + L_D^*) + \frac{1}{2}(L_D - L_D^*))$$

$$A = L_D + G.$$

Let $A \in \mathbb{C}^{n \times m}$ and $D \in \mathbb{C}^{m \times m}, U$ defined in (2.2), Set

$$U^*A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, A_1 \in \mathbb{C}^{r \times m}, A_2 \in \mathbb{C}^{(n-r) \times m}. \tag{2.4}$$

The generalized singular value decomposition (see [11],[12],[13]) of the matrix pair $[A_1^*, A_2^*]$ is

$$A_1^* = M \sum A_1 W^*, A_2^* = M \sum A_2 V^*, \tag{2.5}$$

where $W \in \mathbb{C}^{m \times m}$ is a nonsingular matrix, $W \in \text{OC}^{r \times r}, V \in \text{OC}^{(n-r) \times (n-r)}$ and

$$\sum A_1 = \begin{pmatrix} I_k \\ S_1 \\ O_1 \\ \ldots \\ O \end{pmatrix} \begin{pmatrix} k \\ s \\ t - k - s \\ \ldots \\ m - t \end{pmatrix} \tag{2.6}$$
\sum A_2 = \begin{pmatrix}
O_2 & & & k \\
& S_2 & & s \\
& & I_{t-k-s} & t-k-s \\
& & & \\
& & & \\
& & & m-t \\
O & & & \\
\end{pmatrix}

(2.7)

where

\[ t = \text{rank}(A_1^*, A_2^*), K = t - \text{rank}(A_2^*), \]

\[ S = \text{rank}(A_1^*) + \text{rank}(A_2^*) - t \]

\[ S_1 = \text{diag}(\alpha_1, ..., \alpha_s), S_2 = \text{diag}(\beta_1, ..., \beta_s), \]

with \( 1 > \alpha_1 \geq ... \geq \alpha_s > 0, \ 0 < \beta_1 \leq ... \leq \beta_s < 1, \) and \( \alpha_i^2 + \beta_i^2 = 1, i = 1, ..., s. \)

Theorem 2.5. Given \( A \in \mathbb{C}^{n \times m} \) and \( D \in \mathbb{C}^{m \times m}, U \) defined in (2.2), and \( U^*A \) has the partition form of (2.4), the generalized singular value decomposition of the matrix pair \( [A_1^*, A_2^*] \) as (2.5). Partition the matrix \( M^{-1}DM^{-*} \) as

\[
M^{-1}DM^{-*} = \begin{pmatrix}
D_{11} & D_{12} & D_{13} & D_{14} & k \\
D_{21} & D_{22} & D_{23} & D_{24} & s \\
D_{31} & D_{32} & D_{33} & D_{34} & t-k-s \\
D_{41} & D_{42} & D_{43} & D_{44} & m-t \\
\end{pmatrix}
\]

\[ k \ s \ t-k-s \ m-t \]
then the problem 1 has a solution \( X \in \text{SSC}_P^n \) if and only if
\[
D^* = -D, \quad D_{11} = 0, \quad D_{33} = 0, \quad D_{41} = 0, \quad D_{42} = 0, \quad D_{43} = 0, \quad D_{44} = 0.
\]
In that case it has the general solution
\[
X = U \begin{pmatrix} 0_{r \times r} & F \\ -F^* & 0 \end{pmatrix} U^*,
\]
where
\[
F = W \begin{pmatrix} X_{11} & D_{12} S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1} (L_{D_{22}} + G) S_2^{-1} & S_1^{-1} D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^*,
\]
with \( X_{11} \in C^{r \times (n-r+k-t)}, \quad X_{21} \in C^{s \times (n-r+k-t)}, \quad X_{31} \in C^{(r-k-s) \times (n-r+k-t)}, \quad X_{32} \in C^{(r-k-s) \times s}, \quad X_{33} \in C^{(r-k-s) \times (t-k-s)} \) and \( G \in SC^{s \times s} \) are arbitrary matrices.

Proof. The Necessity:
Assume the equation (1.1) has a solution \( X \in \text{SSC}_P^n \). By the definition of \( \text{SSC}_P^n \), it is easy to verify that
\[
D^* = -D.
\]
Since \( D = A - A^* \)
\[
D^* = (A - A^*)^* = -A + A^* = -(A - A^*) = -D,
\]
and we have from lemma 2.3 that \( X \) can be expressed as
\[
X = U \begin{pmatrix} 0_{r \times r} & F \\ -F^* & 0 \end{pmatrix} U^*,
\]
where \( F \in C^{r \times (n-r)} \).

