Upper Rank of Full Transformation Semigroups on a Finite Set

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Abstract
In this paper we found a lower bound for the upper rank $r_4(T_n)$ of full transformation semigroup $T_n$ on the set $X_n$.

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1 Introduction

The full transformation semigroup $T_X$ consists of all self maps of the set $X$. We consider the finite set $X_n = \{1, 2, \ldots, n\}$ ordered in the standard way and the full transformation semigroup $T_n$ on $X_n$. The full transformation semigroup $T_n$ and some special subsemigroups of $T_n$ have been much studied over the last fifty years (see [3], [8], [9]). For any semigroup $S$, we normally define the rank of $S$ by,

$$\text{rank}(S) = \min\{|U| : U \subseteq S \text{ and } \langle U \rangle = S\}.$$ 

Rank of the full transformation semigroup $T_n$ on $X_n$ has been studied in [1] and [2]. Let $S$ be a finite semigroup. A subset $U$ of $S$ is called independent if, for every $u$ in $U$, the element $u$ does not belong to the semigroup $\langle U \setminus \{u\} \rangle$ generated by the remaining elements of $U$. Recently, Howie and Ribeiro introduced five different type of rank for semigroups, in [5] and [6]. These ranks $r_1(S), r_2(S), r_3(S), r_4(S)$ and $r_5(S)$, are defined as follows:

- $r_1(S) = \max\{k : \text{every subset } U \text{ of } S \text{ cardinality } k \text{ is independent}\}$
- $r_2(S) = \min\{k : \text{there exists a subset } U \text{ of } S \text{ cardinality } k \text{ such that } U \text{ generates } S\}$
• \( r_3(S) = \max \{ k : \text{there exists a subset } U \text{ of } S \text{ cardinality } k \text{ which is independent and generates } S \} \)

• \( r_4(S) = \max \{ k : \text{there exists a subset } U \text{ of } S \text{ cardinality } k \text{ which is independent} \} \)

• \( r_5(S) = \min \{ k : \text{every subset } U \text{ of } S \text{ cardinality } k \text{ generates } S \} \)

It is easily seen that \( r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S) \), and for convenience, in [6], the terminology has been used as \( r_1(S) \) is small rank, \( r_2(S) \) is lower rank, \( r_3(S) \) is intermediate rank, \( r_4(S) \) is upper rank and \( r_5(S) \) is large rank. Here, the lower rank is what is normally called the rank, which has been extensively studied.

It is proved that

\[
\begin{align*}
    r_5(T_n) &= n^n - \frac{n!}{2} + 1 \\
    r_5(B(G, n)) &= (n^2 - n + 1)|G| + 2
\end{align*}
\]

where \( B(G, n) \) is the Brandt semigroup for an finite group \( G \) (see[6]). The upper rank \( r_4(S) \) is calculated when \( S \) is a monogenic semigroup and aperiodic Brandt semigroup. The intermediate rank \( r_3(S) \) is also calculated when \( S \) is a rectangular band and aperiodic Brandt semigroup in [5].

The notion of upper rank turns out to be quite difficult to handle, especially for \( T_n \). It is relatively easy to obtain a lower bound for \( r_4 \), simply by producing an independent set in \( S \), but is usually much harder to show that \( S \) contains no larger independent set.

The main goal of this study is to find a lower bound for the upper rank of full transformation semigroups. For unexplained terms in semigroup theory, see [4] and [7].

2 Preliminaries

Let \( \alpha \in T_n \) and let \( x_0 \in X_n \) be any element. If \( x\alpha = x_0 \) for every element \( x \in X_n \), then \( \alpha \) is called a constant transformation. The image, Defect set, defect and kernel of \( \alpha \) are defined by

\[
\begin{align*}
    \text{im} (\alpha) &= \{ x\alpha : x \in X_n \} \\
    \text{Def} (\alpha) &= X_n \setminus \text{im} (\alpha) \\
    \text{def} (\alpha) &= |\text{Def} (\alpha)| \\
    \text{ker}(\alpha) &= \{ (x, y) \in X_n \times X_n : x\alpha = y\alpha \}
\end{align*}
\]

respectively. Now we state a well known lemma which will be useful throughout this paper.
Lemma 2.1 For any $\alpha, \beta \in T_n$,

(i) $\ker(\alpha) \subseteq \ker(\alpha\beta)$

(ii) $\text{im}(\alpha\beta) \subseteq \text{im}(\beta)$.

