Cofinitely $\delta_M$–Supplemented and Cofinitely $\delta_M$–Semiperfect Modules

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Abstract

Let $M$ be a module over a ring $R$. In this paper cofinitely $\delta_M$-supplemented, $\oplus$-cofinitely $\delta_M$-supplemented, and cofinitely $\delta_M$-semiperfect modules are defined and several properties of these modules are investigated.

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1 Introduction

Throughout this article, $R$ denotes an associative ring with unity and modules $M$ are unitary right $R$-modules. $\text{Mod} - R$ denotes the category of all right $R$-modules. Let $M$ be any $R$-module. Any $R$-module $N$ is $M$-generated (or generated by $M$) if it is a homomorphic image of a direct sum of copies of $M$. An $R$ module $N$ is said to be subgenerated by $M$ if $N$ is isomorphic to a submodule of an $M$-generated module. We denote by $\sigma[M]$ the full subcategory of the right $R$-modules whose objects are all right $R$-modules subgenerated by $M$. Any module $N \in \sigma[M]$ is said to be $M$-singular if $N \cong L/K$ for some $L \in \sigma[M]$ and $K$ is essential in $L$. The class of all $M$-singular modules is closed under submodules, homomorphic images, and direct sums. We denote by $\text{Add} M$ the full subcategory $\sigma[M]$ whose objects are the direct summands of direct sums of copies of $M$. In case of $M = R$, $\sigma[R]=\text{Mod} - R$ and $\text{Add} R$ is just the class of all projective $R$-modules. For any set $I$, we write $M^{(I)}$ for the direct sum of $I$ copies of $M$. The concept of small submodule has been generalized to $\delta$-small submodule by Zhou[15]. Zhou called a submodule $N$ of a module
$M$ is $\delta$-small in $M$ (notation $N \leq_\delta M$) if, whenever $N + X = M$ with $M/X$ singular, we have $X = M$. Özcan and Alkan consider this notation in $\sigma[M]$. For a module $N$ in $\sigma[M]$ Özcan and Alkan [8] call a submodule $L$ of $N$ is $\delta$-

$M$ or $\delta_M$-small submodule, written $L \triangleleft_{\delta_M} N$, in $N$ if $L + K \neq N$ for any proper submodule $K$ of $N$ with $N/K$ $M$-singular. Clearly, if $L$ is $\delta$-small, then $L$ is a $\delta_M$-small submodule. Hence $\delta_M$-small submodules are the generalization of $\delta$-small submodules in the category $\text{Mod} - R$. Also Özcan and Alkan [8] consider the following submodules of a module $N$ in $\sigma[M]$ (see also Zhou[15]), $\delta_M(N) = \cap\{K \leq N : N/K$ is $M -$ singular simple $\}$.

We collect basic properties of $\delta_M$-small submodules in the following lemma which is taken from [8] and [11].

**Lemma 1.1** Let $N \in \sigma[M]$ .

1. For modules $K$ and $L$ with, $K \leq L \leq N$, we have $L \triangleleft_{\delta_M} N$ if and only if $K \triangleleft_{\delta_M} N$ and $L/K \triangleleft_{\delta_M} N/K$.

2. For submodules $K$ and $L$ of $N$, $K + L \triangleleft_{\delta_M} N$ if and only if $K \triangleleft_{\delta_M} N$ and $L \triangleleft_{\delta_M} N$.

3. If $K \triangleleft_{\delta_M} N$ and $f : N \rightarrow L$ is a homomorphism in $\sigma[M]$, then $f(K) \triangleleft_{\delta_M} L$. In particular, if $K \triangleleft_{\delta_M} L$ and $L \leq N$, then $K \triangleleft_{\delta_M} N$.

4. If $K \leq L \leq M$ and $K \triangleleft_{\delta_M} N$ then $K \triangleleft_{\delta_M} L$.

5. $\delta_M(N) = \sum\{L \leq N : L \triangleleft_{\delta_M} N\}$.

6. Let $K$ be a submodule of a module $N$ in $\sigma[M]$. Then $K \triangleleft_{\delta_M} N$ if and only if $N = X \oplus Y$ for a projective semisimple submodule $Y$ in $\sigma[M]$ with $Y \leq K$ whenever $X + K = N$.

Let $L,K$ be two submodules of $M$. Following Kosan [7], $L$ is called a $\delta$-supplement of $K$ in $M$ if $M = L + K$ and $L \cap K \leq L$ . $L$ is called a $\delta$-supplement submodule of $M$ if $L$ is a $\delta$-supplement of some submodule of $M$. $M$ is called a $\delta$-supplemented module if every submodule of $M$ has a $\delta$-supplement in $M$. A submodule $L$ of $M$ has an ample $\delta$-supplement in $M$ if every submodule $K$ of $M$ with $M = L + K$ contains a $\delta$-supplement of $L$ in $M$. A module $M$ is called amply $\delta$-supplemented if every submodule of $M$ has an ample $\delta$-supplement in $M$. These modules are useful in characterizing $\delta$-semiperfect ring. In [15] an epimorphism $f : P \rightarrow M$ is called a projective $\delta$-cover of $M$ if $P$ is projective and $\ker(f) \triangleleft_{\delta} P$. A ring $R$ is called a $\delta$-semiperfect module if every simple $R$-module has a projective $\delta$-cover. Following [7] an $R$-module $M$ is called $\delta$-semiperfect if, for every submodule $K$ of $M$, there exists a decomposition $M = A \oplus B$ such that $A$ is a projective module with $A \leq K$ and $B \cap K \leq M$. Now, let $N \in \sigma[M]$ and $L,K \leq N$. $L$ is called a $\delta_M$-supplement of $K$ in $N$ if $N = K + L$ and $K \cap L \triangleleft_{\delta_M} L$. $L$ is called a $\delta_M$-supplement submodule of $N$ if $L$ is a $\delta_M$-supplement of some submodule of $N$. A submodule $L$ of $N$ has an ample $\delta_M$-supplement in $N$ if every submodule $K$ of $N$ such that $N = L + K$ contains a $\delta_M$-supplement of
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$L$ in $N$. On the other hand $N$ is called (amply, resp.) $\delta_M$-supplemented if every submodule of $N$ has a (an ample, resp.) $\delta_M$-supplement in $N$.

