Ordered Semigroups in which the Left Ideals are Intra-Regular Semigroups

Niovi Kehayopulu

University of Athens
Department of Mathematics
15784 Panepistimiopolis, Athens, Greece
nkehayop@math.uoa.gr

Michael Tsingelis

Hellenic Open University
School of Science and Technology
Studies in Natural Sciences, Greece

Abstract

It is well known that intra-regular semigroups (ordered semigroups) play an essential role in studying the structure of semigroups (ordered semigroups). In this paper we present a structure theorem referring to the decomposition of ordered semigroups into left strongly simple components, that is, into components, which are both simple and left quasi-regular. We prove, among others, that if an ordered semigroup $S$ is a union of its left strongly simple subsemigroups, then every left ideal of $S$ is an intra-regular subsemigroup of $S$. “Conversely” if every left ideal of $S$ is an intra-regular subsemigroup of $S$, then $S$ is a complete semilattice of left strongly simple semigroups. As a consequence, an ordered semigroup $S$ is a semilattice of left strongly simple semigroups if and only if it is a complete semilattice of left strongly simple semigroups. We also characterize the chains of left strongly simple ordered semigroups.

Mathematics Subject Classification: 06F05 (20M10)

Keywords: Ordered semigroup, simple, left quasi-regular, left strongly simple, semisimple, left ideal, intra-regular, (complete) semilattice congruence, (complete) semilattice of left strongly simple semigroups
1 Introduction

Intra-regular semigroups (resp. ordered semigroups) play an essential role in studying the structure of semigroups (resp. ordered semigroups). Decomposition of an intra-regular semigroup into simple components can be found in [1] and [12]. Decomposition of an intra-regular ordered semigroup into simple components can be found in [4]. The present paper deals with the decomposition of ordered semigroups into left strongly simple components, that is, into components which are both simple and left quasi-regular. We prove that an ordered semigroup $S$ is a semilattice of left strongly simple semigroups if and only if every left ideal of $S$ is an intra-regular subsemigroup of $S$. An ordered semigroup $S$ is a semilattice of left strongly simple semigroups if and only if every left ideal of $S$ is a semisimple subsemigroup of $S$. This type of ordered semigroups is the ordered semigroups in which $a \in (Sa^2Sa)$ for every $a \in S$.

Besides, the semilattices of left strongly simple semigroups and the complete semilattices of left strongly simple semigroups are the same. Moreover, we prove that an ordered semigroup is a complete semilattice of left strongly simple semigroups if and only if it is a union of left strongly simple semigroups. Finally, we show that an ordered semigroup $S$ is a chain of left strongly simple semigroups if and only if for every $a, b \in S$, we have $a \in (SabSa)$ or $b \in (SabSb)$. The right analogue of our results also hold.

An ordered semigroup $S$ is called archimedean if for any $a, b \in S$ there exists $k \in N$ such that $b^k \in (SaS)$ [10]. Equivalently, for every $a, b \in S$ there exists a natural number $k \in N$, $k \geq 1$ such that $b^k$ belongs to the ideal of $S$ generated by $a$. It might be noted that every left strongly simple ordered semigroup is an archimedean ordered semigroup. Decomposition of an ordered semigroup into archimedean components has been given in [9]. It has been proved, among others, in [9; Theorem 2.8] that an ordered semigroup $S$ is a complete semilattice of archimedean semigroups if and only if it is a semilattice of archimedean semigroups and this is equivalent to saying that if $b$ is an element of the ideal of $S$ generated by $a$ $(a \in S)$, then there exists a natural number $k$ such that the element $b^k$ belongs to the ideal of $S$ generated by $a^2$. So the Theorem 2.8 in [9] characterizes the (complete) semilattices of archimedean semigroups while in the present paper the (complete) semilattices (of the finer structure) of left strongly simple semigroups are characterized. The chains of left strongly simple semigroups are also characterized. There is an essential difference between semigroups without order and ordered semigroups. While the decomposition of a semigroup (without order) into archimedean (or left strongly simple) components is unique, the decomposition of an ordered semigroup into archimedean (or left strongly simple) components is not unique in general.