The idea behind Green’s equivalences is to sort out the elements of a semi-group. Each $D$-class in a semigroup $S$ is a union of $L$-classes and $R$-classes. The intersection of an $L$-class and an $R$-class is either empty or is an $H$-class. Hence it is convenient to visualize a $D$-class as an eggbox, in which each row represents an $R$-class, each column represents an $L$-class and each cell represents an $H$-class. (It is of course possible for the eggbox to contain a single row or a single column of cells, or even to contains only one cell.) One can then analysis a semigroup, by finding these uniform blocks and describing connections between them. For $\alpha, \beta \in T_n$, these relations defined as:

\[
\begin{align*}
(\alpha, \beta) \in L & \iff \text{im}(\alpha) = \text{im}(\beta) \\
(\alpha, \beta) \in R & \iff \ker(\alpha) = \ker(\beta) \\
(\alpha, \beta) \in H & \iff \text{im}(\alpha) = \text{im}(\beta) \text{ and } \ker(\alpha) = \ker(\beta) \\
(\alpha, \beta) \in D & \iff |\text{im}(\alpha)| = |\text{im}(\beta)|.
\end{align*}
\]

It is well known that $\alpha \in T_n$ is an idempotent element if and only if the restriction of $\alpha$ to $\text{im}(\alpha)$ is the identity map on $\text{im}(\alpha)$. If $e$ is an idempotent in a semigroup $S$, then $H_e$ is a subgroup of $S$. No $H$-class in $S$ can contain more than one idempotent. We denote the $D$-Green class of all self maps of defect $r$ by $D_{n-r}$ ($0 \leq r \leq n-1$). Let $D_k$ be a $D$-class then it is clear that

- $D_k$ has $\binom{n}{k}$ distinct $L$-classes
- $D_k$ has $S(n,k)$ distinct $R$-classes
- $D_k$ has $S(n,k)\binom{n}{k}$ distinct $H$-classes
- Each $H$-class has $k!$ elements
- Each $L$-class has $k^{n-k}$ group $H$-classes.

Here $S(n,k)$, for $1 < k \leq n$ defined by

\[
S(n,k) = \frac{1}{k!} \left( \sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n \right),
\]

is Stirling number of second kind which satisfies recurrence relations below:

$S(1,1) = S(n,n) = 1$ and $S(n+1,k) = S(n,k-1) + kS(n,k)$. 

3 The upper rank of $T_n$

**Theorem 3.1** $r_h(T_n) \geq |A| = [S(n, n - 1) - (n - 1)](n - 1)! + n + 1.$

**Proof.** First we construct an independent subset of $T_n$. For $X_n = \{1, \ldots, n\}$ and $Y = X_n \setminus \{i\}, i = 1, \ldots, n$ we define a set denoted by $A$. Consider the $\mathcal{L}$-class $L_Y$ in $D_{n-1}$. The set $A$ contains only idempotents of group $\mathcal{H}$-classes in $L_Y$ and all elements of non-group $\mathcal{H}$-classes in $L_Y$, and also $\alpha$, is the constant transformation such that $x\alpha = i$ for all $x \in X_n$ and $\beta$, is the identity map of $T_n$.

It is clear that the cardinality of $A$ is

$$[S(n, n - 1) - (n - 1)](n - 1)! + n - 1 + 1 + 1.$$ 

Here $S(n, n - 1)$ is the number of $\mathcal{R}$-classes in $D_{n-1}$, $(n - 1)$ is the number of group $\mathcal{H}$-classes and $(n - 1)!$ is the cardinality of each of the $\mathcal{H}$-classes.

$A$ is an independent subset of $T_n$. Since $\alpha \notin \langle A \setminus \{\alpha\}\rangle$ and $\beta \notin \langle A \setminus \{\beta\}\rangle$. So, without loss of generality we may assume that $A = A \setminus \{\alpha, \beta\}$. For any $\gamma \in A$, it is enough to show that $\gamma \notin \langle A \setminus \{\gamma\}\rangle$. Now suppose that $\gamma \in \langle A \setminus \{\gamma\}\rangle$. Then there exist $\delta_1, \delta_2, \ldots, \delta_k \in A \setminus \{\gamma\}$ such that $\delta_1\delta_2\ldots\delta_k = \gamma$. Since $\delta_1, \delta_2, \ldots, \delta_k, \gamma \in D_{n-1}$, we find from Lemma 2.1

$$\ker(\delta_1) = \ker(\gamma) \text{ and } \text{im}(\gamma) = \text{im}(\delta_k).$$

Especially, since $\delta_1, \delta_2, \ldots, \delta_k, \gamma \in L_Y$ then

$$\text{im}(\gamma) = \text{im}(\delta_j), j = 1, 2, \ldots, k.$$ (2)

From Equation (1) and (2), $\ker(\gamma) = \ker(\delta_1)$ and $\text{im}(\gamma) = \text{im}(\delta_1)$. That is, $\gamma$ and $\delta_1$ are in the same $\mathcal{H}$-class.

(i) if $\gamma$ is an idempotent, $\delta_1$ is not in the $H_\gamma$-class. Because this contradicts with the definition of $A$. So there is no element in $A \setminus \{\gamma\}$ such that $\ker(\delta_1) = \ker(\gamma)$. Hence $\gamma \notin \langle A \setminus \{\gamma\}\rangle$.