A submodule $K$ of a module $M$ is called cofinite in $M$ if $M/K$ is finitely generated. By a module $M$ is called cofinitely (amply resp.) $\delta$-supplemented if every cofinite module in $M$ has a (an ample resp.) $\delta$-supplement $M$ is called $\oplus$-cofinitely $\delta$-supplemented if every cofinite submodule of $M$ has a $\delta$-supplement which is a direct summand of $M$. If every factor module of $M$ by a cofinite submodule has a projective $\delta$-cover. Cofinitely-$\delta$-supplemented modules and cofinitely $\delta$-semiperfect have been studied by K. Al-TKhman[3], L. V. Thuyet, M. T. Kosan and T.C. Quynh [10] and Y. Wang and D. Wu [13]; our main tool in this papers is to consider these concepts in $\sigma[M]$ and investigate their properties by the help of the techniques used in these papers.

For the other definitions and notations in this paper we refer to Anderson and Fuller [4] and Wisbauer [14]. The following Theorem has been proved by Alattass in [1].

**Theorem 1.2** Let $N$ be in $\sigma[M]$ and $K, L, X \leq N$ such that $X \ll_{\delta_M} N$.

1. If $K$ is a $\delta_M$-supplement of $L$ in $N$, then $K$ is a $\delta_M$-supplement of $L + X$ in $N$.
2. If $K$ is a $\delta_M$-supplement of $L + X$ in $N$, then $N$ has a direct summand $Y$ which is semisimple, projective in $\sigma[M]$ and $K + Y$ is a $\delta_M$-supplement of $L$ in $N$.

2 Cofinitely $\delta_M$-Supplemented Modules.

In this section we define and study cofinitely $\delta_M$-supplemented modules.

**Definition 2.1** A module $N$ in $\sigma[M]$ is called cofinitely $\delta_M$-supplemented, briefly, $\text{cof} \cdot \delta_M$-supplemented, if each cofinite submodule of $N$ has a $\delta_M$-supplement in $N$. $N$ is called $\oplus$-cofinitely-$\delta_M$-supplemented, briefly, $\oplus$-cof-$\delta_M$-supplemented, if each cofinite submodule of $N$ has a $\delta_M$-supplement in $N$ which is a direct summand of $N$.

It is clear from the definitions, that a $\delta_M(\oplus-\delta_M)$-supplemented module is $\text{cof} \cdot \delta_M(\oplus \text{cof} \cdot \delta_M)$-supplemented. The converse is true if the module is finitely generated.

**Theorem 2.2** Every factor module of a $\text{cof} \cdot \delta_M$-supplemented module is $\text{cof} \cdot \delta_M$-supplemented and hence homomorphic images and direct summands of $\text{cof} \cdot \delta_M$-supplemented modules are $\text{cof} \cdot \delta_M$-supplemented.

Proof. Let $N \in \sigma[M]$ be $\text{cof} \cdot \delta_M$-supplemented, and let $K \leq N$. If $L/K$ is a cofinite submodule of $N/K$, then $L$ is cofinite in $N$. Hence, since $N$ is $\text{cof} \cdot \delta_M$-supplemented, there exists a submodule $X$ of $N$ such that $N = L + X$ and
\[ L \cap X \leq_{\delta_M} X. \] Then \( N/K = L/K + (X + K)/K. \) Furthermore \( L/K \cap (X + K)/K = (L \cap (X + K))/K = (K + (L \cap X))/K \leq_{\delta_M} (K + X)/K. \) Thus \( N/K \) is \( \text{cof}-\delta_M\)-supplemented.

Next we show that arbitrary sum of submodules of a module in \( \sigma[M] \) is \( \text{cof}-\delta_M\)-supplemented if each submodule is \( \text{cof} - \delta_M\)-supplemented. To do this we need the following Lemma.

**Lemma 2.3** Let \( N \) be in \( \sigma[M] \) and \( K, L \leq N. \) If \( K \) is \( \text{cof}-\delta_M\)-supplemented, \( L \) is cofinite in \( N \) and \( K + L \) has \( \delta_M\)-supplement \( X \) in \( N \), then \( K \cap (X + L) \) has \( \delta_M\)-supplement \( Y \) in \( K \) such that \( X + Y \) is a \( \delta_M\)-supplement of \( L \) in \( N \).

Proof. Since \( K/K \cap (X + L) \cong (N/L)/(X + L)/L, K/K \cap (X + L) \) is finitely generated. Hence \( K \cap (X + L) \) is cofinite in \( K \) and so there exists a \( \delta_M\)-supplement \( Y \) of \( K \cap (X + L) \) in \( K \). Hence \( K = Y + K \cap (X + L) \) and \( Y \cap (K \cap (X + L)) = Y \cap (X + L) \leq_{\delta_M} Y \). Now we show \( X + Y \) is a \( \delta_M\)-supplement of \( L \) in \( N \). It is clear that \( (X + Y) + L = N \). Moreover \( (X + Y) \cap L \leq X \cap (L + Y) + Y \cap (X + L) \leq X \cap (K + L) + Y \cap (X + L) \leq_{\delta_M} X + Y \). This complete the proof. \( \square \)

**Theorem 2.4** Let \( N \) be in \( \sigma[M] \) and let \( \{N_i\}_{i \in I} \) be a family of submodules of \( N \). If each \( N_i \) is \( \text{cof}-\delta_M\)-supplemented, then \( \sum_{i \in I} N_i \) is \( \text{cof}-\delta_M\)-supplemented.

