Minimal Number of Generators of Submodules

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Abstract

We determine a bound for minimal number of generators for a class of submodules of a finitely generated module. We also give several bounds on minimal number of generators of submodules in a finitely generated module.

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1 Introduction

Throughout this paper the ring $R$ is commutative Noetherian with non zero identity and $M$ is a finite (that is, finitely generated) $R$-module. Denote by $\ell(K)$ the length, and by $Z(K)$ the set of zero divisors of an $R$-module $K$. If $(R, \mathfrak{m})$ is a local ring we use the notions $\mu(K)$ and $e(K) = e(\mathfrak{m}, K)$ respectively for the minimal number of generators and the multiplicity of an $R$-module $K$.

For a zero-dimensional local ring $(R, \mathfrak{m})$, Watanabe (1987) proved the following result: For any ideal $I$ of $R$ and any element $x \in \mathfrak{m}$, we have $\mu(I) \leq \ell(R/xR)$.

Matsumura (1990) proved the following rather interesting result:

**Theorem A.** Let $(R, \mathfrak{m})$ be a local ring and let $M$ be a finite $R$-module of dimension at most one. Let $N$ be a submodule of $M$ and let $x \in \mathfrak{m}$. Then $\mu(N) \leq \ell(M/xM)$.

Sharif and Yassemi (2002) extended the Matsumura’s result by proving the following: Let $(R, \mathfrak{m})$ be a local ring and let $M$ be a finite $R$-module of dimension at most one. Let $N$ be a submodule of $M$ and let $x \in \mathfrak{m}$. Then $\ell(N/xN) \leq \ell(M/xM)$. (Recall that $\mu(N) \leq \ell(N/xN)$ for any $x \in \mathfrak{m}$).

In this paper we improve the Sharif-Yassemi’s result by showing the following result (In the other words we can drop the condition of $R$ being local...
and \( \dim M \leq 1 \): Let \( M \) be a finite \( R \)-module. Then for any \( x \in R \) and \( N \) a submodule of \( M \), we have \( \ell(N/xN) \leq \ell(M/xM) \).

Gottlieb (1995) gives a uniform bound for certain class of ideals: Let \( R \) be a local ring of \( \dim d \geq 1 \). Let \( q \) be an ideal generated by a system of parameters. Then \( \mu(I) \leq \ell(R/q) \) for all \( I \) such that \( \depth(R/I) \geq d - 1 \).

The main result of Section 2 is to obtain a uniform bound for certain class of submodules from the following theorem: Let \((R, \mathfrak{m})\) be a local ring and \( I \) be an ideal of \( R \) with \( \mu(I) = n \). Let \( N \) be a submodule of \( M \) such that \( \depth(M/N) \geq n - 1 \). Then \( \ell(N/IN) \leq \ell(M/IM) \).

The notion of superficial element is useful for finding the bounds for number of minimal generators. Good references for the facts about superficial elements are Nagata (1962) and Zariski-Samuel (1960). The following definition is an extension of this concept for modules:

**Definition 1.1.** (see Callejas-Bedregal and Jorge Pérez (2010)) Let \((R, \mathfrak{m})\) be a local ring. An element \( x \in \mathfrak{m} \) is a superficial element of \( R \) with respect to \( M \) if there exists a positive integer \( c \) such that \((\mathfrak{m}^nM:_Mx) \cap \mathfrak{m}^eM = \mathfrak{m}^{n-1}M\), for all \( n > c \).

In Section 3, by using the above definition we generalize some results of Sally (1978) regarding number of generators. (For unexplained terms see Matsumura (1986)).

## 2 Bounding the Number of Generators for a Class of Submodules

The following result is a key lemma for the proof of our main results.