Note that \( U \) is an orthogonal matrix, and the definition of \( A_i (i = 1, 2) \), Equation (1.1) is equivalent to
\[
A_i^* FA_i - A_i^* FA_1 = D.
\]

Substituting (2.5) in (2.12), then we have
\[
M \sum A_1 W^* FA_2 - M \sum A_2 V^* FA_1 = D
\]
\[
M \sum A_1 W^* FV \sum A_2 M^* - M \sum A_2 V^* FA_1 = D
\]
\[
M \sum A_1 W^* FV \sum A_2 M^* - M \sum A_2 V^* FW \sum A_1 M^* = D
\]
\[
M^{-1} M \sum A_1 W^* FV \sum A_2 M^* M^{-*} - M^{-1} M \sum A_2 V^* FW \sum A_1 M^* M^{-*} = M^{-1} DM^{-*}
\]
\[
\sum A_1 (W^* FV) \sum A_2 - \sum A_2 (V^* FW) \sum A_1 = M^{-1} DM^{-*}
\]
\[
\sum A_1 (W^* FV) \sum A_2 - \sum A_2 (W^* FV) \sum A_1 = M^{-1} DM^{-*},
\]
(2.13)
partition the matrix $W^*FV$ as

$$W^*FV = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}, \quad (2.14)$$

where $X_{11} \in C^{r \times (n-r+k-t)}$, $X_{22} \in C^{s \times s}$, $X_{33} \in C^{(r-k-s) \times (t-k-s)}$.

Taking $W^*FV$ and $M^{-1}DM^*$, in (2.13), We have

$$\begin{pmatrix} 0 & X_{12}S_2 & X_{13} & 0 \\ -S_2 X_{21}^* & S_1 X_{22} S_2 - (S_1 X_{22} S_2)^* & S_1 X_{23} & 0 \\ -X_{13}^* & -X_{23}^* S_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix}, \quad (2.15)$$

Therefore (2.15) holds if and only if (2.8) holds and

$$X_{12} = D_{12} S_2^{-1}, X_{13} = D_{13}, X_{23} = S_1^{-1} D_{23}$$

and

$$S_1 X_{22} S_2 - (S_1 X_{22} S_2)^* = D_{22}.$$ 

It follows from Lemma 2.4 that $X_{22} = S_1^{-1} (L_{D_{22}} + G) S_2^{-1}$, where $G \in SC^{S \times S}$ is arbitrary matrix. Substituting the above into (2.14), (2.11), thus we have formulation (2.9) and (2.10).

The sufficiency. Let

$$F_G = W \begin{pmatrix} X_{11} & D_{12} S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1} (L_{D_{22}} + G) S_2^{-1} & S_1^{-1} D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^*.$$
obviously, $F_G \in C^{r \times (n-r)}$. By Lemma 2.3 and

$$X_O = U \left( \begin{array}{cc} 0 & F_G^* \\ -F_G & 0 \end{array} \right) U^*,$$

We have $X_0 \in SSC_p^n$. Hence

$$A^*X_0A = A^*UU^*X_0UU^*A = \left( \begin{array}{cc} A_1^* & A_2^* \end{array} \right) \left( \begin{array}{cc} 0 & F_G^* \\ -F_G & 0 \end{array} \right) \left( \begin{array}{c} A_1 \\ A_2 \end{array} \right)$$

$$= \left( -A_2^*F_G^*A_1^*F_G \right) \left( \begin{array}{c} A_1 \\ A_2 \end{array} \right) \left( -A_2^*F_G^*A_1^*F_G \right) = D,$$

This implies that

$$X_0 = U \left( \begin{array}{cc} 0 & F_G^* \\ -F_G & 0 \end{array} \right) U^* \in SSC_p^n$$

is the skew-symmetric ortho-symmetric solution of equation (1.1). Hence the proof.

### 3 The expression of the solution of Problem 2.

To prepare for an explicit expression for the solution of the matrix nearness problem 2, we first verify the following lemma.