Proof. Let \( L \) be a cofinite submodule of \( K = \sum_{i \in I} N_i \). Then there exist \( i_1, i_2, \ldots, i_n \) in \( I \) such that \( K = L + N_{i_1} + \ldots + N_{i_n} \). Applying Lemma 2.3 inductively, we get \( L \) has \( \delta_M\)-supplement in \( K \). So \( \sum_{i \in I} N_i \) is \( \text{cof}-\delta_M\)-supplemented. \( \square \)

Now we are going to find necessary and sufficient conditions for a module \( N \) in \( \sigma[M] \) to be a cofinite module. Firstly we need the following Lemma.

**Lemma 2.5** Let \( L, K \) be submodules of a module \( N \) in \( \sigma[M] \) such that \( K \) is a \( \delta_M\)-supplement of a maximal submodule of \( N \) and \( K + L \) has a \( \delta_M\)-supplement in \( N \). Then \( L \) has a \( \delta_M\)-supplement in \( N \).

Proof. Let \( K \) be a \( \delta_M\)-supplement of a maximal submodule \( Q \) of \( N \) and \( X \) is \( \delta_M\)-supplement of \( K + L \) in \( N \). Then \( N = K + Q = X + K + L, K \cap Q \leq_{\delta_M} K \) and \( X \cap (K + L) \leq_{\delta_M} X \).

Consider two cases:

Case (i) If \( K \cap (X + L) \leq K \cap Q \), then we will show that \( X + K \) is a \( \delta_M\)-supplement of \( L \) in \( N \). Clearly we have only to prove \( (X + K) \cap L \leq_{\delta_M} X + K \).

Since \( X \cap (L + K) \leq_{\delta_M} X \) and \( K \cap (X + L) \leq K \cap Q \leq_{\delta_M} K \), by Lemma 1.1, \( X \cap (L + K) + K \cap (X + L) \leq_{\delta_M} X + K \). Then \( (X + K) \cap L \leq_{\delta_M} X + K \) as
L \cap (X + K) \subseteq X \cap (L + K) + K \cap (X + L). Thus X + K is a \( \delta_M \)-supplement of L in N.

Case (ii) \( K \cap (X + L) \) is not a subset of \( K \cap Q \). Since \( K/K \cap Q \cong (K+Q)/Q = N/Q \), \( K \cap Q \) is maximal in \( K \). Hence \( K = (K \cap Q) + (K \cap (X + L)) \). Since \( K \cap Q \ll_{\delta_M} K \), by Lemma 1.1, there exists a semisimple submodule \( Y \) of \( N \) which is projective in \( \sigma[M] \) such that \( K = Y \oplus (K \cap (X + L)) \) and \( Y \subseteq K \cap Q \). We will show that \( X + Y \) is a \( \delta_M \)-supplement of \( L \) in \( N \). We have \( X + Y + L = X + K \cap (X + L) + Y + L = N \). Since \( X \cap (L + K) \ll_{\delta_M} X \), \( X \cap (L + Y) \ll_{\delta_M} X \). Since \( Y \cap (L + X) \leq Y \leq K \cap Q \ll_{\delta_M} K \) and \( Y \) is a direct summand of \( K \), by Lemma 1.1, \( Y \cap (L + X) \ll_{\delta_M} Y \). Hence \( X \cap (L + Y) \) is a \( \delta_M \)-supplement in \( N \). This implies \( L \cap (X + Y) \ll_{\delta_M} X + Y \). Since \( L \cap (X + Y) \) is cofinite in \( N \), \( X \cap (L + X) + Y \cap (L + X) \ll_{\delta_M} X + Y \) (Lemma 1.1). So \( X + Y \) is a \( \delta_M \)-supplement of \( L \) in \( N \).

For any module \( N \) in \( \sigma[M] \), let \( \text{Cof}_{\delta_M}(N) \) be the sum of all submodules of \( N \) that are \( \delta_M \)-supplements of maximal submodules of \( N \) and let \( \text{Cof}_{\delta_M}(N) = 0 \) if there is no such submodule.

**Theorem 2.6** For any module \( N \) in \( \sigma[M] \), the following are equivalent:

(a) \( N \) is \( \text{cof-} \delta_M \)-supplemented.

(b) Every maximal submodule of \( N \) has a \( \delta_M \)-supplement in \( N \).

(c) The module \( N/\text{Cof}_{\delta_M}(N) \) has no maximal submodule.

Proof. (a)⇒(b) is obvious.

(b)⇒(c). Suppose to a contrary that \( N/\text{Cof}_{\delta_M}(N) \) has a maximal submodule \( Q/\text{Cof}_{\delta_M}(N) \). Then \( Q \) is a maximal submodule of \( N \) containing \( \text{Cof}_{\delta_M}(N) \). By (b), \( Q \) has a \( \delta_M \)-supplement \( K \) in \( N \). Hence \( N = K + Q \) and \( K \subseteq \text{Cof}_{\delta_M}(N) \). So \( N = Q \), a contradiction. Then \( N/\text{Cof}_{\delta_M}(N) \) has no maximal submodule.

(c)⇒(a). Let \( L \) be a cofinite submodule of \( N \). Then \( L + \text{Cof}_{\delta_M}(N) \) is also cofinite in \( N \). Hence \( L + \text{Cof}_{\delta_M}(N) = N \) otherwise \( N/\text{Cof}_{\delta_M}(N) \) will have a maximal submodule. Then, since \( N/L \) is finitely generated, there exist a finite number of submodules \( K_1, K_2, \ldots, K_n \) of \( N \) that are \( \delta_M \)-supplements of maximal submodules of \( N \) such that \( N = L + K_1 + K_2 + \ldots + K_n \). By Lemma 2.5, \( L + K_1 + K_2 + \ldots + K_{n-1} \) has a \( \delta_M \)-supplement because \( N = (L + K_1 + K_2 + \ldots + K_{n-1}) + K_n \) has 0 as a \( \delta_M \)-supplement in \( N \). By repeated use of Lemma 2.5, \( L \) has a \( \delta_M \)-supplement in \( N \). Hence \( N \) is \( \text{cof-} \delta_M \)-supplemented.

