\((\in, \in \lor q)\)-Fuzzy Lie \(\sim\) Ideal over Fuzzy Field

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Abstract

The property that leads to the definition of \((\in, \in \lor q)\)-fuzzy Lie \(\sim\) ideal over Fuzzy field is investigated. The relation between fuzzy Lie \(\sim\) ideal and \((\in, \in \lor q)\)-fuzzy Lie \(\sim\) ideal over Fuzzy field is checked. A characterization for \((\in, \in \lor q)\)-fuzzy Lie \(\sim\) ideal over Fuzzy field is obtained. The concept of fuzzy coset of \((\in, \in \lor q)\)-fuzzy Lie \(\sim\) ideal over Fuzzy field and the properties of the set of all fuzzy cosets are discussed.

Mathematics Subject Classification: 17B99, 08A72

Keywords: Fuzzy field, Fuzzy Lie algebra, \((\in, \in)\)-fuzzy field, \((\in, \in \lor q)\)-fuzzy field, \((\in, \in \lor q)\)-fuzzy Lie algebra, \((\in, \in \lor q)\)-fuzzy Lie\(\sim\)ideal

1 Introduction

The basic concepts in the theory of Lie algebra have a group theoretic structure. Rosenfeld [7] introduced fuzzy groups. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [6], helped many mathematicians to generate different types of fuzzy subsystems. Bhakat and Das [3] introduced \((\alpha, \beta)\)-fuzzy subgroup. The \((\in, \in \lor q)\)-fuzzy subgroup is an important generalization of fuzzy group. Since the algebraic structures play a prominent role in mathematics with wide range of applications, similar generalization of fuzzy subsystems of other algebraic structures is relevant. In [2], the authors defined the concept of \((\alpha, \beta)\)-fuzzy Lie algebra over \((\alpha, \beta)\)-fuzzy field and discussed some of its properties.
In this paper we check the properties of \((\in, \in \lor q)\)-fuzzy Lie algebra over a fuzzy field to make a definition of \((\in, \in \lor q)\)-fuzzy Lie\(\sim\)ideal over a fuzzy field.

2 Preliminary Notes

By a fuzzy subset of a non-empty set \(X\), we mean a function from \(X\) to \([0, 1]\). Let \(\text{min}\{t, s\}\) be denoted by \(m(t, s)\) and \(\text{max}\{t, s\}\) be denoted by \(M(t, s)\).

**Definition 2.1** (see [4]). A vector space \(L\) over a field \(X\), with an operation \(L \times L \to L\), denoted \((x, y) \to [x, y]\) and called the bracket or commutator of \(x\) and \(y\), is called a Lie algebra over \(X\), if the following axioms are satisfied:

1. The bracket operation is bilinear
2. \([x, x] = 0\), for all \(x \in L\)
3. \(\[x, [y, z]\] + [y, [z, x]] + [z, [x, y]] = 0\) for all \(x, y, z \in L\).

The axiom(3) is called the Jacobi identity.

**Example 2.2**

In the real vector space \(\mathbb{R}^3\), define \([x, y] = x \times y\), where ‘\(\times\)’ is cross product of vectors for all \(x, y \in \mathbb{R}^3\). Then \(\mathbb{R}^3\) is a Lie algebra over the field \(\mathbb{R}\).

**Definition 2.3** (see [4]). A subspace \(G\) of a Lie algebra \(L\) is called an ideal of \(L\) if \(x \in L, y \in G\), together imply \([x, y] \in G\).

**Definition 2.4** Let \(X\) be a field and let \(F\) be a fuzzy subset of \(X\). Then \(F\) is called a fuzzy field of \(X\) if:

(i) for all \(\lambda, \gamma \in X\), \(F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma))\),
(ii) for all \(\lambda, \gamma \neq 0 \in X\), \(F(\lambda^{-1}) \geq m(F(\lambda), F(\gamma))\).

**Remark.** It is seen that if \(F\) is a fuzzy field of \(X\),

\[F(0) \geq F(1) \geq F(\lambda) = F(-\lambda) = F(\lambda^{-1})\] for all \(\lambda \neq 0 \in X\).

**Definition 2.5** (see [1]) Let \(L\) be a Lie algebra over a field \(X\) and \(A\) be a fuzzy subset of \(L\). Then \(A\) is called a fuzzy Lie algebra of \(L\) over a fuzzy field \(F\) of \(X\), if for all \(x, y \in L, \lambda \in X\),

(i) \(A(x - y) \geq m(A(x), A(y))\)
(ii) \(A(\lambda x) \geq m(F(\lambda), A(x))\)
(iii) \(A([x, y]) \geq m(A(x), A(y))\).

A fuzzy Lie algebra of Lie algebra \(L\) over fuzzy field \(F\) is called fuzzy Lie \(F\)-algebra of \(L\).
Definition 2.6 (see [1]) Let $A$ be a fuzzy Lie $F$-algebra. If for all $x, y \in L$ and $\lambda \in X$,

(i) $A(\lambda x) \geq M(F(\lambda), A(x))$

(ii) $A([x, y]) \geq M(A(x), A(y))$

then $A$ is called a fuzzy Lie $F$-ideal of Lie algebra $L$ over fuzzy field $F$, in short fuzzy Lie $F$-ideal.

Remark.

Let $L$ be a Lie algebra over field $X$ and $F$ be a fuzzy field of $X$. If $G$ is an ideal of $L$, then the characteristic function $\delta_G : L \to I$ defined by $\delta_G(x) = 1$ for $x \in G$, $\delta_G(x) = 0$ for $x \notin G$ is a fuzzy Lie $F$-ideal.

In what follows, let $L$ be a Lie algebra over a field $X$, let $A : L \to [0, 1]$ be a fuzzy set on $L$, and $F : X \to [0, 1]$ be a fuzzy set on $X$. The support of fuzzy set $A$, denoted by $\text{supp}A$, is the crisp set that contains all element of $L$ that have non-zero membership grades in $A$.

