Nilpotency of a Finitely Iterated Polynomial Map

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Abstract

Polynomial maps appear in the field of systems and control theory. In this paper, we derive a necessary and sufficient condition for the nilpotency of a polynomial map with respect to iterations.

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1 Introduction

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial map such that \( f(0) = 0 \). The recurrence formula \( x[k+1] = f(x[k]) \) \((x[k] \in \mathbb{R}^n; k \in \mathbb{N})\) is considered in the field of systems and control theory to represent various types of dynamical systems. One of the most fundamental problems in this field is the stability of a dynamical system at the origin. In this paper, we consider the following strong stability that we call nilpotency with respect to iterations: there exists \( m \in \mathbb{N} \) such that \( x[m] = 0 \) for all \( x[0] \in \mathbb{R}^n \). Because \( x[k] = (f \circ \cdots \circ f)(x[0]) \), we can regard the problem as finding \( m \in \mathbb{N} \) such that \( f^m(\mathbb{R}^n) = \{0\} \) for the polynomial map \( f \). On the other hand, in the field of systems and control theory, dead-beat controllability [1] is an important property along with stability. To
test for dead-beat controllability, it is very important to derive a necessary and sufficient condition for nilpotency.

To analyze stability at the origin, i.e., in the case of $k \to \infty$, the Lyapunov stability theorem [2] is a powerful tool. However, generally, this theorem guarantees only sufficiency, and it is difficult to analyze the nilpotency of a finitely iterated map using the Lyapunov stability theorem. It is even difficult to find an upper bound for the number of iterations for a polynomial map to check for nilpotency.

In this paper, we show that the polynomial map $f$ is nilpotent if and only if $f^n(\mathbb{R}^n) = \{0\}$. The result of this paper contributes to not only fields of engineering but also the fields of mathematics of discrete dynamical systems and algebraic geometry.

2 Preliminaries

We use the following results for a polynomial map and an algebraic variety.

Lemma 2.1. [3, p.40] Let $V$ and $W \subset \mathbb{R}^n$ be affine algebraic varieties. A polynomial map $f : V \to W$ is continuous in the Zariski topology.

Lemma 2.2. [4, p.83] Let $X$ and $Y$ be topology spaces, and let a map $f : X \to Y$ be continuous. For a subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$ holds, where $\overline{A}$ is the closure of $A$.

Lemma 2.3. [5, p.73] Let an affine algebraic variety $V \subset \mathbb{R}^n$ be irreducible. For a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$, $\overline{f(V)}$, the Zariski closure of $f(V)$ is also irreducible.

Lemma 2.4. [6, p.93] The algebro-dimension of $\mathbb{R}^n$ is $n$. That is, the maximal length of a descending chain of irreducible subvarieties

$$\mathbb{R}^n \supset V_1 \supset V_2 \supset \cdots \supset V_n \neq \emptyset$$

is $n$.

Our first result is the following lemma.

Lemma 2.5. A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following in the Zariski topology:

$$\mathbb{R}^n \supset \overline{f(\mathbb{R}^n)} \supset \overline{f^2(\mathbb{R}^n)} \supset \cdots \supset \overline{f^n(\mathbb{R}^n)} = \overline{f^{n+1}(\mathbb{R}^n)} = \cdots \quad (1)$$

Proof. From Lemma 2.2, for a subset $A \subset \mathbb{R}^n$, $f(\overline{A}) \subset \overline{f(A)}$ holds in the Zariski topology, where $\overline{A}$ represents the Zariski closure of $A$. On the other
hand, $A \subset \overline{A}$ implies $f(A) \subset f(\overline{A})$. Thus, we have $f(A) \subset f(\overline{A}) \subset \overline{f(A)}$, and consequently, $f(A) \subset \overline{f(A)} \subset f(\overline{A})$. Because $\overline{f(A)} = f(A)$, we obtain

$$f(A) = f(\overline{A}).$$

A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\mathbb{R}^n \supset f(\mathbb{R}^n)$. By iterating the map $f$, we have a descending chain of subsets $\mathbb{R}^n \supset f(\mathbb{R}^n) \supset f^2(\mathbb{R}^n) \supset \cdots$. Thus, we also have a descending chain of affine algebraic varieties

$$\mathbb{R}^n \supset f(\mathbb{R}^n) \supset f^2(\mathbb{R}^n) \supset \cdots. \tag{3}$$

From (2), $\overline{f^k(\mathbb{R}^n)}$ satisfies $\overline{f^k(\mathbb{R}^n)} = \overline{f^{k-1}(f(\mathbb{R}^n))} = \overline{f^{k-2}(f(f(\mathbb{R}^n)))} = \cdots$, which implies

$$\mathbb{R}^n \supset \overline{f(\mathbb{R}^n)} \supset \overline{f(f(\mathbb{R}^n))} \supset \cdots. \tag{4}$$

