Notes on Quasi-Koszul Modules

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Abstract. Let $A$ be a standard graded algebra with graded Jacobson radical $J$. A sufficient condition such that all finitely generated $A$-modules are quasi-Koszul is given. A necessary and sufficient condition for the “minimal Horseshoe Lemma” to be true in the category of quasi-Koszul modules. At last, as an application, we prove that $\text{pd}(M) = \max\{\text{pd}(K), \text{pd}(N)\}$ for an exact sequence $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$ in the category of quasi-Koszul modules with $JK = K \cap JM$, which is a strong version of the classical result in homological algebra: $\text{pd}(M) \leq \max\{\text{pd}(K), \text{pd}(N)\}$.

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1. Introduction

The noncommutative graded algebras play an important role in algebra, topology and mathematical physics. An interesting class of such algebras is Koszul algebras, which are a class of quadratic algebras with a linear resolution and were first introduced by Priddy in 1970 ([14]). Koszul algebras give a nice connection of algebraic objects (quadratic dual algebras) and homological objects (cohomological Yoneda-Ext algebras) ([2]). Many nice homological properties and important applications of Koszul algebras have been shown in research areas of different branches of mathematics ([1]-[3]). Recently, a lot of generalizations of Koszul algebras and modules have been done, see [3] and [6]-[13]. In particular, Green and Martínez-Villa generalized Koszul algebras to the nongraded case, and first studied the Koszulity of Noetherian semiperfect algebras and introduced the notion of quasi-Koszul algebras and quasi-Koszul modules ([4]) in 1996. In 2003, Martínez-Villa and Zacharia also introduced the notion of quasi-Koszul modules in the graded case and we refer [5] for the further details. It should be noted that in this paper, we adore the definitions of quasi-Koszul objects of [5].
From [5], we know that quasi-Koszul modules are a class of finitely generated graded modules over standard graded algebras. But for an arbitrary finitely generated graded module $M$ over a standard graded algebra $A$, we can’t obtain that $M$ is quasi-Koszul in general. One of the aims of this paper is to seek conditions such that all finitely generated graded $A$-modules are quasi-Koszul, where $A$ is a standard graded algebra. We mainly discuss this in Section 2 and prove

**Theorem A** Let $A$ be a standard graded algebra with Jacobson radical $J$. If $A/J$ is a flat $A$-module, then

1. $A$ is a quasi-Koszul algebra, which is equivalent to that $A$ is a Koszul algebra;
2. Any finitely generated $A$-module $M$ is a quasi-Koszul module. In particular, if $M$ is generated in a single degree, then $M$ is a Koszul module.

It is well-known that “Horseshoe Lemma” plays an important role and is a basic tool in homological algebra, but it is a pity that its minimal version is not true in general. In 2008, Wang and Li gave some sufficient conditions for the Horseshoe Lemma to be true in the minimal case in [15]. As an application of quasi-Koszul modules, we will give a necessary and sufficient condition for the minimal Horseshoe Lemma to be true. In particular, the following is the main result of Section 3:

**Theorem B** Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in the category of quasi-Koszul modules. Then $JK = K \cap JM$ if and only if the minimal Horseshoe Lemma holds with respect to such an exact sequence.

As an application of Theorem B, we have

**Corollary C** Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in the category of quasi-Koszul modules such that $JK = K \cap JM$. Then $pd(M) = \max\{pd(K), pd(N)\}$.

Throughout, $k$ denotes an arbitrary fixed ground field. The positively graded $k$-algebra $A = \bigoplus_{i \geq 0} A_i$ will be called standard provided (1) $A_0 = k \times \cdots \times k$ is a finite product of the fixed field $k$; (2) $A$ is generated in degrees 0 and 1; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$ and (3) each $A_i$ is of finite dimension as a $k$-space. The graded Jacobson radical of the standard graded algebra $A$ is obvious $\bigoplus_{i \geq 1} A_i$. Let $Gr(A)$ and $gr(A)$ denote the categories of graded $A$-modules and finitely generated graded $A$-modules, respectively.

2. The proof of Theorem A

We begin with

**Definition 2.1.** ([5]) Let $A$ be a standard graded algebra and $M \in gr(A)$. $M$ will be called a quasi-Koszul module if it admits a minimal graded projective
resolution

\[ \cdots \longrightarrow P_n \overset{d_n}{\longrightarrow} \cdots \longrightarrow P_1 \overset{d_1}{\longrightarrow} P_0 \overset{d_0}{\longrightarrow} M \longrightarrow 0 \]

such that for all \( n \geq 0 \), we have \( J^2 P_n \cap \ker d_n = J \ker d_n \). In particular, \( A \) will be called a \textit{quasi-Koszul algebra} if the trivial \( A \)-module \( A_0 \) is a quasi-Koszul module.