**Lemma 2.1.** Let \( x \in R \) and \( N \) be a submodule of \( M \). Then

\[
\ell \left( \frac{N}{xN} \right) \leq \ell \left( \frac{M}{xM} \right).
\]

**Proof.** If \( \ell(M/xM) = \infty \), the inequality is clear. Therefore assume that \( \ell(M/xM) < \infty \). Since \( \dim(M/xM) = 0 \), we have \( \dim \left( \frac{R}{\Ann(M)+Rx} \right) = 0 \); and hence there are maximal ideals \( \mathfrak{m}_1, \ldots, \mathfrak{m}_t \) (not necessarily distinct) such that \( \mathfrak{m}_1 \cdots \mathfrak{m}_t \subseteq \Ann(M) + Rx \). Thus for any submodule \( L \) of \( M \), we have \( \mathfrak{m}_1 \cdots \mathfrak{m}_t L \subseteq xL \) (In the other words \( \ell(L/xL) < \infty \)).

Now let \( \sim \) denote the quotient module \( xN \) and let \( I := \mathfrak{m}_1 \cdots \mathfrak{m}_t \). It follows from the Artin-Rees Lemma that \( I^nM \cap N = \overline{u} \) for \( n \) sufficiently large, say for
\( n \geq n_0 \). Put \( Q = I^{n_0}M + xN \). Since \( R/I^{n_0} \) is Artinian and \( M/Q \) is annihilated by \( I^{n_0} \), \( \ell(M/Q) \) is finite. Since \( xM/(N + Q) \cong M/(x(N + Q) : M) x \) is isomorphic to a quotient of \( M/(N + Q) \), we have that

\[
\ell\left( \frac{M}{N + Q} \right) = \ell\left( \frac{x(N + Q) : M}{N + Q} \right) + \ell\left( \frac{xM}{x(N + Q)} \right).
\]

By considering the exact sequences

\[
0 \to \frac{N + Q}{x(N + Q)} \to \frac{M}{x(N + Q)} \to \frac{M}{N + Q} \to 0,
\]

\[
0 \to \frac{xM}{x(N + Q)} \to \frac{M}{x(N + Q)} \to \frac{M}{xM} \to 0,
\]

and the facts that \( \ell((N + Q)/x(N + Q)) = \ell((N + Q)/x(N + Q)) < \infty \) and \( \ell(M/(N + Q)) < \infty \), we have

\[
\ell\left( \frac{M}{xM} \right) - \ell\left( \frac{N + Q}{x(N + Q)} \right) = \ell\left( \frac{M}{N + Q} \right) - \ell\left( \frac{xM}{x(N + Q)} \right) = \ell\left( \frac{x(N + Q) : M}{N + Q} \right) \geq 0.
\]

Thus \( \ell((N + Q)/x(N + Q)) \leq \ell(M/xM) \). Since \( N \cap Q = 0 \), it is easy to see that \( \ell(N/xN) + \ell(Q/xQ) = \ell((N + Q)/x(N + Q)) \leq \ell(M/xM) \). Therefore \( \ell(N/xN) \leq \ell(M/xM) \). This proves our assertion. \( \square \)

**Theorem 2.2.** Let \( (R, \mathfrak{m}) \) be a local ring and \( I \) be an ideal of \( R \) with \( \mu(I) = n \). Let \( N \) be a submodule of \( M \) such that \( \text{depth}(M/N) \geq n - 1 \). Then \( \ell(N/IN) \leq \ell(M/IM) \).