Definition 2.7 (see [6]). A fuzzy set $A : L \to [0, 1]$ of the form

$$A(y) = \begin{cases} t \in (0, 1], & \text{if } y = x; \\ 0, & \text{if } y \neq x. \end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$.

A fuzzy point $x_t$ is said to belong to a fuzzy set $A$, written as $x_t \in A$ if $A(x) \geq t$. A fuzzy point $x_t$ is said to be quasi-coincident with a fuzzy set $A$, denoted by $x_t qA$ if $A(x) + t > 1$. This gives the meaning to the symbol $x_t \alpha A$ for $\alpha$ belong to $\{ \in, q, \in \lor q \}$.

For a fuzzy set $A : L \to [0, 1]$ and $t \in (0, 1]$, we denote the level subsets by

$$A_t = \{ x \in L : x_t \in A \}.$$

$$Q(A; t) = \{ x \in L : x_t qA \}.$$

$$J(A; t) = \{ x \in L : x_t \in \lor qA \}.$$

The following notations are used in this paper.

1. $\in \lor q$ means that either belong to or quasi-coincident with

2. $\overline{\alpha}$ means that $\alpha$ does not hold.

Let $\alpha$ and $\beta$ denote any one of $\in, q, \in \lor q$ unless otherwise specified.
Definition 2.8 (see [2]) Let $X$ be a field and $F : X \to [0,1]$ be a fuzzy subset of $X$. Then $F$ is called an $(\alpha, \beta)$-fuzzy field of $X$, if it satisfies the following conditions:

$(i)$ For all $\lambda, \gamma$ in $X$, $\lambda_t \alpha F, \gamma_s \alpha F \Rightarrow (\lambda - \gamma)_m(t,s) \beta F$

$(ii)$ For all $\lambda, \gamma \neq 0$ in $X$, $\lambda_t \alpha F, \gamma_s \alpha F \Rightarrow (\lambda \gamma^{-1})_m(t,s) \beta F$

for all $t, s \in (0, 1)$.

Definition 2.9 (see [2]) Let $L$ be a Lie algebra over a field $X$, $F : X \to [0,1]$ be an $(\alpha, \beta)$-fuzzy field of $X$. Then a fuzzy subset $A : L \to [0,1]$ is called an $(\alpha, \beta)$-fuzzy Lie algebra of $L$ over an $(\alpha, \beta)$-fuzzy field $F$ of $X$, if it satisfies the following conditions:

$(i)$ $x_t \alpha A, y_s \alpha A \Rightarrow (x - y)_m(t,s) \beta A$

$(ii)$ $x_t \alpha A, \lambda_r \alpha F \Rightarrow (\lambda x)_m(t,r) \beta A$

$(iii)$ $x_t \alpha A, y_s \alpha A \Rightarrow ([x, y])_m(t,s) \beta A$

for all $x, y \in L$, for all $\lambda \in X$, for all $t, s, r \in (0, 1)$.

Theorem 2.10 (see [2]) Let $X$ be a field. Then a fuzzy subset $F : X \to [0,1]$ is a fuzzy field if and only if $F$ is an $(\varepsilon, \varepsilon)$-fuzzy field of $X$.

Theorem 2.11 (see [2]) Let $L$ be a Lie algebra over a field $X$. Then a fuzzy subset $A$ of $L$ is a fuzzy Lie algebra over a fuzzy field $F$ of $X$ if and only if $A$ is an $(\varepsilon, \varepsilon)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon)$-fuzzy field $F$ of $X$.

Theorem 2.12 (see [2]) Let $L$ be a Lie algebra over a field $X$. Then a fuzzy subset $A$ of $L$ is an $(\varepsilon, \varepsilon \land q)$-fuzzy Lie algebra of $L$ over a fuzzy field $F$ of $X$ if and only if

$(i)$ for all $x, y \in L$, $A(x - y) \geq m(A(x), A(y), 0.5)$

$(ii)$ for all $x \in L$, $\lambda \in X$, $A(\lambda x) \geq m(F(\lambda), A(x), 0.5)$

$(iii)$ for all $x, y \in L$, $A([x, y]) \geq m(A(x), A(y), 0.5)$

Theorem 2.13 (see [2]) Let $L$ be a Lie algebra over a field $X$. Then every $(\varepsilon, \varepsilon)$-fuzzy Lie algebra of $L$ over a fuzzy field $F$ of $X$ is an $(\varepsilon, \varepsilon \land q)$-fuzzy Lie algebra of $L$ over the fuzzy field $F$.

Proposition 2.14 If $A$ is an $(\varepsilon, \varepsilon \land q)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon \land q)$-fuzzy field $F$, then

$(1)$ $A(0) \geq m(A(x), 0.5)$

$(2)$ $A(-x) \geq m(A(x), 0.5)$

$(3)$ $A(x + y) \geq m(A(x), A(y), 0.5)$. 
Definition 2.15 A fuzzy set $A$ of a set $L$ is said to possess sup property if for every non-empty subset $G$ of $L$, there exists $x \in G$ such that

$$A(x) = \text{Sup}\{A(z) \mid z \in G\}.$$  

Let $f : L \to L'$ be a function. If $A$ and $B$ are fuzzy subsets of $L$ and $L'$ respectively. Then $f(A)$ and $f^{-1}(B)$ are defined using Zadeh’s extension principle (see [5]). If $\alpha$ is one of $\{\in, q, \in \lor q\}$, it follows that $x_t \alpha f^{-1}(B)$ if and only if $(f(x))_t \alpha B$, for all $x \in L$ and for all $t \in (0, 1]$.

Theorem 2.16 Let $L$ and $L'$ be Lie algebras over a field $X$ and $f : L \to L'$ be a homomorphism. If $B$ is an $(\in, q)$-fuzzy Lie algebra of $L'$ over a fuzzy field $F$ of $X$, then $f^{-1}(B)$ is an $(\in, q)$-fuzzy Lie algebra of $L$ over the fuzzy field $F$ of $X$.