(ii) if $\gamma$ is not an idempotent, since each non-group $\mathcal{H}$-class is not closure (indeed is an independent subset) and idempotents of group $\mathcal{H}$-classes are right identity of the same $\mathcal{L}$-classes then $|\text{im}(\gamma)| = |\text{im}(\delta_1\delta_2\ldots\delta_k)| \leq n - 2$. Since $\gamma \in D_{n-1}$, this is in contradiction with $|\text{im}(\gamma)| = n - 1$. So $\gamma \notin \langle A \setminus \{\gamma\}\rangle$. 

$\blacksquare$
Example 3.2 For $n = 3$, $T_3$ has 3 $\mathcal{D}$-Green classes like below:

\[
\begin{array}{cccccc}
D_3 & \begin{pmatrix} 123 \\ 123 \end{pmatrix} & \begin{pmatrix} 123 \\ 132 \end{pmatrix} & \begin{pmatrix} 123 \\ 213 \end{pmatrix} & \begin{pmatrix} 123 \\ 231 \end{pmatrix} & \begin{pmatrix} 123 \\ 312 \end{pmatrix} & \begin{pmatrix} 123 \\ 321 \end{pmatrix} \\
D_2 & \begin{pmatrix} 123 \\ 122 \end{pmatrix} & \begin{pmatrix} 123 \\ 211 \end{pmatrix} & \begin{pmatrix} 123 \\ 133 \end{pmatrix} & \begin{pmatrix} 123 \\ 311 \end{pmatrix} & \begin{pmatrix} 123 \\ 232 \end{pmatrix} & \begin{pmatrix} 123 \\ 323 \end{pmatrix} \\
D_1 & \begin{pmatrix} 123 \\ 111 \end{pmatrix} & \begin{pmatrix} 123 \\ 222 \end{pmatrix} & \begin{pmatrix} 123 \\ 333 \end{pmatrix}
\end{array}
\]

$X_3 = \{1, 2, 3\}$ and let $i = 1$. Since $Y = \{2, 3\}$ consider the $L_Y = L_{\{2,3\}}$ class in top $\mathcal{D}$-class $D_2$. For $A$, we take idempotents of group $\mathcal{H}$-classes and all elements of non-group $\mathcal{H}$-classes in this $L$ class. Also $\alpha$ is constant transformation in $D_1$ and $\beta$ is identity map in $D_3$. Briefly the set $A$ consists of bold elements on the table above.

\[
A = \left\{ \begin{pmatrix} 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 123 \\ 232 \end{pmatrix}, \begin{pmatrix} 123 \\ 323 \end{pmatrix}, \begin{pmatrix} 123 \\ 332 \end{pmatrix}, \begin{pmatrix} 123 \\ 111 \end{pmatrix} \right\}
\]

$A$ is an independent set of $T_3$. The cardinality of $A$ is

\[
|A| = \left[ S(3, 2) - (3 - 1) \right] (3 - 1)! + 2 + 1 + 1 = 6.
\]

Hence $r_4(T_3) \geq 6$.

Remark 3.3 In [6], Howie and Riberio proved that the large rank of $T_n$ is $n^n - (1/2)n! + 1$. For $T_3$, $r_5(T_3) = 3^3 - (1/2)3! + 1 = 25$. Since $r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)$ and $6 \leq r_4(T_3) \leq 25 = r_5(T_3)$, our lower bound is true.

Thus all five ranks of $T_n$, $r_1(T_n) = 1 \leq r_2(T_n) = 3 \leq r_3(T_n) = 3 \leq r_4(T_n) = 6 \leq r_5(T_n) = 25$ were calculated.

Corollary 3.4 The independent set of $T_n$ is not unique. There are $\binom{n}{n-1}$ independent sets. Here $\binom{n}{n-1}$ is the number of distinct $\mathcal{L}$-classes in $D_{n-1}$

In our example, there are $\binom{3}{3-1} = 3$ independent sets of $T_3$. These are below for $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$ respectively.

- $A_1 = \left\{ \begin{pmatrix} 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 123 \\ 122 \end{pmatrix}, \begin{pmatrix} 123 \\ 121 \end{pmatrix}, \begin{pmatrix} 123 \\ 112 \end{pmatrix}, \begin{pmatrix} 123 \\ 221 \end{pmatrix}, \begin{pmatrix} 123 \\ 333 \end{pmatrix} \right\}$
• $A_2 = \left\{ \begin{pmatrix} 123 \\ 123 \\ 123 \\ 123 \\ 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 133 \\ 133 \\ 133 \\ 131 \\ 313 \\ 113 \end{pmatrix}, \begin{pmatrix} 113 \\ 113 \\ 113 \\ 113 \\ 222 \\ 222 \end{pmatrix} \right\}$

• $A_3 = \left\{ \begin{pmatrix} 123 \\ 123 \\ 123 \\ 123 \\ 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 233 \\ 233 \\ 233 \\ 322 \\ 323 \\ 223 \end{pmatrix}, \begin{pmatrix} 323 \\ 323 \\ 323 \\ 223 \\ 111 \\ 111 \end{pmatrix} \right\}$

References


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