**Lemma 3.1.** Suppose that $E, F \in C^{s \times s}$, and let $S_a = \text{diag}(a_1, \ldots, a_s) > 0, S_b = \text{diag}(b_1, \ldots, b_s) > 0$. Then there exists a unique $S_s \in SC^{s \times s}$ and a unique $S_r \in SSC^{s \times s}$ such that

$$\|S_aS_b - E\|^2 + \|S_aS_b - F\|^2 = \text{min.}$$

(3.1)

and

$$S_s = \Phi \ast [S_a(E + F)S_b + S_b(E + F)^*S_a],$$

(3.2)

$$S_r = \Phi \ast [S_a(E + F)S_b - S_b(E + F)^*S_a],$$

(3.3)

where

$$\Phi = (\psi_{ij}) \in SC^{s \times s}, \quad \psi_{ij} = \frac{1}{2(a_i^2b_j^2 + a_j^2b_i^2)}, \quad 1 \leq i, j \leq s.$$  \hspace{1cm} (3.4)

**Proof.** We prove only the existence of $S_r$ and (3.3). For any $S = (S_{ij}) \in SSC^{s \times s}$, $E = (e_{ij}), F = (f_{ij}) \in C^{s \times s}$, since $S_{ii} = 0, S_{ij} = -S_{ji}$,

$$\|S_aS_b - E\|^2 + \|S_aS_b - F\|^2 = \sum_{1 \leq i, j \leq s} [(a_i b_j S_{ij} - e_{ij})^2 + (a_i b_j S_{ij} - f_{ij})^2]$$

Proof complete.
Hence there exists a unique solution $S_{ij}$

Using $\frac{\partial g(s)}{\partial S_{ij}} = 0 \quad (1 \leq i, j \leq n)$, We have

$$= 2(a_i b_j S_{ij} - e_{ij})(a_i b_j) + 2(-a_j b_i S_{ij} - e_{ji})(-a_j b_i) + 2(a_i b_j S_{ij} - f_{ij})(a_i b_j) + 2(-a_j b_i S_{ij} - f_{ji})(-a_j b_i)
$$

$$= 2a_i^2 b_j^2 s_{ij} - 2a_i b_j e_{ij} + 2a_j^2 b_i^2 S_{ij} + 2a_j b_i e_{ji} + 2a_i^2 b_j^2 S_{ij} - 2a_i b_j f_{ij} + 2a_j^2 b_i^2 S_{ij} + 2a_j b_i f_{ji}
$$

$$= 4a_i^2 b_j^2 s_{ij} + 4a_j^2 b_i^2 S_{ij} - 2a_i b_j (e_{ij} + f_{ij}) + 2a_j b_i (e_{ji} + f_{ji})
$$

$$- 4S_{ij}(a_i^2 b_j^2 + a_j^2 b_i^2) = -2a_i b_j (e_{ij} + f_{ij}) + 2a_j b_i (e_{ji} + f_{ji})$$

$$S_{ij} = \frac{a_i b_j (e_{ij} + f_{ij}) - a_j b_i (e_{ji} + f_{ji})}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, \quad 1 \leq i, j \leq s$$

Here

$$\Phi = \Phi_{ij} = \frac{1}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, \quad 1 \leq i, j \leq s$$

Hence there exists a unique solution $S_r = S_{ij} \in SSC^{s \times s}$ for (3.1) such that

$$S_r = \Phi * [S_a(E + F)S_b - S_b(E + F)^*S_a].$$

**Theorem 3.2.** Let $\tilde{X} \in C^{m \times n}$, the generalized singular value decomposition of the matrix pair $[A_1^*, A_2^*]$ as (2.5), Let

$$U^* \tilde{X} U = (Z_{11}^*, Z_{12}^*, Z_{21}^*, Z_{22}^*),$$

$$W^* Z_{12}^* V = \left( \begin{array}{cccc}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{array} \right),$$

$$W^* Z_{21}^* V = \left( \begin{array}{cccc}
Y_{11} & Y_{12} & Y_{13} \\
Y_{21} & Y_{22} & Y_{23} \\
Y_{31} & Y_{32} & Y_{33}
\end{array} \right).$$

(3.5)
If problem 1 is solvable, then problem 2 has a unique solution $\hat{X}$, which can be expressed as

$$\hat{X} = U \left( \begin{array}{c|c} 0 & \tilde{F} \\ \hline -\tilde{F}^* & 0 \end{array} \right) U^*,$$

where

$$\tilde{F} = W \begin{pmatrix}
\frac{1}{2}(X_{11}^*-Y_{11}^*) & & & D_{12}S_2^{-1} & D_{13} \\
\frac{1}{2}(X_{21}^*-Y_{21}^*) & S_1^{-1}(L_{D_{22}} + \tilde{G})S_2^{-1} & S_1^{-1}D_{23} \\
\frac{1}{2}(X_{31}^*-Y_{31}^*) & \frac{1}{2}(X_{32}^*-Y_{32}^*) & \frac{1}{2}(X_{33}^*-Y_{33}^*) \\
\end{pmatrix} V^*,
$$

$$\tilde{G} = \phi [S_1^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})S_2^{-1} + S_2^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})^*S_1^{-1}],$$

with

$$\phi = (\psi_{ij}) \in SC^{s \times s}, \quad \psi_{ij} = \frac{a_i^2a_j^2b_i^2b_j^2}{2(a_i^2b_j^2 + a_j^2b_i^2)}, \quad 1 \leq i, j \leq s.$$