**Proof.** We use induction on \( n \). For \( n = 1 \), the assertion follows from Lemma 2.1. Suppose that \( n > 1 \), and let \( \text{Ass}(M/N) = \{p_1, \ldots, p_k\} \). Since \( \text{depth}(M/N) > 0 \), we have \( p_i \neq \mathfrak{m} \) for each \( i \). If \( \ell(M/IM) = \infty \), the inequality is clear. Therefore assume that \( \ell(M/IM) < \infty \). Since \( \text{dim}(M/IM) = 0 \), we have that \( \text{dim}(\text{Ann}(M)+I) = 0 \); and hence there exists \( s \in \mathbb{N} \) such that \( \mathfrak{m}^s \subseteq \text{Ann}(M) + I \). Therefore \( \text{Supp}(M/IM) = \{\mathfrak{m}\} \). Suppose \( I \subseteq p_i, 1 \leq i \leq k \). Then \( \text{Ann}(M) + I \subseteq \text{Ann}(M/N) + p_i \subseteq p_i \), and hence \( \sqrt{\text{Ann}(M)} + I \subseteq p_i \). Thus \( p_i = \mathfrak{m} \). Since \( \text{depth}(M/N) > 0 \), this is contradiction. Therefore \( I \not\subseteq (\bigcup_{i=1}^{k} p_i) \bigcup \mathfrak{m} \). Choose \( x \in I \setminus (\bigcup_{i=1}^{k} p_i) \bigcup \mathfrak{m} \). Now let \( \overline{N} = (N + xM)/xM, \overline{M} = M/xM, \overline{R} = R/xR \) and \( \overline{T} = I/xR \). Since \( x \not\in Z(M/N) \), we have

\[
\text{depth}_R\left( \frac{M}{N} \right) = \text{depth}_R\left( \frac{M}{N} \right) = \text{depth}_R\left( \frac{M}{x(M/N)} \right) = \text{depth}_R\left( \frac{M}{N} \right) - 1.
\]
Therefore $\text{depth}(M/N) \geq n-2$. Since $\mu(T) = \mu(I) - \mu(xR) = n-1$, it is enough to show that $\ell_R(M/IM) = \ell_R(M/IM)$ and $\ell_R(N/IN) = \ell_R(N/IN)$. The first statement is trivial, we prove the second statement. We have $\ell_R(N) = \ell_R(N/IM) = \ell_R(N/IM)$. Let $\ell_R(N/IN) = t$. Then there is a chain

$$0 = \frac{N_0}{IN} \subset \frac{N_1}{IN} \subset \cdots \subset \frac{N_t}{IN} = \frac{N}{IN}$$

with $N_i/N_{i-1} \cong R/m$ for $1 \leq i \leq t$. We claim that

$$0 = \frac{N_0 + xM}{IN + xM} \subset \frac{N_1 + xM}{IN + xM} \subset \cdots \subset \frac{N_t + xM}{IN + xM} = \frac{N + xM}{IN + xM}$$

is also strictly increasing and necessarily with simple factor. (Note that if $N$ and $K$ are submodules of $M$ such that $M/N$ is a simple $R$-module, then the module $(M + K)/(N + K)$ is zero or simple). If not, we have $N_i \subset N_{i-1} + xM$ for some $i \geq 1$. Let $z \in N_i$. Then $z \in N_{i-1} + xM$ and hence $z = w + xy$ for some $w \in N_{i-1}$ and $y \in M$. Since $xy \in N$ and $x \not\in Z(M/N)$ we have that $y \in N$. Therefore $N_i \subset N_{i-1} + xN \subset N_{i-1} + IN \subset N_{i-1}$. This is a contradiction.

**Corollary 2.3.** Let $(R, m)$ be a local ring and $I$ be a proper ideal of $R$ with $\mu(I) = n$. Then, for any submodule $N$ of $M$ with $\text{depth}(M/N) \geq n - 1$, we have $\mu(N) \leq \ell(M/IM)$.

**Proof.** Since $I \subseteq m$, we have that $\mu(N) = \ell(N/mN) \leq \ell(N/IN) \leq \ell(M/IM)$. 

The following corollary was given in Sharif-Yassemi (2002) where the authors use the assumption $\dim(R) \geq 1$ in their paper. In the following it is shown that the assumption $\dim(R) \geq 1$ does not need any more.