\textbf{Remark 2.2.} It should be noted that the notion of \textit{quasi-Koszul module} here is different from that of [4] and we are still in the graded case.

\textbf{Lemma 2.3.} Let \( A \) be a standard graded algebra and

\[ 0 \longrightarrow K \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0 \]

be an exact sequence in \( \text{Gr}(A) \). Then the reduced sequence

\[ 0 \longrightarrow K \cap J^k M \overset{\tilde{f}}{\longrightarrow} J^k M \overset{\tilde{g}}{\longrightarrow} J^k N \longrightarrow 0 \]

is exact, where \( k \geq 1 \), \( \tilde{f} \) and \( \tilde{g} \) are reduced by \( f \) and \( g \), respectively.

\textit{Proof.} It is a routine check and we omit the details. \hfill \Box

\textbf{Lemma 2.4.} Let \( A \) be a standard graded algebra and \( M \in \text{gr}(A) \). Let \( P \overset{f}{\longrightarrow} M \longrightarrow 0 \) be a graded projective cover of \( M \), where \( P \) is a graded projective module. Then we have the following exact sequence

\[ 0 \longrightarrow \ker f \longrightarrow JP \overset{\tilde{f}}{\longrightarrow} JM \longrightarrow 0. \]

\textit{Proof.} Straightforward. \hfill \Box

\textbf{Lemma 2.5.} Let \( A \) be a standard graded algebra and

\[ 0 \longrightarrow K \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0 \]

be an exact sequence in \( \text{Gr}(A) \). Then the following are equivalent:

(a) \( J^k K = K \cap J^k M \) for all \( k \geq 0 \);
(b) \( A/J^k \otimes_A K \rightarrow A/J^k \otimes_A M \) is a monomorphism for all \( k \geq 0 \);
(c) \( 0 \rightarrow J^k K \rightarrow J^k M \rightarrow J^k N \rightarrow 0 \) is exact for all \( k \geq 0 \);
(d) \( 0 \rightarrow J^k K/J^{k+1} K \rightarrow J^k M/J^{k+1} M \rightarrow J^k N/J^{k+1} N \rightarrow 0 \) is exact for all \( k \geq 0 \);
(e) \( 0 \rightarrow J^k K/J^m K \rightarrow J^k M/J^m M \rightarrow J^k N/J^m N \rightarrow 0 \) is exact for all \( m > k \).

\textit{Proof.} The equivalence of (a) and (b) has been proved in [5]. We only prove (a) \( \Leftrightarrow \) (c) \( \Leftrightarrow \) (d) \( \Leftrightarrow \) (e). For all \( k \geq 0 \), by Lemma 2.3, \( J^k K = K \cap J^k M \) is equivalent to that the sequence

\[ 0 \longrightarrow J^k K \longrightarrow J^k M \longrightarrow J^k N \longrightarrow 0 \]
is exact, which is equivalent to that the following commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \longrightarrow & J^nK \\
\downarrow & & \downarrow \\
0 & \longrightarrow & J^nM \\
\downarrow & & \downarrow \\
0 & \longrightarrow & J^nN \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & J^kK \\
\downarrow & & \downarrow \\
0 & \longrightarrow & J^kM \\
\downarrow & & \downarrow \\
0 & \longrightarrow & J^kN \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & J^kK/J^nK \\
\downarrow & & \downarrow \\
0 & \longrightarrow & J^kM/J^nM \\
\downarrow & & \downarrow \\
0 & \longrightarrow & J^kN/J^nN \\
\end{array}
\]

where \( m > k \) and \( m \geq 0 \). Now by “3 × 3” Lemma, we are done.

Now we are ready to prove Theorem A.

**Proof.** First we prove the “quasi-Koszul” case. It suffices to prove (2) since (1) is a special case of (2). Let \( A \) be a standard graded algebra and \( M \) an arbitrary finitely generated graded \( A \)-module. Then we have the exact sequence

\[
0 \longrightarrow \ker d_0 \longrightarrow P_0 \overset{d_0}{\longrightarrow} M \longrightarrow 0,
\]

where \( P_0 \overset{d_0}{\longrightarrow} M \longrightarrow 0 \) is a graded projective cover of \( M \). By Lemma 2.4, we have the following reduced exact sequence

\[
0 \longrightarrow \ker d_0 \longrightarrow JP_0 \overset{\widetilde{d_0}}{\longrightarrow} JM \longrightarrow 0.
\]

Note that \( A/J \) is flat as an \( A \)-module, which implies the following exact sequence

\[
0 \longrightarrow A/J \otimes_A \ker d_0 \longrightarrow A/J \otimes_A JP_0 \overset{\widetilde{d_0}}{\longrightarrow} A/J \otimes_A JM \longrightarrow 0.
\]