**Example 2.7** The \( \mathbb{Z} \)-module \( \mathbb{Q} \) of rational numbers has no maximal submodule, so it is \( \text{cof-} \delta_2 \)-supplemented. But it is not \( \delta_2 \)-supplemented.

**Definition 2.8** Let \( N \) be in \( \sigma[M] \). \( N \) is called an amply cofinitely \( \delta_M \)-supplemented, briefly, amply \( \text{cof-} \delta_M \)-supplemented, if every cofinite submodule of \( N \) has an ample \( \delta_M \)-supplement in \( N \).
Theorem 2.9  A module $N \in \sigma[M]$ whose all submodules are cof-$\delta_M$-supplemented modules is amply cof-$\delta_M$-supplemented.

Proof. Let $K$ be a cofinite submodule of $N$ and $L \leq N$ such that $N = L + K$. Then $K \cap L$ is a cofinite submodule of $L$ as $L/(L \cap K) \cong N/K$ and $N/K$ is finitely generated. Since $L$ is cof-$\delta_M$-supplemented, $K \cap L$ has a $\delta_M$-supplement $H$ in $L$. Hence $L = (L \cap K) + H$ and $L \cap K \cap H \ll_{\delta_M} H$, imply that $N = L + K = L \cap K + H + K = K + H$ and $H \cap K \ll_{\delta_M} H$ that is $L$ contains a $\delta_M$-suplement of $K$ in $N$. So $N$ is an amply cofinitely $\delta_M$-supplemented module.

Corollary 2.10  The following are equivalent for any module $M$

(a) Every module in $\sigma[M]$ is an amply cof-$\delta_M$-supplemented module.

(b) Every module in $\sigma[M]$ is a cof-$\delta_M$-supplemented module.

Proof. (a) $\Rightarrow$ (b). This obvious as every amply cof-$\delta_M$-supplemented module is cof-$\delta_M$-supplemented.

(b) $\Rightarrow$ (a). Since $\sigma[M]$ is closed under submodules, (a) follows directly from Theorem 2.9.

If we take $M = R$ in Corollary 2.10, then we get the following corollary. □

Corollary 2.11  ([13], Corollary 3.5) The following are equivalent for a ring $R$.

(a) Every module is an amply cof-$\delta$-supplemented module.

(b) Every module is a cof-$\delta$-supplemented module.

Following [14 page. 359], a module $M$ is called $\pi$-projective if for every two submodules $L, K$ of $M$ with $K + L = M$ there exists $f \in \text{End}(M)$ with $\text{Im}(f) \subseteq K$ and $\text{Im}(1-f) \subseteq L$.

Theorem 2.12  If $N$ is a $\pi$-projective cof-$\delta_M$-supplemented module in $\sigma[M]$, then $N$ is an amply cof-$\delta_M$-supplemented module.

Proof. Let $K$ be a cofinite submodule of $N$ and $L \leq N$ such that $N = K + L$. Since $N$ is $\pi$-projective, there exists $f \in \text{End}(N)$ such that $f(N) \leq K$ and $(1-f)(N) \leq L$. Note that $(1-f)(K) \leq K$ and $N = f(N) + (1-f)(N)$. If $H$ is a $\delta_M$-supplement of $K$ in $N$, then $(1-f)(N) = (1-f)(K) + (1-f)(H) \leq K + (1-f)(H)$. So $N = K + (1-f)(H)$. We claim that $K \cap (1-f)(H) \ll_{\delta_M} (1-f)(H)$. Clearly $(1-f)(H) \leq L$. Since $K \cap H \ll_{\delta_M} K$ by Lemma 1.1, $(1-f)(K \cap H) \ll_{\delta_M} (1-f)(H)$. Now, let $k \in K \cap (1-f)(H)$. Then $k \in K$ and $k = h - f(h)$ for some $h \in H$. Hence $h = k + f(h)$ in $K$ as $k, f(h) \in K$. So $k \in (1-f)(K \cap H)$. Thus $K \cap (1-f)(H) \leq (1-f)(K \cap H) \ll_{\delta_M} (1-f)(H)$. It follows that $N$ is an amply cof-$\delta_M$-supplemented module. □
**Theorem 2.13** Let $N \in \sigma[M]$ be $\text{cof-}\delta_M$ supplemented. Then every cofinite submodule of $N/\delta_M(N)$ is a direct summand.

Proof. Let $K/\delta_M(N)$ be a cofinite submodule of $N/\delta_M(N)$. Hence $N/K$ is finitely generated since $(N/\delta_M(N))/(K/\delta_M(N)) \cong N/K$. So $K$ is a cofinite submodule of $N$. Since $N$ is a $\oplus-\text{cof-}\delta_M$ supplemented module, there exists a submodule $L$ of $N$ such that $N = K + L$ and $K \cap L \leq_{\delta_M} L$. Hence $N/\delta_M(N) = (K + L)/\delta_M(N)$ and $K \cap L \leq_{\delta_M} N$. Thus $N/\delta_M(N) = K/\delta_M(N) \oplus (L + \delta_M(N))/\delta_M(N)$ because $K \cap L \leq_{\delta_M} N$. Hence $K/\delta_M(N)$ is a direct summand of $N/\delta_M(N)$.

**Corollary 2.14** Let $N$ be cof-\(\delta_M\)-supplemented. Then:

1. $N/\delta_M(N)$ is $\oplus$-cof-\(\delta_M\)$-supplemented.
2. If $\delta_M(N)$ is a cofinite submodule of $N$, then $N/\delta_M(N)$ is a semisimple module.

We show that arbitrary direct sum of $\oplus$-cof-\(\delta_M\)-supplemented modules is $\oplus$-cof-\(\delta_M\)$-supplemented. First we prove the following Lemma.