Proof: The result is proved by imitating the arguments as in the proof of Theorem 4.14 in [2].

Theorem 2.17 Let $L$ and $L'$ be Lie algebras over a field $X$ and $f : L \to L'$ be an onto homomorphism. Let $A$ be an $(\in, \lor q)$-fuzzy Lie algebra of $L$ over a fuzzy field $F$ of $X$, and let $A$ satisfy the sup property. Then $f(A)$ is an $(\in, q)$-fuzzy Lie algebra of $L'$ over the fuzzy field $F$ of $X$.

Proof: The result is proved by imitating the arguments as in the proof of Theorem 4.16 in [2].

3 Main Results

The fuzzy Lie $F \sim$-ideals have been defined (see Definition 2.6) by changing the conditions of fuzzy Lie $F$-algebra from $\text{min}$ to $\text{max}$. If we try to define $(\in, \lor q)$-fuzzy Lie $\sim$-ideal over a fuzzy field like wise, we obtain

1. $x_t \in A$, $y_s \in A \Rightarrow (x - y)_{m(t,s)} \in \lor q A$
2. $x_t \in A$, $\lambda_r \in F \Rightarrow (\lambda x)_{m(r,t)} \in \lor q A$
3. $x_t \in A$, $y_s \in A \Rightarrow ([x, y])_{M(t,s)} \in \lor q A$.

We can see that the condition (iii)' is not equivalent to condition

$$A([x, y]) \geq M(A(x), A(y), 0.5), \text{ for all } x, y \in L.$$  

It can also be observed that the condition (iii)' is not equivalent to condition

$$A([x, y]) \geq m(M(A(x), A(y)), 0.5), \text{ for all } x, y \in L.$$
Example 3.1

Let \( L = \mathbb{R}^3 \) and \( [x, y] = x \times y \), where ‘\( \times \)' is cross product for all \( x, y \in L \). Then \( L \) is a Lie algebra over the field \( \mathbb{R} \).
Define \( A : \mathbb{R}^3 \to [0, 1] \) for all \( x = (a, b, c) \in \mathbb{R}^3 \), by

\[
A(a, b, c) = \begin{cases} 
0.6 & \text{if } a = b = c = 0 \\
0.8 & \text{if } a \neq 0, b = 0, c = 0 \\
0.4 & \text{otherwise}
\end{cases}
\]

and define \( F : \mathbb{R} \to [0, 1] \) for all \( \lambda \in \mathbb{R} \), by

\[
F(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \mathbb{Q} \\
0.5 & \text{if } \lambda \in \mathbb{Q}(\sqrt{2}) - \mathbb{Q} \\
0 & \text{if } \lambda \in \mathbb{R} - \mathbb{Q}(\sqrt{2})
\end{cases}
\]

Then \( A \) is an \((\in, \in \triangledown)\)-fuzzy Lie algebra of \( \mathbb{R}^3 \) over fuzzy field \( F \) of \( \mathbb{R} \).

Take \( x = (1, 0, 0), \ y = (2, 0, 0) \) in \( \mathbb{R}^3 \). Then \( [x, y] = 0 \) and \( x_{0.1} \in A, y_{0.7} \in A \Rightarrow ([x, y])_{M(0.1, 0.7)} \in \triangledown q A \). But \( A([x, y]) = 0.6 < M(A(x), A(y), 0.5) \).

Take \( x = (2, 0, 0), \ y = (1, 2, 0) \) in \( \mathbb{R}^3 \). Then \( [x, y] = (0, 0, 4) \) and \( x_{0.4} \in A, y_{0.1} \in A \Rightarrow ([x, y])_{M(0.4, 0.1)} \in \triangledown q A \).

But \( A([x, y]) = 0.4 < m(M(A(x), A(y)), 0.5) = 0.5 \).

By observing the properties of fuzzy points, it is possible to formulate a definition of \((\in, \in \triangledown)\)-fuzzy Lie \( \sim \)-ideal similar to the definition of ideal in Lie algebra.

Definition 3.2 An \((\in, \in \triangledown)\)-fuzzy Lie algebra of \( L \) over a fuzzy field \( F \) is called an \((\in, \in \triangledown)\)-fuzzy Lie \( F \sim \)-ideal of \( L \), if

\[
x_t \in A, \ y \in L \Rightarrow ([x, y])_t \in \triangledown q A, \text{ for all } x \in L, t \in (0, 1].
\]

Theorem 3.3 An \((\in, \in \triangledown)\)-fuzzy Lie algebra of \( L \) over a fuzzy field \( F \) is an \((\in, \in \triangledown)\)-fuzzy Lie \( F \sim \)-ideal of \( L \) if and only if

\[
A([x, y]) \geq m(A(x), 0.5) \text{ for all } x, y \in L.
\]

Proof: Suppose that \( A \) is an \((\in, \in \triangledown)\)-fuzzy Lie \( F \sim \)-ideal of \( L \) and \( x \in L \).
Case 1. \( A(x) \geq 0.5 \). Then \( m(A(x), 0.5) = 0.5 \). If possible let \( A([x, y]) < m(A(x), 0.5) \) for some \( y \in L \). Then \( A([x, y]) < 0.5 \leq A(x) \) shows that \( x_{0.5} \in A, A([x, y]) < 0.5, \text{and } A([x, y]) + 0.5 < 1 \). So, \( [x, y]_{0.5} \in \triangledown q A \). This is a contradiction.
Case 2. \( A(x) < 0.5 \). Then \( m(A(x), 0.5) = A(x) \). If possible, let \( A([x, y]) < m(A(x), 0.5) = A(x) \) for some \( y \in L \). Let \( t \in (0, 1] \) be such that \( A([x, y]) < t < A(x) < 0.5 \). Then \( x_t \in A \) and \( A([x, y]) < t, A([x, y]) + t < 1 \). This
shows that \([x, y]_t \in \overline{\eta} \mathcal{A}\) and \(x_t \in A\) for some \(y \in L\), a contradiction. Thus \(A([x, y]) \geq m(A(x), 0.5)\) for all \(x, y \in L\).