Suppose $S$ is a semilattice of left strongly simple semigroups and that $\sigma$ is a semilattice congruence on $S$ such that $(x)_\sigma$ is a left strongly simple sub-
semigroup of $S$ for every $x \in S$. Then $S$ is a semilattice of archimedean semigroups and, by [9; Theorem 2.8], $S$ is a complete semilattice of archimedean semigroups. That is, there exists a complete semilattice congruence $\rho$ on $S$ such that $(x)_{\rho}$ is an archimedean subsemigroup of $S$ for every $x \in S$. In the particular case in which $\rho = \sigma$, we immediately have that $S$ is a complete semilattice of left strongly simple semigroups. Being $\rho$ different from $\sigma$, in general, it is natural to ask if a semilattice of left strongly simple semigroups is a complete semilattice of left strongly simple semigroups (the converse being obvious).

The corresponding results for semigroups (without order) can be also obtained as application of the results of this paper, and this is because every semigroup endowed with the equality relation is an ordered semigroup. Our terminology of left quasi-regular, semisimple and left strongly simple ordered semigroups is the same as in semigroups (without order) in [11].

Let $(S, \leq)$ be an ordered semigroup. For a subsemigroup $T$ of $S$ and a subset $H$ of $T$, we denote by $(H)_T$ the subset of $T$ defined by

$$(H)_T := \{ t \in T \mid t \leq h \text{ for some } h \in H \}.$$ 

In particular, for $T = S$, we write $(H)$ instead of $(H)_S$. So, for $H \subseteq S$, we have

$$(H) := \{ t \in S \mid t \leq h \text{ for some } h \in H \}.$$ 

A nonempty subset $A$ of $S$ is called a left (resp. right) ideal of $S$ if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) If $a \in A$ and $b \in S$, $b \leq a$, then $b \in A$. $A$ is called an ideal of $S$ if it is both a left and a right ideal of $S$. We denote by $L(a)$ (resp. $R(a)$) the left (resp. right) ideal of $S$ generated by $a$, and by $I(a)$ the ideal of $S$ generated by $a$ ($a \in S$). We have $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$ and $I(a) = (a \cup Sa \cup aS \cup SaS]$ for every $a \in S$. A left (resp. right) ideal $A$ of $S$ is clearly a subsemigroup of $S$ i.e. $A^2 \subseteq A$. $S$ is called simple if $S$ is the only ideal of $S$ (i.e. $S$ does not contain proper ideals). A subsemigroup $L$ of $S$ is called intra-regular if for each $a \in L$ there exist $x, y \in L$ such that $a \leq xa^2y$, in symbol, if $a \in (La^2L)_L$ for every $a \in L$. An equivalence relation $\sigma$ on $S$ is called congruence if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence $\sigma$ on $S$ is called semilattice congruence if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$. If $\sigma$ is a semilattice congruence on $S$, then the $\sigma$-class $(x)_{\sigma}$ of $S$ containing $x$ is a subsemigroup of $S$ for every $x \in S$. A semilattice congruence $\sigma$ on $S$ is called complete if $a \leq b$ implies $(a, ab) \in \sigma$ [6]. Recall that if $\sigma$ is a complete semilattice congruence on $S$ then, the relation $a \leq a$ implies $(a^2, a) \in \sigma$, so the complete semilattice congruences on $S$ can be also defined as the congruences on $S$ such that $(ab, ba) \in \sigma$ and $a \leq b$ implies $(a, ab) \in \sigma$ for every $a, b \in S$. A subsemigroup $F$ of $S$ is called a filter of $S$ if (1) $a, b \in S$ such that $ab \in F$ implies $a \in F$ or $b \in F$ and (2) if $a \in F$ and
\( b \in S \) such that \( b \geq a \), then \( b \in F \). Denote by \( \mathcal{N} \) the relation on \( S \) defined by \( \mathcal{N} := \{(x, y) \mid N(x) = N(y)\} \) where \( N(a) \) denotes the filter of \( S \) generated by \( a \) (\( a \in S \)). The relation \( \mathcal{N} \) is the least complete semilattice congruence on \( S \) [6]. We say that \( S \) is a semilattice of left strongly simple semigroups (resp. complete semilattice of left strongly simple semigroups) if there exists a semilattice congruence (resp. complete semilattice congruence) \( \sigma \) on \( S \) such that the \( \sigma \)-class \( (x)_\sigma \) of \( S \) containing \( x \) is a left strongly simple subsemigroup of \( S \) for every \( x \in S \). An equivalent definition is the following: The ordered semigroup \( S \) is a semilattice of left strongly simple semigroups if there exists a semilattice \( \mathcal{Y} \) and a nonempty family \( \{S_\alpha \mid \alpha \in \mathcal{Y}\} \) of left strongly simple subsemigroups of \( S \) such that