**Corollary 2.4.** Let $(R, m)$ be a local Cohen-Macaulay ring, let $q$ be a parameter ideal of, and let $t = \ell(R/q)$. Let $M$ be a maximal Cohen-Macaulay module of rank $r$ (that means $M_p$ is free of constant rank $r$ for $p \in \text{Ass}(R)$). Then $\mu(M) \leq rt$.

**Proof.** It is easy to see that the module $M$ is isomorphic to a submodule of $R^r$. Now let $\dim(M) = \dim(R) = \mu(q) = n$ and apply Corollary 2.3.

### 3 Number of Generators of Submodules

From the Definition 1.1 one can easily prove the following facts.

**Fact 1.** Let $(R, m)$ be a local ring, and let $x \in m$ be a superficial element of $R$ with respect to $M$ that is not zero divisor of $M$. Then
(i) \((m^n M :_M x) = m^{n-1} M\) for \(n \gg 0\),
(ii) \(e(M/xM) = e(M)\), if \(\dim(M) = d > 0\).

**Fact 2.** (Sally (1978, Proposition 3.2)) Let \((R, m)\) be a local ring such that \(R/m\) is infinite. Let \(I, J_1, \ldots, J_s\) be distinct ideals of \(R\) which are also distinct from \(m\). Then there exists \(x \in R\) such that

(i) \(x \notin J_i, i = 1, \ldots, s\).

(ii) \(x\) is a superficial element of \(R\) with respect to \(M\).

(iii) If \(N\) is a submodule of \(M\) and \(I = \text{Ann}(M/N)\), then the image of \(x\) in \(R/I\) is a superficial element of \(R/I\) with respect to \(M/N\).

Now we are ready to give some of boundedness results.

**Theorem 3.1.** Let \((R, m)\) be a local ring and \(M\) be a finite Cohen-Macaulay \(R\)-module with \(\dim(M) \leq 1\). Let \(N\) be a submodule of \(M\). Then the following hold:

(i) If \(\dim(M/N) = 1\), then \(\mu(N) \leq e(M) - e(M/N)\),
(ii) if \(\dim(M/N) = 0\), then \(\mu(N) \leq e(M)\).

**Proof.** We divided the proof in two cases:

**Case 1:** \(\dim(M) = 0\). In this case we have \(\ell(M) < \infty, \ell(M/N) < \infty, e(M) = \ell(M)\) and \(e(M/N) = \ell(M/N)\). Now consider the following short exact sequence:

\[
0 \to N \to M \to \frac{M}{N} \to 0.
\]

So, \(\mu(N) = \ell(N/mN) \leq \ell(N) = \ell(M) - \ell(M/N) = e(M) - e(M/N)\).

**Case 2:** \(\dim(M) = 1\). By the standard trick of passing to \(R[u]_{mR[u]}\), where \(u\) is an indeterminate, we may assume that \(R/m\) is infinite. So, there exists a superficial element \(x \in m\) with respect to \(M\) such that \(x \notin Z(M)\). Since \(\dim(M/xM) = 0\), there exists \(t \in \mathbb{N}\) such that \(m^t \subseteq xR + \text{Ann}(M)\). Let \(c \geq t\) be a positive integer number such that \((m^{2c+1} M :_M x) \cap m^c M = m^{2c} M\). We claim that \(xm^{2c} M = m^{2c+1} M\). Let \(y \in m^{2c+1} M\). Then \(y \in x m^{c+1} M\), therefore \(y = xw\), where \(w \in m^c M\). Since \((m^{2c+1} M :_M x) \cap m^c M = m^{2c} M\), we have \(w \in m^{2c} M\) and therefore \(y \in xm^{2c} M\). Thus \(m^{2c+1} M \subseteq xm^{2c} M\) and hence \(xm^{2c} M = m^{2c+1} M\). Now we consider two cases:

**Case A:** \(\dim(M/N) = 0\). Since \(x \notin Z(M)\), we have \(xM/xN \cong M/N\) and hence \(\dim(M/xM) = 0\). By the following exact sequences:

\[
0 \to \frac{N}{xN} \xrightarrow{N} \frac{M}{xN} \xrightarrow{M} \frac{M}{N} \to 0,
\]

\[
0 \to \frac{xM}{xN} \xrightarrow{xM} \frac{M}{xN} \xrightarrow{M} \frac{M}{xM} \to 0
\]
we have
\[
\ell \left( \frac{M}{xM} \right) = \ell \left( \frac{M}{xN} \right) - \ell \left( \frac{xM}{xN} \right) = \ell \left( \frac{M}{N} \right) - \ell \left( \frac{M}{xN} \right) = \ell \left( \frac{N}{xN} \right).
\]
On the other hand, we can also have
\[
\ell \left( \frac{M}{xM} \right) = \ell \left( \frac{m^t M}{x m^t M} \right) = \ell \left( \frac{m^t M}{m^{t+1} M} \right) \text{ for } t \gg 0.
\]
Now consider the following exact sequence:
\[
0 \rightarrow \frac{mN}{xN} \rightarrow \frac{N}{xN} \rightarrow \frac{N}{mN} \rightarrow 0
\]
to see that $\mu(N) = \ell(N/mN) = \ell(N/xN) - \ell(mN/xN) = e(M) - \ell(mN/xN)$. So $\mu(N) \leq e(M)$.

Case $B$: dim$(M/N) = 1$. By Artin-Rees Lemma there exists $s_0 \in \mathbb{N}$ such that $m^s M \cap N \subseteq m^{s-s_0} N \subseteq mN$ for all $s \gg s_0$. We have
\[
\frac{N}{mN} = \frac{N}{mN + (m^s M \cap N)} = \frac{N}{N \cap (mN + m^s M)} \approx \frac{N + m^s M}{mN + m^s M}.
\]
Consider the following exact sequence
\[
0 \rightarrow \frac{N + m^{s+1} M}{mN + m^{s+1} M} \rightarrow \frac{N + m^s M}{mN + m^{s+1} M} \rightarrow \frac{N + m^s M}{N + m^{s+1} M} \rightarrow 0.
\]
We deduce
\[
\mu(N + m^s M) = \ell \left( \frac{N + m^s M}{m(N + m^s M)} \right) = \ell \left( \frac{N + m^{s+1} M}{mN + m^{s+1} M} \right) + \ell \left( \frac{N + m^s M}{N + m^{s+1} M} \right) = \mu(N) + \ell \left( \frac{m^s (M/N)}{m^{s+1} (M/N)} \right).
\]
Since dim$(\frac{M}{N+m^s M}) = 0$, by Case $A$ we have
\[
\mu(N) + e \left( \frac{M}{N} \right) = \mu(N + m^s M) \leq e(M).
\]
This completes the proof. \qed
Theorem 3.2. Let \((R, \mathfrak{m})\) be a local ring and \(M\) be a finite Cohen-Macaulay \(R\)-module with \(\dim(M) = d > 0\). Let \(N\) be a submodule of \(M\) such that \(\ell(M/N) < \infty\). If \(t\) is the smallest number that \(\mathfrak{m}^t M \subseteq N\), then
\[
\mu(N) \leq t^{d-1} e(M) + (d - 1) \mu(M).
\]

Proof. The proof is by induction on \(d\). If \(d = 1\), the assertion follows from Theorem 3.1. Assume that \(d > 1\). Again we may suppose that \(R/\mathfrak{m}\) is infinite.

There exists a superficial element \(x \in \mathfrak{m}\) with respect to \(M\) such that \(x \not\in Z(M)\). Since \(x^t \not\in Z(M)\), we have \(\dim(M/x^t M) = \dim(M) - 1\) and \(\text{depth}(M/x^t M) = \text{depth}(M) - 1\). Hence \(M/x^t M\) is a Cohen-Macaulay \(R\)-module (see Bruns-Herzog (1993, Theorem 2.1.3)). Let \(s\) be the number that \(\mathfrak{m}s(M/x^t M) \subseteq N/x^t M\). Then \(\mathfrak{m}s M \subseteq N\) and hence \(t \leq s\), so we can use induction.