**Lemma 2.15** Let $L$ and $K$ be submodules of a module $N$ in $\sigma[M]$ such that $K + L$ has a $\delta_M$-supplement $X$ in $N$ and $K \cap (X + L)$ has a $\delta_M$-supplement $Y$ in $K$. Then $X + Y$ is a $\delta_M$-supplement of $L$ in $N$.

Proof. We have $N = X + (K + L)$ such that $X \cap (K + L) \leq_{\delta_M} X$ and $K = Y + (K \cap (X + L))$ such that $Y \cap (K \cap (X + L)) \leq_{\delta_M} Y$. Hence, by Lemma 1.1, $X \cap (K + L) + Y \cap (K \cap (X + L)) \leq_{\delta_M} X + Y$. Then $L \cap (X + Y) \leq_{\delta_M} X + Y$. Moreover $N = X + Y + L$. Thus $X + Y$ is a $\delta_M$-supplement of $L$ in $N$. \(\square\)

**Lemma 2.16** Let $N_1$ and $N_2$ be submodules of a module $N \in \sigma[M]$ such that $N = N_1 \oplus N_2$. If $N_1$ and $N_2$ are $\oplus$-cof-\(\delta_M\)$-supplemented modules, then $N$ is also $\oplus$-cof-\(\delta_M\)$-supplemented.

Proof. Let $K$ be a cofinite submodule of $N$. Then $N = N_1 + N_2 + K$ has 0 as a $\delta_M$-supplement in $N$. We have $N_2/(N_2 \cap (N_1 + K)) \cong (N_1 + N_2 + K)/(N_1 + K) = N/(N_1 + K)$. So $N_2 \cap (N_1 + K)$ is a cofinite submodule of $N_2$. Since $N_2$ is $\oplus$-cof-\(\delta_M\)$-supplemented, there exists $L \leq_{\oplus} N_2$ such that $L$ is a $\delta_M$-supplement of $N_2 \cap (N_1 + K)$ in $N_2$. By Lemma 2.15, $L$ is a $\delta_M$-supplement of $N_1 + K$. Similarly since $N_1$ is $\oplus$-cof-\(\delta_M\)$-supplemented, there exists $H \leq_{\oplus} N_1$ such that $H$ is a $\delta_M$-supplement of $N_1 \cap (L + K)$ in $N_1$. Again applying Lemma 2.15, $L + H$ is a $\delta_M$-supplement of $K$ in $N$. Since $H \leq_{\oplus} N_1$ and $L \leq_{\oplus} N_2$, $H + L = H \oplus L$ is a direct summand of $N$. \(\square\)

**Theorem 2.17** A direct sum $\oplus_{i \in I} N_i$ of $\oplus$-cof-\(\delta_M\)$-supplemented modules is a $\oplus$-cof-\(\delta_M\)$-supplemented module.
Proof. Let \( N = \oplus_{i \in I} N_i \) and let \( K \) be a cofinite submodule of \( N \). Then there exists a finitely generated submodule \( L \) of \( N \) such that \( N = K + L \). Hence there exists a finite subset \( J \) of \( I \) such that \( L \leq \oplus_{j \in J} N_j \) and so \( N = K + (\oplus_{j \in J} N_j) \). By Lemma 2.16, \( \oplus_{j \in J} N_j \) is a \( \oplus \)-cof-\( \delta_M \)-supplemented module. Let \( Q = \oplus_{j \in J} N_j \). Then \( N = K + Q \). Note that \( N/K \cong (K + Q)/K \cong Q/(K \cap Q) \). So \( K \cap Q \) is a cofinite submodule of \( Q \). Since \( Q \) is a \( \oplus \)-cof-\( \delta_M \)-supplemented, there exists \( H \leq \oplus Q \) such that \( Q = H + (K \cap Q) \) and \( K \cap Q \leq \delta_M H \). Now \( N = K + Q = K + H \) and \( H \cap Q \leq \delta_M H \). Hence \( H \) is a \( \delta_M \)-supplement of \( K \) in \( N \) and \( H \leq \oplus N \) because \( Q \leq \oplus N \).

The next example gives a \( \oplus \)-cof-\( \delta_M \)-supplemented module which is not \( \oplus \)-cof-supplemented.

**Example 2.18** Let \( M = \mathbb{Z} = R \) and \( N_i = \mathbb{Z} (P^\infty) \) be the Prüfer \( p \)-group for all \( i \in \mathbb{N} \). Then each \( N_i \) is supplemented module. Suppose that \( N = \oplus_{i \in \mathbb{N}} N_i \). By Theorem 2.17 \( N \) is \( \oplus \)-cof-\( \delta_M \)-supplemented but not \( \oplus \)-cof-supplemented by ([7], Example 2.14).

Now we give sufficient coditions for a factor module of a \( \oplus \)-cof-\( \delta_M \)-supplemented module to be \( \oplus \)-cof-\( \delta_M \)-supplemented.

**Theorem 2.19** Let \( N \in \sigma[M] \) be a \( \oplus \)-cof-\( \delta_M \)-supplemented module and \( K \) a submodule of \( N \). If for every direct summand \( L \) of \( N \), \( (K + L)/K \) is a direct summand of \( N/K \), then \( N/K \) is a \( \oplus \)-cof-\( \delta_M \)-supplemented module.

Proof: Suppose that \( L/K \) is a cofinite submodule of \( N/K \). Then \( N/L \) is finitely generated since \( (N/K)/(L/K) \cong N/L \). Since \( N \) is a \( \oplus \)-cof-\( \delta_M \)-supplemented module, there exists a direct summand \( X \) of \( N \) such that \( N = X \oplus Y = L + X \) and \( X \cap L \leq \delta_M X \) for some submodule \( Y \) of \( N \). Now \( N/K = (L/K + (X + K)/K \). By the hypothesis, \( (X + K)/K \) is a direct summand of \( N/K \). Note that \( (L/K) \cap (X + K)/K = (L \cap (X + K))/K \). Since \( X \cap L \leq \delta_M X \), \( (K + (X \cap L))/K \leq \delta_M (X + K)/K \). This implies \( (X + K)/K \) is a \( \delta_M \)-supplement of \( L/K \) in \( N/K \) which is a direct summand.