\[
\begin{align*}
(1) & \quad S_\alpha \cap S_\beta = \emptyset \text{ for every } \alpha, \beta \in \mathcal{Y}, \alpha \neq \beta \\
(2) & \quad S = \bigcup_{\alpha \in \mathcal{Y}} S_\alpha \\
(3) & \quad S_\alpha S_\beta \subseteq S_{\alpha \beta} \text{ for every } \alpha, \beta \in \mathcal{Y}.
\end{align*}
\]

In ordered semigroups, the semilattice congruences are defined exactly as in semigroups (without order) so the two definitions are equivalent. As we have already seen in [8], an ordered semigroups \( S \) is a complete semilattice of left simple semigroups if and only if in addition to (1), (2) and (3) above, we have the following:

\[
(4) S_\beta \cap (S_\alpha) \neq \emptyset \text{ implies } \beta = \alpha \beta.
\]

We say that \( S \) is a chain (resp. complete chain) of left strongly simple semigroups if there exists a semilattice congruence (resp. complete semilattice congruence) \( \sigma \) on \( S \) such that the \( \sigma \)-class \( (x)_\sigma \) of \( S \) containing \( x \) is a left strongly simple subsemigroup of \( S \) for every \( x \in S \), and the set \( S/\sigma \) of (all) \( (x)_\sigma \)-classes of \( S \) endowed with the relation \( (x)_\sigma \preceq (y)_\sigma \iff (x)_\sigma = (xy)_\sigma \) is a chain (cf., for example [5]).

In case we need to be more specific, we could also say that \( S \) is a semilattice (complete semilattice) or chain \( \mathcal{Y} \) of left strongly semigroups \( S_\alpha \); \( \alpha \in \mathcal{Y} \).

## 2 Main results

**Lemma 1** (cf. also [10; Lemma 4]) A subsemigroup \( L \) of an ordered semigroup \( S \) is simple if and only if \( (LaL)_L = L \) for every \( a \in L \).

In other words, a subsemigroup \( L \) of \( S \) is simple if and only if for every \( a, b \in L \) there exist \( x, y \in L \) such that \( a \leq xby \).

**Lemma 2** (cf. also [2; Lemma 1]) If \( S \) is an ordered semigroup, then we have the following:

\[
\begin{align*}
(1) & \quad A \subseteq (A) \text{ for every } A \subseteq S. \\
(2) & \quad If A \subseteq B \subseteq S, then (A) \subseteq (B). \\
(3) & \quad (A)(B) \subseteq (AB) \text{ for every } A, B \subseteq S.
\end{align*}
\]
(4) \((A] = [A]\) for every \(A \subseteq S\).

(5) \(((A[B] = (A)(B)] = ([A][(B)] = (AB)\) for every \(A, B \subseteq S\).

(6) The set \((S[a]\) (resp. \((aS]\) is a left (resp. right) ideal of \(S\), the set \((SaS]\) is an ideal of \(S\) for every \(a \in S\).

(7) For every left (right) ideal or ideal \(T\) of \(S\), we have \((T]\) = \(T\).

**Definition 3** A subsemigroup \(L\) of an ordered semigroup \(S\) is called left (resp. right) quasi-regular if \(a \in (LaLa]_{L}\) (resp. \(a \in (aLaL]_{L}\) for every \(a \in L\).

**Definition 4** An ordered semigroup \(S\) is called left (resp. right) strongly simple if it is simple and left (resp. right) quasi-regular.