**Proof.** Using the invariance of the Frobenius norm under unitary transformations, from (2.9), (3.5) and (3.6) we have,

(2.9) implies that

$$X = U \left( \begin{array}{c|c} 0 & \tilde{F} \\ \hline -\tilde{F}^* & 0 \end{array} \right) U^*,$$

(3.5) implies that

$$U^*\tilde{X}U = \begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix}$$

(3.6) implies that

$$W^*Z_{12}^*V = \begin{pmatrix} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix}$$

$$W^*Z_{21}^*V = \begin{pmatrix} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & Y_{22}^* & Y_{23}^* \\ Y_{31}^* & Y_{32}^* & Y_{33}^* \end{pmatrix}$$

$$\tilde{X} = U \left( \begin{array}{c|c} Z_{11}^* & Z_{12}^* \\ \hline Z_{21}^* & Z_{22}^* \end{array} \right) U^{-1}$$

$$X - \tilde{X} = U \left( \begin{array}{c|c} 0 & \tilde{F} \\ \hline -\tilde{F}^* & 0 \end{array} \right) U^* - U \left( \begin{array}{c|c} Z_{11}^* & Z_{12}^* \\ \hline Z_{21}^* & Z_{22}^* \end{array} \right) U^* = U \left( \begin{array}{c|c} -Z_{11}^* & \tilde{F} - Z_{12}^* \\ \hline -\tilde{F}^* - Z_{21}^* & -Z_{22}^* \end{array} \right) U^*$$
\[
\left\| X - \hat{X} \right\|^2 = \left\| Z_{11}^* \right\|^2 + \left\| F - Z_{12}^* \right\|^2 + \left\| -F^* - Z_{21}^* \right\|^2 + \left\| Z_{22}^* \right\|^2
\]

Thus

\[
\left\| \tilde{X} - \hat{X} \right\| = \inf_{X \in \mathcal{S}_G} \left\| \tilde{X} - X \right\|
\]
is equivalent to

\[
\left\| X_{11} - X_{11}^* \right\|^2 + \left\| X_{11} + Y_{11}^* \right\|^2 = \min, \quad \left\| X_{21} - X_{21}^* \right\|^2 + \left\| X_{21} + Y_{21}^* \right\|^2 = \min,
\]

\[
\left\| X_{31} - X_{31}^* \right\|^2 + \left\| X_{31} + Y_{31}^* \right\|^2 = \min, \quad \left\| X_{32} - X_{32}^* \right\|^2 + \left\| X_{32} + Y_{32}^* \right\|^2 = \min,
\]

\[
\left\| X_{33} - X_{33}^* \right\|^2 + \left\| X_{33} + Y_{33}^* \right\|^2 = \min,
\]

\[
\left\| S_1^{-1}G_2^{-1} - (X_{22}^* - S_1^{-1}L_2 S_2^{-1}) \right\|^2 + \left\| S_1^{-1}G_2^{-1} + (Y_{22}^* + S_1^{-1})L_2 S_2^{-1} \right\|^2 = \min.
\]

From Lemma 3.1. We have,

\[
X_{11} = \frac{1}{2}(X_{11}^* - Y_{11}^*), \quad X_{21} = \frac{1}{2}(X_{21}^* - Y_{21}^*),
\]

\[
X_{31} = \frac{1}{2}(X_{31}^* - Y_{31}^*), \quad X_{32} = \frac{1}{2}(X_{32}^* - Y_{32}^*), \quad X_{33} = \frac{1}{2}(X_{33}^* - Y_{33}^*)
\]

and

\[
G = \Phi \ast [S_1^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_2 S_2^{-1})S_2^{-1} + S_2^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_2 S_2^{-1})S_1^{-1}].
\]

Taking \(X_{11}, X_{21}, X_{31}, X_{32}, X_{33}\) and \(G\) into (2.9), (2.10), we obtain that the solution of (the matrix nearness) Problem 2 can be expressed as

\[
\hat{X} = U \left( \begin{array}{cc} 0 & \hat{F}^* \\ -\hat{F}^* & 0 \end{array} \right) U^*.
\]
References


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