Note that the corresponding terms between (3) and (4) are equivalent sets. Since $\mathbb{R}^n$ is an irreducible variety, Lemma 2.3 implies that $\overline{f(\mathbb{R}^n)}$ is also an irreducible variety. Moreover, Lemma 2.3 implies that $f(\overline{f(\mathbb{R}^n)})$ is also an irreducible variety. Thus, (4) is a descending chain of irreducible varieties. In fact, (4) is a strictly descending chain. If, for an affine algebraic variety $V \subset \mathbb{R}^n$, $f(V) = V$ holds, then $\overline{f(V)} = \overline{f(V)} \supset V$ also holds. Since the corresponding terms between (3) and (4) are equivalent sets, (3) is also a strictly descending chain of irreducible varieties. According to Lemma 2.4, the maximal length of a strictly descending chain (3) is $n$. Thus, (1) holds. \hfill \Box

### 3 Main Result

Our main result is the following theorem.

**Theorem 3.1.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map such that $f(0) = 0$. There exists $m \in \mathbb{N}$ such that $f^m(\mathbb{R}^n) = \{0\}$ if and only if $f^n(\mathbb{R}^n) = \{0\}$.

**Proof.** (Sufficiency) We simply set $m = n$.

(Necessity) First, we consider the case of $m \leq n$. The conditions $f^m(\mathbb{R}^n) = \{0\}$ and $f(\{0\}) = \{0\}$ lead to $f^n(\mathbb{R}^n) = f^{n-m}(f^m(\mathbb{R}^n)) = f^{n-m}(\{0\}) = \{0\}$.

Next, we consider the case of $m > n$. Since $\{0\} \subset \mathbb{R}^n$ is an affine algebraic variety, $f^m(\mathbb{R}^n) = \{0\}$ leads to $\overline{f^m(\mathbb{R}^n)} = \{0\} = \{0\}$. By using (1), which means that $\overline{f^n(\mathbb{R}^n)} = \overline{f^m(\mathbb{R}^n)}$ holds for $m \geq n$, we have $\overline{f^n(\mathbb{R}^n)} = \overline{f^m(\mathbb{R}^n)} = \{0\}$, which implies $f^n(\mathbb{R}^n) = \{0\}$. \hfill \Box
**Example 3.2.** Let a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ be represented by

$$f(x_1, \ldots, x_n) = (f_1(x_2, \ldots, x_n), f_2(x_3, \ldots, x_n), \ldots, f_{n-1}(x_n), 0), \tag{5}$$

where $f_1(0, \ldots, 0) = 0, f_2(0, \ldots, 0) = 0, \ldots, f_{n-1}(0) = 0$. By iterating the map $f$, we have

$$(f \circ f)(x_1, \ldots, x_n) = (f_1(f_2(x_3, \ldots, x_n), \ldots, f_{n-1}(x_n), 0), \ldots, f_{n-2}(f_{n-1}(x_n), 0, 0)$$

\[ \vdots \]

$$(f \circ \cdots \circ f)(x_1, \ldots, x_n) = (0, \ldots, 0).$$

Thus, the map represented by (5) satisfies the condition of Theorem 3.1.

In Example 3.2, let polynomial functions $f_i(x_1, \ldots, x_n)$ ($i = 1, \ldots, n$) be linear combinations of $x_1, \ldots, x_n$. We thus have a nilpotent matrix. This is a well-known result for a linear map.

**Example 3.3.** Let a polynomial map $f : \mathbb{R}^3 \to \mathbb{R}^3$ be represented by

$$f(x_1, x_2, x_3) = (-2x_1 + 4x_2 + x_1 x_2 + 2x_3 - x_2 x_3, -x_1 + 2x_2 + x_3, x_1 x_2 - x_2 x_3).$$

We compute $(f \circ f \circ f)(x_1, x_2, x_3)$ as follows:

$$(f \circ f)(x_1, x_2, x_3) = (2x_1^2 - 8x_1x_2 + 8x_2^2 - 4x_1x_3 + 8x_2x_3 + 2x_3^2, 0, 2x_1^2 - 8x_1x_2 + 8x_2^2 - 4x_1x_3 + 8x_2x_3 + 2x_3^2),$$

$$(f \circ f \circ f)(x_1, x_2, x_3) = (0, 0, 0).$$

Thus, the polynomial map $f$ satisfies $f^3(\mathbb{R}^3) = 0$. In fact, by the coordinate transformation given by $(y_1, y_2, y_3) = \phi(x_1, x_2, x_3) := (x_1 + x_2, x_2 + x_3, x_1 - x_2)$, we have

$$g(y_1, y_2, y_3) := (f_1(\phi^{-1}(y_1, y_2, y_3)) + f_2(\phi^{-1}(y_1, y_2, y_3)), f_2(\phi^{-1}(y_1, y_2, y_3)) + f_3(\phi^{-1}(y_1, y_2, y_3)),\]

$$f_1(\phi^{-1}(y_1, y_2, y_3)) - f_2(\phi^{-1}(y_1, y_2, y_3)) = (2y_2^2 + 2y_2y_3 + 2y_3, 2y_3, 0).$$

This is a special case of (5).
References


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