**Definition 5** A subsemigroup \(L\) of an ordered semigroup \(S\) is called semisimple if \(a \in (LaLaL]\) for every \(a \in L\).

It might be noted that an ordered semigroup \(S\) is semisimple if and only if the ideals of \(S\) are idempotent, that is, for every ideal \(A\) of \(S\), we have \((A^2] = A]\) [2].

**Lemma 6** An ordered semigroup \(S\) is left strongly simple if and only if \(a \in (SbSa]\) for every \(a, b \in S\).

**Proof.** \(\implies\). Let \(a, b \in S\). Since \(S\) is simple, by Lemma 1, we have \((SbS]\) = \(S]\), then \(a \in (SbS]\). Since \(S\) is left quasi-regular, we have \(a \in (SaSa]\). Thus we get \(a \in (SaSa] \subseteq (SbS][Sa] = (SbS)Sa \subseteq (SbSsa]\). \(\leftarrow\). If \(a \in S\), by hypothesis, we have \(a \in (SaSa]\), so \(S\) is left quasi-regular. Let now \(a, b \in S\). By hypothesis, we have \(a \in (SbSa] \subseteq (SbS[SbSsa]\) = (SbS(SbSsa]) \subseteq (SbS]\), and \(S\) is simple. \(\square\)

**Lemma 7** (cf. [4; the Theorem] and [7; Lemma 1]) If \(S\) is an intra-regular ordered semigroup, then for the complete semilattice congruence \(N\) on \(S\), the class \((x]\)\(N\) is a simple subsemigroup of \(S\) for every \(x \in S\).

**Theorem 8** Let \((S, .., \leq)\) be an ordered semigroup and \(\sigma\) a complete semilattice congruence on \(S\). Then \(S\) is left quasi-regular if and only if \((a)_{\sigma}\) is a left quasi-regular subsemigroup of \(S\) for every \(a \in S\).

**Proof.** \(\implies\). Let \(b \in (a)_{\sigma}\). Then there exist \(u, v \in (a)_{\sigma}\) such that \(b \leq uvbv\). In fact: Since \(b \in S\) and \(S\) is left quasi-regular, \(b \leq sbtb\) for some \(s, t \in S\). Then we have

\[b \leq sbt(sbtb) \leq sbtsbt(sbtb) = (sbts)b(tsbtb)b.\]

Moreover we have \(sbts, tsbt \in (a)_{\sigma}\). In fact, since \(b \leq sbtb\) and \(\sigma\) is a complete semilattice congruence on \(S\), we have \((b, sbtb) \in \sigma\), then \((b, sbtb) \in \sigma\). Since
(a, b) ∈ σ, we have (a, sbtb) ∈ σ. Since (tb, bt) ∈ σ, we have (sbtb, sb^2t) ∈ σ, then (a, sbt) ∈ σ, (a, sbs) ∈ σ, and sbs ∈ (a)_σ. Moreover, since (a, sbt) ∈ σ, we have (a, sb^2t) ∈ σ, (a, tsb) ∈ σ, and tsbt ∈ (a)_σ.

Let a ∈ S. Since (a)_σ is left quasi-regular, we have a ∈ ((a)_σa(a)_σa)_σ ⊆ (SaSa), so S is left quasi-regular.

\[ \text{Theorem 9} \] \text{Let } (S, , \leq) \text{ be an ordered semigroup. If } S \text{ is a union of left strongly simple subsemigroups of } S, \text{ then every left ideal of } S \text{ is an intra-regular subsemigroup of } S. \text{ "Conversely", if every left ideal of } S \text{ is an intra-regular subsemigroup of } S, \text{ then } S \text{ is a complete semilattice of left strongly simple semigroups.}

\text{Proof.} \text{ Suppose } S \text{ is the union of the left strongly simple subsemigroups } S_α, α ∈ Y. \text{ Then we have } a ∈ (Sa^2Sa) \text{ for every } a ∈ S. \text{ In fact: Let } a ∈ S, \text{ and let } a ∈ S_α \text{ for some } α ∈ Y. \text{ Since } S_α \text{ is a left strongly simple ordered semigroup and } a, a^2 ∈ S_α, \text{ by Lemma 6, we have } a ∈ (S_αa^2S_αa)|_{S_α} ⊆ (Sa^2Sa).