Conversely suppose that the condition is satisfied by an \((\in, \in \sim \eta)\)-fuzzy Lie algebra over a fuzzy field \(F\). Let \(x_t \in A\) for \(x \in L, t \in (0, 1]\) and \(y\) be any element of \(L\). Then \(A(x) \geq t\) and \(A([x, y]) \geq m(A(x), 0.5) \geq m(t, 0.5)\). Suppose \(t \leq 0.5\). Then \(A([x, y]) \geq t\) and so \([x, y]_t \in A\). If \(t > 0.5\), then \(A([x, y]) \geq 0.5\) and hence \(A([x, y]) + t \geq 0.5 + t > 0.5 + 0.5 = 1\). In this case \([x, y]_t q A\). Therefore, it follows that \([x, y]_t \in \overline{\eta} A\). Hence \(A\) is an \((\in, \in \sim \eta)\)-fuzzy Lie \(FA\sim\) ideal of \(L\).

□

Remarks.

1. The above theorem shows that the condition in the definition of \((\in, \in \sim \eta)\)-fuzzy Lie \(F\sim\) ideal is equivalent to the condition
\[
A([x, y]) \geq m(A(x), 0.5) \text{ for all } x, y \in L.
\]

2. Suppose \([x, y]_t \in \overline{\eta} A\). Then \(A([x, y]) \geq m(A(x), 0.5)\). Now \(A([y, x]) = A([-[x, y]]) \geq m(A([x, y]), 0.5) \geq m(A(x), 0.5)\). This shows that \([y, x]_t \in \overline{\eta} A\)

**Theorem 3.4** Every fuzzy Lie \(F\sim\) ideal of Lie algebra \(L\) is an \((\in, \in \sim \eta)\)-fuzzy Lie \(F\sim\) ideal.

**Proof:** Let \(A\) be a fuzzy Lie \(FA\sim\) ideal of Lie algebra \(L\). Then \(A\) is a fuzzy Lie algebra over a fuzzy field \(F\). Now by Theorems 2.11 and 2.13, \(A\) is an \((\in, \in \sim \eta)\)-fuzzy Lie algebra over the fuzzy field \(F\). Let \(x_t \in A\) for \(x \in L, t \in (0, 1]\) and \(y\) be any element of \(L\). Then \(A(x) \geq t\). If \(A(y) = 0\), then \(A(x) \geq A(y)\) and hence \(M(A(x), A(y)) = A(x) \geq t\). Since \(A\) is a fuzzy Lie \(F\sim\) ideal of \(L\), \(A([x, y]) \geq M(A(x), A(y)) \geq t\). Thus \([x, y]_t \in A\). If \(A(y) > 0\), then \(M(A(x), A(y)) \geq M(t, 0) = t\) and so \([x, y]_t \in A\). Therefore, \(A\) is an \((\in, \in \sim \eta)\)-fuzzy Lie \(F\sim\) ideal.

□

**Example 3.5**

Consider \(\mathbb{R}^3\) and \([x, y] = x \times y\), where ‘\(\times\)’ is cross product for all \(x, y \in \mathbb{R}^3\). Then \(\mathbb{R}^3\) is a Lie algebra over the field \(\mathbb{R}\).

Define \(A : \mathbb{R}^3 \to [0, 1]\) for all \(x = (a, b, c) \in \mathbb{R}^3\), by
\[
A(a, b, c) = \begin{cases} 
0.6 & \text{if } a = b = c = 0 \\
0.8 & \text{if } a \neq 0, b = 0, c = 0 \\
0.5 & \text{otherwise}
\end{cases}
\]
and define $F : \mathbb{R} \to [0, 1]$ for all $\lambda \in \mathbb{R}$, by

$$F(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \mathbb{Q} \\ 0.5 & \text{if } \lambda \in \mathbb{Q}(\sqrt{2}) - \mathbb{Q} \\ 0 & \text{if } \lambda \in \mathbb{R} - \mathbb{Q}(\sqrt{2}) \end{cases}$$

Applying Theorem 3.3, it can be seen that $\mathbb{R}^3$ is an $(\in, \in \lor q)$-fuzzy Lie $\sim$ideal over the fuzzy field $F$.

Take $x = (2, 0, 0), y = (1, 2, 0)$ in $\mathbb{R}^3$. Then $A(x) = 0.8, A(y) = 0.5$, and $A([x, y]) = 0.5$. Thus, $A([x, y]) < M(A(x), A(y))$. By Definition 2.6, it is not a fuzzy Lie $F \sim$ideal.

**Theorem 3.6** Let $L$ be a Lie algebra over a field $X$ and $G$ be a non-empty subset of $L$. Let $\delta$ denote the characteristic function. Then,

(i) $G$ is a Lie subalgebra of $L$ if and only if $\delta_G$ is an $(\in, \in \lor q)$-fuzzy Lie algebra over the fuzzy field $\delta_X$.

(ii) $G$ is an ideal of $L$ if and only if $\delta_G$ is an $(\in, \in \lor q)$-fuzzy Lie $\delta X \sim$ideal.

**Proof:**

(i). By Definition 2.5, $\delta_G$ is a fuzzy Lie algebra over the fuzzy field $\delta_X$. Then, by Theorems 2.11 and 2.13, $\delta_G$ is an $(\in, \in \lor q)$-fuzzy Lie algebra over the fuzzy field $\delta_X$.

To prove the converse, take $x, y \in G$. Then, by assumption,

$\delta_G(x - y) \geq m(\delta_G(x), \delta_G(y), 0.5) = 0.5$ and $\delta_G([x, y]) \geq m(\delta_G(x), \delta_G(y), 0.5) = 0.5$. So, $x - y \in G$ and $[x, y] \in G$. Let $\lambda \in X$ and $x \in G$. Then, $\delta_G(\lambda x) \geq m(\delta_X(\lambda), \delta_G(x), 0.5) = 0.5$ and hence $\lambda x \in G$. Therefore, $G$ is a Lie subalgebra.