First of all, we show that \(\mu(M/x^t M) = \mu(M)\) and \(e(M/x^t M) = te(M)\).

Since the first statement is trivial, it is enough to show that the second one:

\[
\begin{align*}
\frac{x M}{(x M \cap \mathfrak{m}^n M) + x^t M} &= \frac{x M}{x M \cap (\mathfrak{m}^n M + x^t M)} \\
&= \frac{x M}{(x M \cap \mathfrak{m}^n M) + x^t M}.
\end{align*}
\]

On the other hand, \((\mathfrak{m}^n M :_M x) = \mathfrak{m}^{n-1} M\) for \(n \gg 0\). So

\[
\begin{align*}
\frac{x M}{(x M \cap \mathfrak{m}^n M) + x^t M} &= \frac{x M}{x (\mathfrak{m}^n M :_M x) + x^t M} \\
&= \frac{x M}{x \mathfrak{m}^{n-1} M + x^t M} \\
&= \frac{x M}{\mathfrak{m}^{n-1} M + x^t M}.
\end{align*}
\]

Consider the following exact sequence:

\[
0 \to \frac{\mathfrak{m}^n M + x M}{\mathfrak{m}^n M + x^t M} \to \frac{M}{\mathfrak{m}^n M + x^t M} \to \frac{M}{\mathfrak{m}^n M + x M} \to 0.
\]

Thus, \(\ell(M/(\mathfrak{m}^n M + x^t M)) = \ell(M/(\mathfrak{m}^n M + x M)) + \ell(M/(\mathfrak{m}^{n-1} M + x^{t-1} M))\) and hence \(e(M/x^t M) = e(M/x M) + \ell(M/x^{t-1} M)\). Therefore, we can deduce that \(e(M/x^t M) = te(M)\).

Now by induction we have \(\mu(N/x^t M) \leq t^{d-2} e(M/x^t M) + (d - 2) \mu(M/x^t M)\). Consider the following exact sequence:

\[
0 \to x^t M \to N \to \frac{N}{x^t M} \to 0.
\]
So we get the following sequence:
\[ x^t M \otimes_R \frac{R}{m} \to N \otimes_R \frac{R}{m} \to \frac{N}{x^t M} \otimes_R \frac{R}{m} \to 0. \]

Therefore
\[ \mu \left( \frac{N}{mN} \right) \leq \ell \left( \frac{N}{x^t M} \otimes_R \frac{R}{m} \right) + \ell \left( x^t M \otimes_R \frac{R}{m} \right) \]
\[ = \mu \left( \frac{N}{x^t M} \right) + \mu(x^t M) \leq \mu \left( \frac{N}{x^t M} \right) + \mu(M) \]
and hence
\[ \mu(N) \leq t^{d-2}e(M) + (d-2)\mu \left( \frac{M}{x^t M} \right) + \mu(M) \leq t^{d-1}e(M) + (d-1)\mu(M). \]

\[ \square \]

**Corollary 3.3.** Let \((R, m)\) be a local ring and \(M\) be a finite \(R\)-module with \(\dim(M) = d > 0\). Then for all submodule \(N\) of \(M\), we have \(\mu(N) \leq a^{d-1}e(M) + (d-1)\mu(M)\), where \(a\) is the least integer that \(N \cap m^a M \subseteq mN\).