\text{Let now } L \text{ be a left ideal of } S \text{ and } a ∈ L. \text{ Since } a, a^2 ∈ S, \text{ we have}

\[ a ∈ (Sa^2Sa) ⊆ (S(Sa^3Sa))Sa = (S(Sa^3Sa))Sa ⊆ (Sa^2)a^2(Sa^2Sa). \]

\text{Since } a^2 ∈ L, \text{ we have } Sa^2 ⊆ SL ⊆ L \text{ and } Sa^2Sa ⊆ SL ⊆ L. \text{ Thus we have}

\[ a ∈ (La^2L) = (La^2L)_L, \text{ and } L \text{ is intra-regular.}

\text{The converse statement: Suppose every left ideal of } S \text{ is an intra-regular subsemigroup of } S. \text{ Then every left ideal of } S \text{ is a semisimple subsemigroup of } S. \text{ Indeed: Let } L \text{ be a left ideal of } S \text{ and } a ∈ L. \text{ Since } L \text{ is intra-regular, we have}

\[ a ∈ (La^2L)_L \subseteq (La(La^2L)_L)_L = (La(La^2L)L)_L \subseteq (LaLa(La)L) ⊆ (LaLaL)_L. \]

\text{and } L \text{ is semisimple. Let now } a ∈ S. \text{ Then } L(a) \text{ is a semisimple subsemigroup of } S \text{ i.e. } x ∈ (L(a)xL(a)xL(a))_{L(a)} = (L(a)xL(a)xL(a)) \text{ for every } x ∈ L(a). \text{ Thus we have}

\[ a ∈ (L(a)aL(a)aL(a)) = ((a ∪ Sa)a(a ∪ Sa)a(a ∪ Sa)) = (a^2Sa^3 ∪ Sa^2Sa). \]

\text{Then } a^2 ∈ (a^2Sa^3 ∪ Sa^2Sa)[a] ⊆ (a^2Sa^4 ∪ Sa^2Sa^2), \text{ and}

\[ a^2Sa^3 \subseteq (a^2Sa^4 ∪ Sa^2Sa^2)(Sa^3) ⊆ ((a^2Sa^4 ∪ Sa^2Sa^2)(Sa^3) \subseteq (a^2Sa^4 ∪ Sa^2Sa^2)(Sa^3)]. \]

\text{Thus we have } a ∈ ((Sa^2Sa) ∪ Sa^2Sa) = ((Sa^2Sa)] = (Sa^2Sa).

Since } a ∈ (Sa^2Sa) ⊆ (Sa^2S), (SaSa) \text{ for every } a ∈ S, \text{ S is both intra-regular and left quasi-regular. Since } S \text{ is intra-regular, by Lemma 7, } (x)_N \text{ is a simple}
subsemigroup of $S$ for every $x \in S$. Since $S$ is left quasi-regular and $\mathcal{N}$ a complete semilattice congruence of $S$, by Theorem 8, $(x)_{\mathcal{N}}$ is a left quasi-regular subsemigroup of $S$ for every $x \in S$. Since $\mathcal{N}$ is a complete semilattice congruence on $S$ and $(x)_{\mathcal{N}}$ a left strongly simple subsemigroup of $S$ for every $x \in S$, $S$ is a complete semilattice of left strongly simple semigroups. 

As an illustrative example of the Theorem we give the following:

**Example 10** [3; the Example] We consider the ordered semigroup $S = \{a, b, c, d, f\}$ defined by the multiplication and the order below:

\[
\begin{array}{cccccc}
  . & a & b & c & d & f \\
  a & a & b & c & d & f \\
  b & b & a & c & d & f \\
  c & c & c & c & c & c \\
  d & c & c & c & c & c \\
  f & f & f & c & d & f \\
\end{array}
\]

$\leq = \{(a, a), (b, b), (c, c), (d, c), (d, d), (f, a), (f, b), (f, c), (f, f)\}$. We give the covering relation and the figure of $S$.