(ii). Let $G$ be an ideal of $L$. Then, $\delta_G$ is a fuzzy Lie $\delta X \sim$ideal and hence, by Theorem 3.4, $\delta_G$ is an $(\in, \in \lor q)$-fuzzy Lie $\delta X \sim$ideal.

Conversely suppose that the condition holds. By (i), $G$ is a Lie subalgebra. Let $x \in G, y \in L$. By assumption, $\delta_G([x, y]) \geq m(\delta_G(x), 0.5) = 0.5$. This shows that $[x, y] \in G$ and hence $G$ is an ideal of $L$.

**Theorem 3.7** Let $L$ be a Lie algebra over a field $X$, $F$ be fuzzy field of $X$, and $A$ be a fuzzy subset of $L$. If $A_{0.5} = L$, then $A$ is an $(\in, \in \lor q)$-fuzzy Lie $F \sim$ideal.

**Proof:** $x - y, [x, y]$ and $\lambda x$ are members of $L$ for $x, y \in L$ and $\lambda \in X$. Then, $A(x) \geq 0.5, A(y) \geq 0.5, A(x - y) \geq 0.5, A([x, y]) \geq 0.5, A(\lambda x) \geq 0.5$ and $m(A(x), A(y), 0.5) = 0.5$. Thus, $A(x - y) \geq m(A(x), A(y), 0.5)$, $A([x, y]) \geq m(A(x), A(y), 0.5)$ and $A([x, y]) \geq m(A(x), A(y), 0.5)$. If $F(\lambda) \geq 0.5$, then $m(F(\lambda), A(x), 0.5) = 0.5$ and hence

$$A(\lambda x) \geq m(F(\lambda), A(x), 0.5).$$
If $F(\lambda) < 0.5$, then $m(F(\lambda), A(x), 0.5) = F(\lambda)$ and so

$$A(\lambda x) \geq 0.5 > F(\lambda) = m(F(\lambda), A(x), 0.5).$$

Therefore, $A$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie $F \sim$ideal.

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\begin{proof}
Suppose that $A$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie $F \sim$ideal. Let $x_t \in A$, $y_s \in A$ for $x, y \in L$ and $t, s \in (0, 1]$. If $M(t, s) = t$, take $x_t \in A, y \in L$. Then, by assumption, $[x, y]_t \in \vee qA$ and hence $[x, y]_{M(t, s)} \in \vee qA$. If $M(t, s) = s$, take $x \in L, y_s \in A$. Then, $[x, y]_s \in \vee qA$ imply that $[x, y]_{M(t, s)} \in \vee qA$.

Conversely, suppose that the condition holds. For $t \in (0, 1]$, $x \in L$ consider $x_t \in A$ and any $y \in L$. Since $\text{supp} A = L$, we have $A(y) > 0$. Then, there exists $s \in (0, 1]$ satisfying $A(y) \geq s > 0$ and $M(t, s) = t$. In this case, by assumption, $x_t \in A$ and $y_s \in A$ together imply that $[x, y]_{M(t, s)} \in \vee qA$ and hence $[x, y]_t \in \vee qA$. Therefore, $A$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie $F \sim$ideal.
\end{proof}

The $(\varepsilon, \in \vee q)$-fuzzy Lie $F \sim$ideals can be characterized by their level ideals.

\begin{theorem}
Let $L$ be a Lie algebra over a field $X$ and $F$ be a fuzzy field of $X$ such that $F_t$ contains at least two elements for all $t \in (0, 0.5]$. Then:

(i) A fuzzy subset $A$ of $L$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie algebra over the fuzzy field $F$ if and only if the non-empty subset $A_t$ of $L$ is a Lie subalgebra over $F_t$ for all $t \in (0, 0.5]$.

(ii) $A$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie $F \sim$ideal if and only if the non-empty subset $A_t$ of $L$ is an ideal for all $t \in (0, 0.5]$.
\end{theorem}

\begin{proof}
(i). Suppose that $A$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie algebra over the fuzzy field $F$. Let $\lambda, \gamma \in F_t$ for $t \in (0, 0.5]$. Since $F$ is a fuzzy field, $F(\lambda) \geq t, F(\gamma) \geq t$ together imply that $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma)) \geq t$. This shows that $(\lambda - \gamma) \in F_t$. Similarly, if $\gamma \neq 0, F(\lambda \gamma^{-1}) \geq t$ and hence $\lambda \gamma^{-1} \in F_t$. Therefore, $F_t$ is a subfield of $X$.

Let $x, y \in A_t$. Then $m(A(x), A(y), 0.5) \geq m(t, 0.5) = t$. Since $A(x + y) \geq m(A(x), A(y), 0.5)$, we have $A(x + y) \geq t$ and hence $x + y \in A_t$. Similarly $[x, y] \in A_t$. Let $\lambda \in F_t, x \in A_t$. Then $m(F(\lambda), A(x), 0.5) \geq m(t, 0.5) = t$. Since $A(\lambda x) \geq m(F(\lambda), A(x), 0.5)$, we have $A(\lambda x) \geq t$. Thus, $\lambda x \in A_t$. Therefore, $A_t$ is a Lie subalgebra over $F_t$.