By Lemma 2.5, we have \( J \ker d_0 = \ker d_0 \cap J(JP_0) = \ker d_0 \cap J^2P_0 \). Now replacing \( M \) by \( \ker d_0 \) and repeating the above argument, we can get the following exact sequence

\[
0 \longrightarrow A/J \otimes_A \ker d_1 \longrightarrow A/J \otimes_A JP_1 \overset{\widetilde{d_1}}{\longrightarrow} A/J \otimes_A J \ker d_0 \longrightarrow 0.
\]

Now by Lemma 2.5 again, we get \( J \ker d_1 = \ker d_1 \cap J^2P_1 \). By an induction, we can obtain a minimal graded projective resolution of \( M \)

\[
\cdots \longrightarrow P_n \overset{d_n}{\longrightarrow} \cdots \longrightarrow P_1 \overset{d_1}{\longrightarrow} P_0 \overset{d_0}{\longrightarrow} M \longrightarrow 0
\]
such that for all $n \geq 0$, we have $J^2P_n \cap \ker d_n = J \ker d_n$. Therefore, $M$ is quasi-Koszul.

For the latter part of Theorem A, i.e., the “Koszul” case, which is immediate from the following result (see [4]).

- Let $M$ be a graded module over a standard graded algebra and be generated in a single degree. Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

be a minimal graded projective resolution of $M$. Then $M$ is Koszul if and only if for all $n \geq 0$, we have $J^2P_n \cap \ker d_n = J \ker d_n$.

\[ \square \]

Remark 2.6. (1) Usually, in the minimal graded projective resolution $P_* \xrightarrow{d_*} M \longrightarrow 0$, $\ker d_i$ is usually denoted by $\Omega^{i+1}(M)$ and called the $(i+1)^{th}$ syzygy of $M$.

(2) Unfortunately, the author can’t find any nontrivial examples of graded algebras satisfying the condition of Theorem A until now. Now we only have:

Example 2.7. Graded semisimple algebras obviously satisfy the condition of Theorem A.

3. The proofs of Theorem B and Corollary C

We will prove Theorem B and Corollary C in this section.

Lemma 3.1. Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of finitely generated graded $A$-modules. Then $JK = K \cap JM$ if and only if we have the following commutative diagram with exact rows and columns

$\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \Omega^1(K) & \Omega^1(M) & \Omega^1(N) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
P_0 & Q_0 & L_0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
K & M & N & 0, \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}$

where $P_0 \rightarrow K \rightarrow 0$, $Q_0 \rightarrow M \rightarrow 0$ and $L_0 \rightarrow N \rightarrow 0$ are projective covers.
Proof. (⇒) Clearly, we obtain the exact sequence $0 \mapsto K/JK \mapsto M/JM \mapsto N/JN \mapsto 0$.

Note that for any finitely generated $A$-module $M$, $A \otimes_{A_0} M/JM \mapsto M \mapsto 0$ is a projective cover. Now set $P_0 := A \otimes_{A_0} K/JK$, $Q_0 := A \otimes_{A_0} M/JM$ and $L_0 := A \otimes_{A_0} N/JN$. We have the following exact sequence $0 \mapsto P_0 \mapsto Q_0 \mapsto L_0 \mapsto 0$ since $A_0$ is semisimple. Therefore, we have the desired diagram.

(⇐) Suppose that we have the above commutative diagram. Note that the projective cover of a module is unique up to isomorphisms. We may assume that $P_0 := A \otimes_{A_0} K/JK$, $Q_0 := A \otimes_{A_0} M/JM$ and $L_0 := A \otimes_{A_0} N/JN$. From the middle row of the diagram, we have the following exact sequence

$$0 \mapsto A \otimes_{A_0} K/JK \mapsto A \otimes_{A_0} M/JM \mapsto A \otimes_{A_0} N/JN \mapsto 0.$$ 

Thus, we have the following short exact sequence as $A_0$-modules

$$0 \mapsto K/JK \mapsto M/JM \mapsto N/JN \mapsto 0$$

since $A_0$ is semisimple, which implies $JK = K \cap JM$. □

Lemma 3.2. Let $0 \mapsto K \mapsto M \mapsto N \mapsto 0$ be a short exact sequence of finitely generated graded $A$-modules. Then $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$ if and only if, for any given diagram

\[
\begin{array}{ccccccccc}
& & & & \mathcal{P}_* & & \mathcal{L}_* & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \mapsto & K & \mapsto & M & \mapsto & N & \mapsto & 0 \\
& & & & \downarrow & & \downarrow & & \\
0 & \mapsto & 0 & \mapsto & 0 & \mapsto & 0 & \mapsto & 0 \\
\end{array}
\]

with both sides being minimal graded projective resolutions. Then it can be perfected as the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & \mapsto & \mathcal{P}_* & \mapsto & \mathcal{Q}_* & \mapsto & \mathcal{L}_* & \mapsto & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \mapsto & K & \mapsto & M & \mapsto & N & \mapsto & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \mapsto & 0 & \mapsto & 0 & \mapsto & 0 & \mapsto & 0 \\
\end{array}
\]

such that the middle row is also a minimal graded projective resolution and for all $n \geq 0$, we have $Q_n = P_n \oplus L_n$. That is, the minimal Horseshoe Lemma holds.