**Proof.** We have
\[ \frac{N}{mN} = \frac{N}{(N \cap m^a M) + mN} = \frac{N}{N \cap (mN + m^a M)} \cong \frac{N}{mN + m^a M}. \]

Therefore
\[ \mu(N) = \ell \left( \frac{N}{mN} \right) \]
\[ = \ell \left( \frac{N + m^a M}{mN + m^a M} \right) \]
\[ \leq \ell \left( \frac{N + m^a M}{m(N + m^a M)} \right) \]
\[ = \mu(N + m^a M). \]

On the other hand, if \(m^t M \subseteq N + m^a M\), then
\[ N \cap m^t M \subseteq N \cap (N + m^a M) \]
\[ = N + N \cap m^a M \]
\[ = N \cap m^a M \]
\[ \subseteq mN. \]

Therefore \(a \leq t\). So, by Theorem 3.2 we have
\[ \mu(N + m^a M) \leq a^{d-1}e(M) + (d-1)\mu(M). \]

Now the assertion follows immediately. \[ \square \]
Theorem 3.4. Let \((R, m)\) be a Cohen-Macaulay ring and \(M\) be a finite \(R\)-module with \(\dim(M) = d > 0\). Then for all Cohen-Macaulay submodule \(N\) of \(M\) we have \(\mu(N) \leq e(M/N)^{h-1}e(M) + (h-1)\mu(M)\), where \(h = \dim(M) - \dim(M/N)\).

**Proof.** We may assume that \(R/m\) is infinite. We use induction on \(s = \dim(M/N)\).

If \(s = 0\), then \(e(M/N) = \ell(M/N), h = d\) and \(t \leq \ell(M/N)\) (\(t\) is the smallest number that \(m' M \subseteq N\)). So

\[
\mu(N) \leq t^{d-1}e(M) + (d-1)\mu(M) \leq e\left(\frac{M}{N}\right)^{h-1}e(M) + (h-1)\mu(M).
\]

Now let \(s > 0\). Since \(\dim(M) \geq \dim(M/N) > 0\), we have \(\text{depth}(M/N) > 0\) and \(\text{depth}(R) > 0\). Therefore, there exists a superficial element \(x \in m\) with respect to \(M\) such that \(x \notin Z(R), x \notin Z(M/N)\) and \(x + I\) is a superficial element of \(R/I\) with respect to \(M/N\), where \(I = \text{Ann}(M/N)\). We have

\[
\mu\left(\frac{xM + N}{xM}\right) \leq e\left(\frac{M/N}{x(M/N)}\right)^{h-1}e\left(\frac{M}{xM}\right) + (h-1)\mu\left(\frac{M}{xM}\right).
\]

Therefore \(\mu((xM + N)/xM) \leq e(M/N)^{h-1}e(M) + (h-1)\mu(M)\). It is enough to show that \(\mu((xM + N)/xM) = \mu(N)\). Let \(\mu(N) = n\) and \(N = Rx_1 + \cdots + Rx_n\) for some \(x_1, \cdots, x_n \in N\). We claim that the set \(\{x_1 + xM, \cdots, x_n + xM\}\) is a minimal generator for \((xM + N)/xM\). It is easy to see that \(\{x_1 + xM, \cdots, x_n + xM\}\) is a generator set for \((xM + N)/xM\). Suppose on the contrary that there are \(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n \in R\) such that

\[
x_i + xM = \sum_{\substack{1 \leq j \leq n \\text{ such that } j \neq i}} (r_j + xR)(x_j + xM).
\]

Thus there exists \(y \in M\) such \(x_i - \sum_{\substack{1 \leq j \leq n \\text{ such that } j \neq i}} r_j x_j = xy\). Since \(xy \in N\) and \(x \notin Z(M/N)\), we have \(y \in N\) and hence \(y = \sum_{j=1}^{n} s_j x_j\) for some \(s_1, \cdots, s_n \in R\). It follows that

\[
x_i = (1 - s_i x)^{-1} \sum_{\substack{1 \leq j \leq n \\text{ such that } j \neq i}} (r_j + s_j)x_j,
\]

which is a contradiction. Therefore \(\mu((xM + N)/N) = n\). This concludes the proof. 

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References


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