(ii).
Conversely suppose that the condition holds. Let \( x, y \in L \). If \( A(x) = 0 \) or \( A(y) = 0 \), then \( m(A(x), A(y)) = 0 \) and \( A(x - y) \geq 0 = m(A(x), A(y)) \geq m(A(x), A(y), 0.5) \). If \( A(x) \neq 0, A(y) \neq 0 \), take \( m(A(x), A(y), 0.5) = s \). Then \( s \in (0, 0.5] \) and \( A(x) \geq m(A(x), A(y)) \geq m(A(x), A(y), 0.5) = s \). Similarly, \( A(y) \geq s \). Therefore, \( x \in A_s, y \in A_s \). Then, by assumption, \( x - y \in A_s \), 
\[ A(x - y) \geq s = m(A(x), A(y), 0.5), \quad A([x, y]) = s = m(A(x), A(y), 0.5). \]

Let \( \lambda \in X, x \in L \). If \( F(\lambda) = 0 \) or \( A(x) = 0 \), then \( m(F(\lambda), A(x), 0.5) = 0 \) and \( A(\lambda x) \geq 0 = m(F(\lambda), A(x), 0.5) \). If \( F(\lambda) \neq 0, A(x) \neq 0 \), take \( m(F(\lambda), A(x), 0.5) = r \). Then \( r \in (0, 0.5] \) and \( F(\lambda) \geq r, A(x) \geq r \) and so \( \lambda \in F_r, x \in A_r \). Then, by assumption, \( \lambda x \in A_r \). This shows that 
\[ A(\lambda x) \geq r = m(F(\lambda), A(x), 0.5). \]

Hence, \( A \) is an \((\varepsilon, \in \vee q)\)-fuzzy Lie algebra over fuzzy field \( F \).

(ii). Let \( A \) be an \((\varepsilon, \in \vee q)\)-fuzzy Lie \( F \)-ideal. Let \( x \in A_t, y \in L \). By assumption, \([x, y]_t \in \vee q A\). If \([x, y]_t \in A_t \), then \([x, y]_t \in A_t \). If \([x, y]_t q A\), then \( A([x, y]) + t > 1 \) and since \( t \in (0, 0.5] \), it follows that \( A([x, y]) > t \). Thus, \([x, y] \in A_t \). Therefore, \( A_t \) is an ideal for \( t \in (0, 0.5) \).

To prove the converse suppose that \( A_t \) is an ideal of \( L \) over \( F_t \) for all \( t \in (0, 0.5] \). Let \( x, y \in L \). Take \( m(A(x), 0.5) = t \). Then \( t \in (0, 0.5] \) and \( x \in A_t \).

Thus, \( x \in A_t, y \in L \) together imply that \([x, y] \in A_t \) and so,
\[ A([x, y]) \geq t = m(A(x), 0.5). \]

This shows that \( A \) is an \((\varepsilon, \in \vee q)\)-fuzzy Lie \( F \)-ideal.

\[ \square \]

**Theorem 3.10** Let \( A \) be an \((\varepsilon, \in \vee q)\)-fuzzy Lie algebra of \( L \) over a fuzzy field \( F \). Then, for \( t \in (0, 1] \), \( J(A; t) \) is a Lie subalgebra of \( L \) over \( F_t \), when \( F_t \) contains at least two elements.

**Proof:** \( F_t \) is a subfield of \( X \) for all \( t \in (0, 1] \).

Case 1: Suppose \( t \leq 0.5 \)

Let \( x \in Q(A; t) \). Then \( A(x) + t > 1 \). This shows that \( A(x) > 1 - t \geq 0.5 \geq t \) and hence \( x \in A_t \). Therefore, \( Q(A; t) \subseteq A_t \) and \( J(A; t) = Q(A; t) \cup A_t = A_t \).

By Theorem 3.9, in this case, \( J(A; t) \) is a Lie subalgebra over \( F_t \).

Case 2: Suppose \( t > 0.5 \)

Let \( x, y \in J(A; t) \). We consider three different possibilities, namely

\[ x, y \in A_t, \quad x, y \in Q(A; t) \quad \text{and} \quad x \in Q(A; t), y \in A_t. \]
(i):- \( x, y \in A_t \). Then, \( m(A(x), A(y), 0.5) \geq m(t, 0.5) = 0.5 \). But \( A([x, y]) \geq m(A(x), A(y), 0.5) \) shows that \( A([x, y]) \geq 0.5 \) and hence \( A([x, y]) + t > 0.5 + 0.5 = 1 \). Thus, \([x, y] \in Q(A; t) \subseteq J(A; t)\). Similarly \( x+y \in J(A; t)\).

(ii):- \( x, y \in Q(A; t) \). Then, \( A(x) + t > 1, A(y) + t > 1 \). Since \( A([x, y]) \geq m(A(x), A(y), 0.5) \), it follows that \( A([x, y]) > 1-t \). If not, \( A([x, y]) \leq 1-t \) and so \( m(A(x), A(y), 0.5) \leq 1-t \). Then, either \( A(x) \leq 1-t \) or \( A(y) \leq 1-t \). This would imply that \( x \in qA \) or \( y \in qA \) which is a contradiction. Therefore, \( A([x, y]) > 1-t \) and hence \([x, y] \in J(A; t)\). Similarly \( x+y \in J(A; t)\).

(iii):- \( x \in Q(A; t), y \in A_t \). Then \( A(x) + t > 1 \) and \( A(y) \geq t \). Now it follows that \( A(x+y) > 1-t \). If not, since \( A(x+y) \geq m(A(x), A(y), 0.5) \), we would have \( m(A(x), A(y), 0.5) \leq 1-t < t \) and that would imply either \( x \in qA \) or \( y \in A \), which is a contradiction. Thus, \( A(x+y) + t > 1 \) and hence \( x+y \in Q(A; t) \subseteq J(A; t) \). Similarly \( [x, y] \in J(A; t) \). So, for different possibilities (i),(ii) and (iii) of \( x, y \in J(A; t) \), we have \( x+y \in J(A; t) \) and \([x, y] \in J(A; t)\).

Let \( \lambda \in F_t \) and \( x \in J(A; t) \). Since \( A \) is an \((\varepsilon, \in \vee q)\)-fuzzy Lie algebra over fuzzy field \( F \), \( A(\lambda x) \geq m(F(\lambda), A(x), 0.5) \). Then, \( A(\lambda x) > 1-t \). Otherwise, \( m(F(\lambda), A(x), 0.5) \leq 1-t \) would imply that either \( F(\lambda) \leq 1-t < t \) or \( A(x) \leq 1-t < t \), which is a contradiction. Thus, \( A(\lambda x) + t > 1 \) and so \( \lambda x \in Q(A; t) \subseteq J(A; t) \).