Proof. By Lemma 3.1, $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$ if and only if, for all $i \geq 0$, we have the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
& & & & \mathcal{P}_* & & \mathcal{L}_* & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \mapsto & K & \mapsto & M & \mapsto & N & \mapsto & 0 \\
& & & & \downarrow & & \downarrow & & \\
0 & \mapsto & 0 & \mapsto & 0 & \mapsto & 0 & \mapsto & 0 \\
\end{array}
\]
where $P_i$, $Q_i$ and $L_i$ are projective covers of $\Omega^i(K)$, $\Omega^i(M)$ and $\Omega^i(N)$, respectively. Now putting this commutative diagrams together, we finish the proof.

Now we can prove Theorem B.

*Proof. ($\Rightarrow$) By Lemma 3.1, we have the following commutative diagram with exact rows and columns

$$
\begin{array}{c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\Omega^{i+1}(K) & \Omega^{i+1}(M) & \Omega^{i+1}(N) \\
\downarrow & \downarrow & \downarrow \\
P_i & Q_i & L_i \\
\downarrow & \downarrow & \downarrow \\
\Omega^i(K) & \Omega^i(M) & \Omega^i(N) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
$$

such that $P_0 \to K \to 0$, $Q_0 \to M \to 0$ and $L_0 \to N \to 0$ are projective covers. Now applying the functor $A/J \otimes_A -$ to the above diagram, note that $K$, $M$ and $N$ are quasi-Koszul modules and by Lemma 2.5, we have the following commutative diagram with exact rows and columns
which implies that $\alpha_1$ is a monomorphism. By Lemma 2.5 again, we have $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$. Now repeating the above argument, we get the following commutative diagram with exact rows and columns

$$
\begin{array}{c}
\begin{array}{c}
0 \\
A/J \otimes_A \Omega^1(K) \xrightarrow{\alpha_1} A/J \otimes_A \Omega^1(M) \rightarrow A/J \otimes_A \Omega^1(N) \rightarrow 0 \\
0 \\
0 \\
A/J \otimes_A JP_0 \\
0 \\
A/J \otimes_A JK \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0 \\
A/J \otimes_A \Omega^1(K) \xrightarrow{\alpha_{i+1}} A/J \otimes_A \Omega^1(M) \rightarrow A/J \otimes_A \Omega^1(N) \rightarrow 0 \\
0 \\
0 \\
A/J \otimes_A JP_i \\
0 \\
A/J \otimes_A JK^i \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\end{array}
\end{array}
$$

for all $i \geq 1$, which implies that $\alpha_{i+1}$ is a monomorphism for all $i \geq 1$. By Lemma 2.5 again and again, we have $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. By Lemma 3.2, we finish the proof of this direction.

$(\Leftarrow)$ By Lemma 3.2, the minimal Horseshoe Lemma is true if and only if $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. In particular, we have $JK = K \cap JM$.

As an application of Theorem B, we can prove Corollary C.

**Proof.** By Theorem B, we have $P_* \longrightarrow K \longrightarrow 0$, $Q_* \longrightarrow M \longrightarrow 0$ and $L_* \longrightarrow N \longrightarrow 0$ are minimal graded projective resolution of $K$, $M$ and $N$, respectively, and moreover, for all $n \geq 0$, we have $Q_n = P_n \oplus L_n$. \qed
Notes on quasi-Koszul modules

If $\text{pd}(M) = \infty$, then there exists an infinite minimal graded projective resolution of $M$

$$
\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0.
$$

Note that the minimal graded projective resolution of a module is unique up to isomorphisms. Thus at least one of

$$
\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow K \longrightarrow 0
$$

and

$$
\cdots \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow N \longrightarrow 0
$$

is infinite, which implies $\text{pd}(M) = \max\{\text{pd}(K), \text{pd}(N)\}$.

If $\text{pd}(M) = n < \infty$, then there exists a minimal graded projective resolution of $M$

$$
0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,
$$

which implies that $K$ and $N$ possess the following minimal projective resolutions

$$
0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow K \longrightarrow 0,
$$

$$
0 \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow N \longrightarrow 0
$$

such that at least one of $P_n$ and $L_n$ isn’t zero, which implies $\text{pd}(M) = \max\{\text{pd}(K), \text{pd}(N)\}$ as well.

\[\square\]

\textbf{References}


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