Recall that a submodule \( X \) of a module \( N \) is called fully invariant if for every \( f \in \text{End}(N) \), \( f(X) \subseteq X \). The module \( N \) is called duo, if every submodule of \( N \) is fully invariant. It is well known that if \( N = N_1 \oplus N_2 \) is a duo module, then \( X = (X \cap N_1) \oplus (X \cap N_2) \), for each submodule \( X \) of \( N \). A module \( N \) is called distributive if its lattice of submodules is a distributive lattice.

**Theorem 2.20** Let \( N \in \sigma[M] \) be a \( \oplus \)-cof-\( \delta_M \)-supplemented. Assume that either \( N \) is

1. a duo module, or
(2) a distributive module. Then every factor module and hence every direct summand of \( N \) is \( \oplus \)-cof-\( \delta_M \)-supplemented.

Proof. Let \( K \leq N \). We show that \( N/K \) is \( \oplus \)-cof-\( \delta_M \)-supplemented.

(1) Suppose that \( N \) is a duo module. As any cofinite submodule of \( N/K \) has the form \( L/K \) where \( L \) is a cofinite submodule of \( N \) and \( K \leq L \), there exist submodules \( X \) and \( Y \) such that \( N = L + X = X \oplus Y \) and \( L \cap X \leq \delta_M X \). Note that \( N/K = (X + L)/K = (X + K)/K + L/K \), by modularity, \( L \cap (X + K) = (L \cap X) + K \). Since \( L \cap X \leq \delta_M X \), we have \( (L/K) \cap (X + K)/K = ((L \cap X) + K)/K \leq \delta_M (X + K)/K \) by Lemma 1.1. This implies \( (X + K)/K \) is a \( \delta_M \)-supplement of \( L/K \) in \( N/K \). Now \( K = (K \cap X) \oplus (K \cap Y) \) implies \((X + K) \cap (Y + K) \leq K\) and \((X + (X \cap K) + (K \cap Y)) \cap Y\). It follows that \((X + K) \cap (X + K) \leq \delta_M (X + K)/K \) and \( N/K = [(X + K)/K] \oplus [(Y + K)/K] \). Then \((X + K)/K \) is a direct summand of \( N/K \). Hence by Theorem 2.19, \( N/K \) is \( \oplus \)-cof-\( \delta_M \)-supplemented.

(2) If \( N \) is distributive, let \( X \) be a direct summand of \( N \). Then \( N = X \oplus Y \), for some \( Y \leq N \). Then \( N = (X + K)/K \oplus (Y + K)/K \) and \( K = K + X \cap Y = (K + X) \cap (K + Y) \) by distributivity of \( N \). This implies that \( N = (X + K)K \oplus (Y + K)/K \). By Theorem 2.19, \( N/K \) is a \( \oplus \)-cof-\( \delta_M \)-supplemented module.

A module \( N \) is said to have the summand sum property (SSP, for short), if the sum of any two direct summands of \( N \) is again a direct summand of \( N \).

**Theorem 2.21.** Every direct summand of a \( \oplus \)-cof-\( \delta_M \) supplemented module with SSP is \( \oplus \)-cof-\( \delta_M \) supplemented.

Proof. Let \( N \in \sigma[M] \) be a \( \oplus \)-cof-\( \delta_M \)-supplemented module with PSS. Let \( K \) be a direct summand of \( N \). Then \( N = K \oplus L \) for some \( L \leq N \). It is suffice to show that \( N/L \) is \( \oplus \)-cof-\( \delta_M \) -supplemented. Suppose that \( X \) is a direct summand of \( N \). Since \( N \) has SSP, \( L + X \) is a direct summand of \( N \). Then \( N = (L + X) \oplus Y \), for some \( Y \leq N \). Then \( N/L = (L + X)/L \oplus (Y + L)/L \). Therefore \( N/L \) is a \( \oplus \)-cof-\( \delta_M \)-supplemented module by Theorem 2.19. □

## 3 Cofinitely \( \delta_M \)-semiperfect Modules.

By the semiperfect in \( \sigma[M] \) and cofinitely \( \delta \)-semiperfect we introduce the notation cofinitely \( \delta_M \)-semiperfect modules and study their basic properties. We also characterize these modules using the notation cofinitely \( \delta_M \)-supplemented modules.

Let \( N, P \in \sigma[M] \). An epimorphism \( f : P \to N \) is called a \( \delta_M \)-cover if \( \text{Ker}(f) \leq \delta_M P \). A \( \delta_M \)-cover \( f : P \to N \) is called a projective \( \delta_M \)-cover in case \( P \) is projective in \( \sigma[M] \).
Definition 3.1 A module $N$ in $\sigma[M]$ is called a cofinitely $\delta_M$-semiperfect module, briefly cof-$\delta_M$-semiperfect, if factor module of $N$ by a cofinite submodule has a projective $\delta_M$-cover.

Next we give some properties of cof-$\delta_M$-semiperfect modules, but first we need the following lemma:

Lemma 3.2 Let $N, P$ and $Q$ be in $\sigma[M]$. Suppose that $f : P \rightarrow N, g : P \rightarrow Q$ and $h : Q \rightarrow N$ are homomorphisms such that $hg = f$. Then $f$ is a projective $\delta_M$-cover if and only if $g(P)$ is a $\delta_M$-supplement of $\text{Ker}(h)$ in $Q$ and $\text{Ker}(g) \leq \delta_M P$.

Proof. Suppose that $f$ is a projective $\delta_M$ -cover. Then $Q = \text{Ker}(h) + g(P)$. Note that $g(\text{Ker}(f)) = \text{Ker}(h) \cap g(P)$. Since $\text{ker}(f) \leq \delta_M P$, by lemma 1.1, $g(\text{Ker}(f)) \leq \delta_M g(P)$. Thus $g(P)$ is a $\delta_M$-supplement of $\text{Ker}(h)$ in $Q$.