Therefore, \( J(A; t) \) is a Lie subalgebra over \( F_t \) for \( t \in (0, 1] \).

\[\square\]

**Theorem 3.11** Let \( f : L \rightarrow L' \) be a homomorphism of Lie algebras \( L, L' \) over a field \( X \) and \( F \) be a fuzzy field of \( X \). If \( B \) is an \((\varepsilon, \in \vee q)\)-fuzzy Lie \( F \sim \)ideal of \( L' \) then \( f^{-1}(B) \) is an \((\varepsilon, \in \vee q)\)-fuzzy \( F \sim \)ideal of \( L \).

**Proof:** By Theorem 2.16, \( f^{-1}(B) \) is an \((\varepsilon, \in \vee q)\)-fuzzy Lie algebra of \( L \) over the fuzzy field \( F \). Let \( x_t \in f^{-1}(B) \) for \( x \in L, t \in (0, 1] \) and \( y \) be any element of \( L \). Then \( (f(x))_t \in B \) and \( f(y) \in L' \). Since \( B \) is an \((\varepsilon, \in \vee q)\)-fuzzy \( F \sim \)ideal of \( L' \), \([f(x), f(y)]_t \in \vee qB \) and hence \( (f([x, y]))_t \in \vee qB \). This shows that \([x, y]_t \in \vee qf^{-1}(B) \). Hence \( f^{-1}(B) \) is an \((\varepsilon, \in \vee q)\)-fuzzy \( F \sim \)ideal.

\[\square\]

**Theorem 3.12** Let \( L \) and \( L' \) be Lie algebras over a field \( X \), \( F \) be a fuzzy field of \( X \), and \( f : L \rightarrow L' \) be an epimorphism. Let \( A \) be an \((\varepsilon, \in \vee q)\)-fuzzy \( F \sim \)ideal of \( L \) which satisfies the sup property. Then \( f(A) \) is an \((\varepsilon, \in \vee q)\)-fuzzy \( F \sim \)ideal of \( L' \).

**Proof:** By Theorem 2.17, \( f(A) \) is an \((\varepsilon, \in \vee q)\)-fuzzy Lie algebra of \( L' \) over the fuzzy field \( F \). Let \( a_t \in f(A) \) for \( a \in L', t \in (0, 1] \) and \( b \in L' \). Therefore,

\[\text{Sup}\{A(z) \mid z \in f^{-1}(a)\} \geq t \quad \text{and there exists } y \in L \text{ such that } f(y) = b.\]
By \textit{sup property} of $A$, there exists 

$$x \in f^{-1}(a) \text{ such that } A(x) = \text{Sup}\{A(z) \mid z \in f^{-1}(a)\}. $$

Then $A(x) \geq t$ and $y \in L$. Since $A$ is an $(\in, \in \vee q)$-fuzzy Lie $L$-$\sim$ideal of $L$, 

$$[x, y]_c \in \vee q \phi. $$

So, $A([x, y]) \geq t$ or $A([x, y]) + t > 1$.

Now $f(x) = a$, $f(y) = b$ and $[x, y] \in f^{-1}([a, b])$. Thus,

$$f(A)[a, b] = \text{Sup}\{A(z) \mid z \in f^{-1}([a, b])\} \geq A([x, y])$$

and hence $f(A)[a, b] \geq t$ or $f(A)[a, b] + t > 1$. This shows that 

$[a, b]_c \in \vee q f(A)$. Therefore, $f(A)$ is an $(\in, \in \vee q)$-fuzzy Lie $L$-$\sim$ideal of $L'$.

\hfill \Box

**Definition 3.13** Let $A$ be an $(\in, \in \vee q)$-fuzzy Lie $L$-$\sim$ideal of a Lie algebra $L$ and $x \in L$. The fuzzy subset $\tilde{A}_x$ of $L$ defined by $\tilde{A}_x(y) = m(A(y - x), 0.5)$ for all $y \in L$ is called the fuzzy coset determined by $x$ and $A$.

**Remark.** Let $A$ be an $(\in, \in \vee q)$-fuzzy Lie algebra of $L$ over a fuzzy field $F$. Suppose that $A_{0.5} \neq \phi$. So, there exists $z \in L$ such that $A(z) \geq 0.5$. Now, by Proposition 2.14, $A(0) \geq m(A(z), 0.5) = 0.5$. Therefore, $A(0) \geq 0.5$.

**Lemma 3.14** Let $A$ be an $(\in, \in \vee q)$-fuzzy Lie $L$-$\sim$ideal of Lie algebra $L$ with $A_{0.5} \neq \phi$. Then $\tilde{A}_x = \tilde{A}_y$ if and only if $x - y \in A_{0.5}$ for $x, y \in L$.

**Proof:** Suppose that $\tilde{A}_x = \tilde{A}_y$ for $x, y \in L$. Then $\tilde{A}_x(z) = \tilde{A}_y(z)$ for all $z \in L$. So, by definition, $m(A(z - x), 0.5) = m(A(z - y), 0.5)$. Since $A(0) \geq 0.5$, taking $z = x$, we obtain

$$m(A(x - y), 0.5) = m(A(0), 0.5) = 0.5$$

This shows that $A(x - y) \geq 0.5$ and hence $x - y \in A_{0.5}$.

Conversely suppose that, for $x, y \in L$, $x - y \in A_{0.5}$. Then for all $z \in L$,

$$\tilde{A}_x(z) = m(A(z - x), 0.5) = m(A((z - y) - (x - y)), 0.5)$$

$$\geq m(A(z - y), A(x - y), 0.5) = m(A(z - y), m(A(x - y), 0.5))$$

$$= m(A(z - y), 0.5), \text{ by assumption}$$

$$= \tilde{A}_y(z).$$

$$\tilde{A}_y(z) = m(A(z - y), 0.5) = m(A((z - x) + (x - y)), 0.5)$$

$$\geq m(A(z - x), A(x - y), 0.5) = m(A(z - x), m(A(x - y), 0.5))$$

$$= m(A(z - x), 0.5) = \tilde{A}_x(z).$$

Therefore, $\tilde{A}_x = \tilde{A}_y$ for $x, y \in L$.