$\prec = \{(d, c), (f, a), (f, b), (f, c)\}$.

For an easy way to check that this is an ordered semigroup we refer to [3].

We have $x \in (Sx^2Sx)$ for every $x \in S$ so, by Theorem 9, $S$ is a semilattice, also a complete semilattice of left strongly simple semigroups.

The relation

$\sigma_1 = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d), (f, f)\}$

is a semilattice congruence on $S$ and $(x)_{\sigma_1}$ is a simple and left quasi-regular subsemigroup of $S$ for every $x \in S$.

The relation

$\sigma_2 = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (c, f), (d, c), (d, d), (d, f), (f, c), (f, d), (f, f)\}$

is a complete semilattice congruence on $S$ and $(x)_{\sigma_2}$ is a simple and left quasi-regular subsemigroup of $S$ for every $x \in S$. Note that $\sigma_2$ is equal to the relation $\mathcal{N}$. Finally, there are two left ideals of $S$ and these are the sets $\{c, d, f\}$ and $S$. 
**Theorem 11** An ordered semigroup $S$ is a chain of left strongly simple semigroups if and only if, for every $a, b \in S$, we have

$$ a \in (SabSa) \text{ or } b \in (SabSb). $$

**Proof.** $\implies$. Suppose $\sigma$ is a semilattice congruence on $S$ such that $(x)_\sigma$ is a left strongly simple subsemigroup of $S$ for every $x \in S$ and the set $S/\sigma$ endowed with the relation

$$ (x)_\sigma \preceq (y)_\sigma \iff (x)_\sigma = (xy)_\sigma $$

is a chain. Let now $a, b \in S$. Since $(S/\sigma, \preceq)$ is a chain, we have $(a)_\sigma \preceq (b)_\sigma$ or $(b)_\sigma \preceq (a)_\sigma$. Let $(a)_\sigma \preceq (b)_\sigma$. Then $(a)_\sigma = (ab)_\sigma$ and $a, ab \in (a)_\sigma$. Since $(a)_\sigma$ is a left strongly simple ordered semigroup and $a, ab \in (a)_\sigma$, by Lemma 6, we have $a \in ((a)_\sigma ab(a)_\sigma a)_\sigma \subseteq (SabSa)$. If $(b)_\sigma \preceq (a)_\sigma$, similarly we obtain $b \in (SabSb)$.

$\impliedby$. Let $a \in S$. By hypothesis, we have $a \in (Sa^2Sa)$. For the semilattice congruence $\mathcal{N}$, the $\mathcal{N}$-class $(x)_{\mathcal{N}}$ is a left strongly simple subsemigroup of $S$ for every $x \in S$ (cf. the proof of $(6) \implies (1)$ in Theorem 9). Let now $x, y \in S$ such that $(x)_{\mathcal{N}}, (y)_{\mathcal{N}} \subseteq S/\mathcal{N}$. By hypothesis, we have $x \in (SxySx)$ or $y \in (SxySy)$. Let $x \in (SxySx)$. Since $x \in N(x)$ and $x \leq txyhx$ for some $t, h \in S$, we have $xy \in N(x)$, then $N(xy) \subseteq N(x)$. Let $y \in (SxySy)$. Since $y \in N(y)$ and $y \leq zxyky$ for some $z, k \in S$, we have $xy \in N(y)$, so $N(xy) \subseteq N(y)$. On the other hand, $xy \in N(xy)$ implies $x, y \in N(xy)$, then $N(x) \subseteq N(xy)$ and $N(y) \subseteq N(xy)$. Hence we have $N(xy) = N(x)$ or $N(xy) = N(y)$. Thus $(x)_{\mathcal{N}} = (xy)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} = (xy)_{\mathcal{N}} = (yx)_{\mathcal{N}}$, that is, $(x)_{\mathcal{N}} \preceq (y)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} \preceq (x)_{\mathcal{N}}$. \qed

**Remark 12** An ordered semigroup is a chain of left strongly simple semigroups if and only if it is a complete chain of left strongly simple semigroups.

The right analogue of our results also hold.

For the example of the paper we used computer programs.

**References**


\textbf{Received: October 8, 2011}