Conversely, suppose that $g(P)$ is a $\delta_M$-supplement of $\text{Ker}(h)$ in $Q$ and $\text{Ker}(g) \leq \delta_M P$. Hence $Q = \text{Ker}(h) + g(P)$ and $\text{Ker}(h) \cap g(P) \leq \delta_M g(P)$. Clearly $f$ is epimorphism. It remain to show that $\text{Ker}(f) \leq \delta_M P$. Let $P = \text{Ker}(f) + S$ with $P/S$ $M$-singular. So $g(P) = g(\text{Ker}(f)) + g(S)$. Since $g(P)/g(S)$ is a homomorphic image of $M$-singular module $P/S$ and $g(\text{Ker}(f)) = g(\text{Ker}(h) \cap g(P)) \leq \delta_M g(P), g(P) = g(S)$. So $P = S + \text{Ker}(g)$. But $\text{Ker}(g) \leq \delta_M P$ and $P/S$ is $M$ -singular. Hence $P = S$ and thus $\text{Ker}(f) \leq \delta_M P$. □

Theorem 3.3 Let $N$ be a cof-$\delta_M$-semiperfect module in $\sigma[M]$. Then

1. $N$ is cof-$\delta_M$-supplemented.
2. Homomorphic images of $N$ are cofinitely $\delta_M$ -semiperfect and hence factor modules and direct summands of $N$ are cofinitely $\delta_M$-semiperfect modules.

Proof. (1) Let $K$ be a cofinite submodule of $N$. Then there exists a projective $\delta_M$-cover $f : P \rightarrow N/K$. If $\eta : N \rightarrow N/K$ is the canonical map, then, since $P$ is projective in $\sigma[M]$, there exists a homomorphism $g : P \rightarrow N$ such that $\eta g = f$. Thus, by Lemma 3.2, $g(P)$ is a $\delta_M$ supplement of $K$ in $N$. Therefore $N$ is cof-$\delta_M$ -supplemented.

(2) Let $f : N \rightarrow Q$ be an epimorphism and $X$ be a cofinite submodule of $Q$. Then $f^{-1}(X)$ is cofinite in $N$ as $Q/X \cong N/f^{-1}(X)$. By assumption $N/f^{-1}(X)$ and hence $Q/X$ has a projective $\delta_M$-cover. Then, $Q$ is cof-$\delta_M$-semiperfect. □

Theorem 3.4 (see[1], Theorem 3.2) Let $N$ be in $\sigma[M]$ and $K \leq N$. Then the following are equivalent :

(a) $N/K$ has a projective $\delta_M$-cover.
(b) If $L \leq N$ and $N = K + L$ , then $K$ has a $\delta_M$-supplement $K' \leq L$ such that $K'$ has a projective $\delta_M$-cover.
(c) $K$ has a $\delta_M$-supplement $K'$ which has a projective $\delta_M$-cover.
As immediate consequence of Theorem 3.4 we get the following theorem.

**Theorem 3.5** Let \( N \) be a module in \( \sigma[M] \). Then the following are equivalent:

(a) \( N \) is cof-\( \delta_M \)-semiperfect.
(b) \( N \) is amply cof-\( \delta_M \)-supplemented by \( \delta_M \)-supplements which have a projective \( \delta_M \)-cover.
(c) \( N \) is cof-\( \delta_M \)-supplemented by \( \delta_M \)-supplements which have a projective \( \delta_M \)-cover.

Take \( M = R \) in 3.5 we get the following Corollary.

**Corollary 3.6** ([3], Theorem 3.4) The following are equivalent for any module \( N \):

(a) \( N \) is cof-\( \delta \)-semiperfect.
(b) \( N \) is amply cof-\( \delta \)-supplemented by \( \delta \)-supplements which have a projective \( \delta \)-cover.
(c) \( N \) is cof-\( \delta \)-supplemented by \( \delta \)-supplements which have a projective \( \delta \)-cover.

**Theorem 3.7** Let \( N \in \sigma[M] \) be a projective module in \( \sigma[M] \). Then \( N \) is cof\( \delta_M \)-semiperfect if and only if \( N \) is \( \oplus \)-cof\( \delta_M \)-supplemented.

Proof. First suppose that \( N \) is cof\( \delta_M \)-semiperfect.

Let \( K \) be a cofinite submodule of \( N \). Then, by assumption, \( N/K \) has a projective \( \delta_M \)-cover. Hence, by [8, Lemma 2.14], there are \( N_1, N_2 \leq N \) such that \( N = N_1 \oplus N_2 \) with \( N_1 \leq K \) and \( N_2 \cap K \ll_{\delta_M} N \). Hence, by Lemma 1.1, \( N_2 \cap K \ll_{\delta_M} N_2 \). Hence \( N_2 \) is a \( \delta_M \)-supplement of \( K \) in \( N \).

Conversely, assume that \( K \) is a cofinite submodule of \( N \). Since \( N \) is \( \oplus \)-cof\( \delta_M \)-supplemented, there exists submodule \( K_1 \) and \( K_2 \) of \( N \) such that \( N = K + K_1 \), \( K \cap K_1 \ll_{\delta_M} K_1 \) and \( N = K_1 \oplus K_2 \), clearly, \( K_1 \) is projective in \( \sigma[M] \). For the inclusion homomorphism \( i : K_1 \to N \) and the canonical epimorphism \( \sigma : N \to N/K, \text{Ker}(\sigma i) = K \cap K_1 \ll_{\delta_M} K_1 \). Thus \( \sigma i : K_1 \to N/K \) is a projective \( \delta_M \)-cover. \( \square \)