\hfill \Box
Theorem 3.15 Let \( A \) be an \((\varepsilon, \in \vee q)\)-fuzzy Lie \( F \sim \) ideal of Lie algebra \( L \) over field \( X \) with \( A_{0.5} \neq \phi \) and \( F_{0.5} = X \). Let \( \tilde{L}_A \) denote the set of all fuzzy cosets of \((\varepsilon, \in \vee q)\)-fuzzy Lie \( F \sim \) ideal \( A \) of \( L \). For \( x, y \in L, \lambda \in X \), define \( \tilde{A}_x + \tilde{A}_y = \tilde{A}_{x+y} \); \( \lambda \tilde{A}_x = \tilde{A}_{\lambda x} \); \([\tilde{A}_x, \tilde{A}_y] = \tilde{A}_{[x,y]}\). Then \( \tilde{L}_A \) is a Lie algebra over \( X \).

**Proof:** First we prove that the operations are well defined. Suppose that \( \tilde{A}_{x_1} = \tilde{A}_{y_1} \) and \( \tilde{A}_{x_2} = \tilde{A}_{y_2} \) for \( x_1, x_2, y_1, y_2 \in L \). Then, by Lemma 3.14, \( A(x_1 - y_1) \geq 0.5, A(x_2 - y_2) \geq 0.5 \).

\[
A((x_1 + x_2) - (y_1 + y_2)) = A((x_1 - y_1) + (x_2 - y_2)) \\
\geq m(A(x_1 - y_1), A(x_2 - y_2), 0.5) = 0.5
\]

Therefore, \( \tilde{A}_{x_1 + x_2} = \tilde{A}_{y_1 + y_2} \).

Let \( \lambda \in X \). Since \( A(\lambda(x_1 - y_1)) \geq m(F(\lambda), A(x_1 - y_1), 0.5) = 0.5 \), we have \( \tilde{A}_{\lambda x_1} = \tilde{A}_{\lambda y_1} \).

\[
[x_1 - y_1, x_2] - [x_2 - y_2, y_1] = [x_1, x_2] - [y_1, y_2].
\]

Then \( A([x_1, x_2] - [y_1, y_2]) = A([x_1 - y_1, x_2] - [x_2 - y_2, y_1]) \)

\[
\geq m(A([x_1 - y_1, x_2]), A([x_2 - y_2, y_1]), 0.5) \\
\geq m(m(A(x_1 - y_1), 0.5), m(A(x_2 - y_2), 0.5), 0.5) \\
= 0.5
\]

Therefore \( \tilde{A}_{[x_1, x_2]} = \tilde{A}_{[y_1, y_2]} \).

Thus, the compositions are well defined. For \( x \in L, \tilde{A}_x + \tilde{A}_0 = \tilde{A}_x \) and \( \tilde{A}_x + \tilde{A}_{-x} = \tilde{A}_0 \). So, \( \tilde{A}_0 \) is the zero of \( \tilde{L}_A \).

For \( \lambda, \gamma \in X \) and \( x, y \in L \),

\[
(\lambda + \gamma)\tilde{A}_x = \tilde{A}_{(\lambda + \gamma)x} = \lambda \tilde{A}_x + \gamma \tilde{A}_x, \\
\lambda(\tilde{A}_x + \tilde{A}_y) = \lambda(\tilde{A}_{x+y}) = \tilde{A}_x + \lambda \tilde{A}_y, \\
1 \tilde{A}_x = \tilde{A}_{1x} = \tilde{A}_x.
\]

Therefore, \( \tilde{L}_A \) is a vector space over \( X \). It is easy to verify that the bracket operation is bilinear and it satisfies the Jacobi identity.

\[
[\tilde{A}_x, \tilde{A}_x] = \tilde{A}_{[x,x]} = \tilde{A}_0.
\]

Hence, \( \tilde{L}_A \) is a Lie algebra over \( X \). 

\[\square\]
Theorem 3.16 If \( A \) is an \((\varepsilon, \in, \vee, q)\)-fuzzy Lie \( F \sim \) ideal of Lie algebra \( L \) with non-empty \( A_{0.5} \) and \( F_{0.5} = X \), then the map \( \Phi : L \to \tilde{L}_A \) defined by 
\[
\Phi(x) = \tilde{A}_x \quad \text{for every} \ x \in L
\]
is a homomorphism with kernel \( A_{0.5} \).

Proof: By Theorem 3.15, \( \tilde{L}_A \) is a Lie algebra over \( X \). Let \( x, y \in L \) and \( \lambda, \gamma \in X \).
\[
\Phi(\lambda x + \gamma y) = \tilde{A}_{\lambda x + \gamma y} = \lambda \tilde{A}_x + \gamma \tilde{A}_y = \lambda \Phi(x) + \gamma \Phi(y).
\]
\[
\Phi([x, y]) = \tilde{A}_{[x,y]} = [\tilde{A}_x, \tilde{A}_y] = [\Phi(x), \Phi(y)].
\]

Therefore, \( \Phi \) is a homomorphism of Lie algebras.
\[
Ker \ \Phi = \{x \in L : \Phi(x) = \tilde{A}_0\} = \{x \in L : \tilde{A}_x = \tilde{A}_0\}.
\]
\[
x \in Ker \ \Phi \iff \tilde{A}_x = \tilde{A}_0 \iff A(x - 0) \geq 0.5 \iff x \in A_{0.5}
\]
Thus, \( Ker \ \Phi = A_{0.5} \).

\[\square\]

References


Received: August, 2011