**Theorem 3.8** Let \( N \in \sigma[M] \) and \( f : Q \to N \) be a \( \delta_M \)-cover. Then

1. If \( N \) is cof\( \delta_M \)-semiperfect, then so is \( Q \).
2. If \( Q \) is a projective in \( \sigma[M] \), then the following are equivalent:
   (a) \( N \) is cof\( \delta_M \)-semiperfect.
   (b) \( Q \) is cof\( \delta_M \)-semiperfect.
   (c) \( Q \) is \( \oplus \)-cof\( \delta_M \)-supplemented.
Proof. (1) Let $N$ be a $\text{cof-} \delta_M$-semiperfect module in $\sigma[M]$ and $f : Q \rightarrow N$ be a $\delta_M$-cover. Suppose that $X$ is a cofinite submodule of $Q$. Then $f(X)$ is a cofinite submodule of $N$. Hence $N/f(X)$ has a projective $\delta_M$-cover

$$g : P \rightarrow N/f(X).$$

The homomorphism $\varphi : Q/X \rightarrow N/f(X)$ defined by $q + X \mapsto f(q) + f(X)$ is an epimorphism. We claim that $\text{Ker}(\varphi) \ll_{\delta_M} Q/X$. Let $Y/X \leq Q/X$ such that $Q/X = \text{Ker}(\varphi) + Y/X$ and $(Q/X)/(Y/X)$ is $M$-singular. So $Q/X = (\text{Ker}(f) + X)/X + Y/X$ as $\text{Ker}(\varphi) = (X + \text{Ker}(f))/X$. Hence $Q = \text{Ker}(f) + Y$ and $Q/Y$ is $M$-singular. This implies $Q = Y$ since $\text{Ker}(f) \ll_{\delta_M} Q$. We have $\text{Ker}(\varphi) = (X + \text{Ker}(f))/X$. Since $P$ is projective in $\sigma[M]$, there is a homomorphism $h : P \rightarrow N/X$ such that $g = \varphi h$. Then $N/X = h(P) + \text{Ker}(\varphi)$.

Let $\eta : P \oplus Z \rightarrow N/X$, defined by $h(p, A) = h(p) + A$. Then $\eta$ is an epimorphism and $\text{Ker}(\eta) = \text{Ker}(h) \oplus 0$. Moreover $\text{Ker}(h) \leq \text{Ker}(g) \leq_{\delta_M} A$. Hence $\text{Ker}(h) \oplus 0 \ll_{\delta_M} P \oplus Z$. Thus $\eta : P \oplus Z \rightarrow N/X$ is a projective $\delta_M$-cover of $N/X$.

(2) $(a) \Leftrightarrow (b)$ By (1) and Theorem 3.3(2).

$(b) \Leftrightarrow (c)$ By Theorem 3.7.

**Theorem 3.9** An arbitrary direct sum of projective modules $P_i, i \in I$ in $\sigma[M]$ is $\text{cof-} \delta_M$-semiperfect if and only if each $P_i$ is $\text{cof-} \delta_M$-semiperfect.

Proof. Let $\{P_i\}_{i \in I}$ be a collection of projective modules in $\sigma[M]$. Suppose that $P = \bigoplus_{i \in I} P_i$ is a $\text{cof-} \delta_M$-semiperfect. Since the projection map $\pi_j : P \rightarrow P_j$ is an epimorphism for every $j$ in $I$, by Theorem 3.3 $\pi_j(P) = P_j$ is $\text{cof-} \delta_M$-semiperfect.

Conversely, assume that each $P_i$ is $\text{cof-} \delta_M$-semiperfect. Hence, by Theorem 3.7, each $P_i$ is $\text{cof-} \delta_M$-supplemented and so $P$ is $\text{cof-} \delta_M$-supplemented by Theorem 2.17. Thus $P$ is $\text{cof-} \delta_M$-semiperfect by Theorem 3.7.

Using 3.9, 3.7, 3.3 and 2.7, we get the following Theorem:

**Theorem 3.10** Let $N \in \sigma[M]$ be a projective module in $\sigma[M]$. Then the following are equivalent:

(a) $N$ is $\text{cof-} \delta_M$-semiperfect.
(b) $N^{(\Lambda)}$ is $\text{cof-} \delta_M$-semiperfect, for any index set $\Lambda$.
(c) $N^{(\Lambda)}$ is $\text{cof-} \delta_M$-supplemented, for any index set $\Lambda$.
(d) $N^{(\Lambda)}$ is $\text{cof-} \delta_M$-semiperfect, for any finite set $\Lambda$.
(e) $N^{(\Lambda)}$ is $\text{cof-} \delta_M$-supplemented for any finite set $\Lambda$.
(f) Every module in $\text{Add } N$ is $\text{cof-} \delta_M$-semiperfect.
(g) Every module in $\text{Add } N$ is $\text{cof-} \delta_M$-supplemented.
(h) Every $N$-generated $R$-module is $\text{cof-} \delta_M$-semiperfect.
(i) $N$ is $\text{cof-} \delta_M$-supplemented.
Recall that $\text{Add} R$ is the class of all projective $R$-modules and an $R$-module is a free $R$-module if and only if it is a direct sum of copies of $R$. Moreover $R$ generates every $R$-module. Hence taking $M = N = R$ in Theorem 3.10 and using the fact finitely generated cof-$\delta$-semiperfect ($\oplus$-cof-$\delta$-supplemented) modules are $\delta$-semiperfect ($\oplus$-$\delta$-supplemented)modules we obtain the following known characterization of a $\delta$-semiperfect ring.

**Corollary 3.11** For any ring $R$, the following are equivalent:

(a) $R$ is $\delta$-semiperfect.
(b) Every free $R$-module is cof-$\delta$-semiperfect.
(c) Every free $R$-module is $\oplus$-cof-$\delta$-supplemented.
(d) Every finitely generated free $R$-module is $\delta$-semiperfect.
(e) Every finitely generated free $R$-module is $\oplus$-$\delta$-supplemented.
(f) Every projective $R$-module is cof-$\delta$-semiperfect.
(g) Every projective $R$-module is $\oplus$-cof-$\delta$-supplemented.
(h) Every $R$-module is cof-$\delta$-supplemented.
(i) $R$ is $\oplus$-$\delta$-supplemented.

**References**


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