A Note on Linear Homomorphisms in R-Vector Spaces

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Abstract

In this paper we give necessary and sufficient condition for a linear transformation to be strongly linear. Further we prove that if $G^*$ is a basis for $V$ then $T(G^*)$ is a basis for $T(V)$

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Introduction

The concept of Bounded Boolean extension of an group is due to A.L.Foster[1]. He demonstrated that each element of a p-ring $R$ with unity can be represented as a type of Boolean vector over the Boolean algebra of all the idempotent elements of $R$. Later Penning [4] and Zemmer [9] have simplified the proof of A. L. Foster concerning a basis consisting of non zero elements of the additive subgroup of $R$ by
its unity element. Subrahmanyam [5] motivated by the concept of A.L.Foster has introduced the notion of abstract vector space over a Boolean algebra (simply Boolean vector space). In fact Boolean vector space is a natural generalization of this idea of A.L.Foster. The contribution of Subrahmanyam’s work is in [5,6,7]. Later Raja Gopala rao [2] has generalized the concept of B-vector spaces to vector spaces over regular rings (simply R-vector spaces). He studied several properties of these spaces in [2,3], generalizing the results of Subrahmanayam. Also Venkateswarlu [8] has introduced the concept of direct sums in R-vector spaces and has proved that \((\sum \nabla G_i)^*\) is a basis for \((\sum \nabla V_i)\) if \(V_1, \ldots, V_n\) are vector spaces over the same regular ring \(R\) having bases \(G_1^*, \ldots, G_n^*\) respectively.

In this paper we introduce the concept of strong linear homomorphism from an R-Vector space \(V\) into another R-Vector space \(W\) and give a necessary and sufficient condition of a linear homomorphism to be strongly linear homomorphism (see theorem 2.5). Also we prove that \(G^*\) is a basis for \(V\) then \(T(G^*)\) is a basis for \(T(V)\) where \(T : V \rightarrow W\) is an isomorphism (see theorem 2.6)

1. Preliminary Results

We collect certain definitions and results concerning R-Vector spaces of Raja Gopala rao [2,3]

**Definition 1.1.** A ring \(R\) is called a regular ring \(\iff\) to each \(a \in R\) there is an element \(x \in R\) such that \(axa = a\).

If \(a \in R\) and \(a \times a = a\) then we put \(a^* = xax\)

For any \(a \in R\), define norm of an element ‘\(a\)’ which will be denoted by \(\|a\|\), by putting \(a = a^{**}\). \(B\) denotes the set of all idempotent elements of regular ring \(R\).

**Lemma 1.2.** 1. \(\|a\| = a \iff a \in B\)

2. If \(ab = 0\) then \(b \|a\| = a \|b\| = \|a\| \|b\| = 0\)

3. \(\|a\| a = a \|a\|\)

4. \(\|a\| (1 + a - \|a\|) = a\)
Linear homomorphisms in $R$-vector spaces

Let $(V,+)$ be any group, $R = (R, +, .)$ be a commutative regular ring with unity element 1

$B = (B, \cup \cap ,')$ denotes the Boolean algebra of idempotents of $R$ (for $a, b \in B$, $a \cup b = a + b - ab$, $a \cap b = ab$, $a' = 1 - a$) and $U$ denotes the multiplicative group of invertible elements of $R$

**Definition 1.3.** $V$ is said to be a vector space over $R$ (or an abstract $R$-vector space) if and only if there exists a mapping $: R \times V \to V$ (the image of any $(a, x)$ in $R \times V$ will be denoted by $ax$ such that for all $x, y \in V$ and $a, b \in R$, the following properties hold

1. $a^2 (x + y) = ax + ay$
2. $a(bx) = (ab)x$ if $a^2 = a$
3. $1x = x$
4. $(a + b)x = ax + bx$ if $ab = 0$
5. $r(sx) = (rs)x$ if $r$ and $s$ are invertible elements of $R$ and
6. $a0 = 0$ implies $a^2 = a$

We denote 0 element of $R$ and 0 element of $V$ as ‘0’

**Remark 1.4.** Any $R$-vector space can be treated as a Boolean vector space (simply $B$-vector space) where $B$ is the Boolean algebra of idempotents of $R$ with respect to the scalar multiplication as defined in $R$-vector space.

**Definition 1.5.** A $R$-vector space $V$ is said to be $B$-normed (or simply normed) $\iff$ there exists a mapping $\| \cdot \| : V \to B$ satisfying the properties

1) $\| x \| = 0 \iff x = 0$
2) $\| ax \| = a \| x \|$ for all $x \in V$, $a \in B$

**Definition 1.6.** A finite subset of nonzero elements $x_1, x_2, \ldots, x_n$ of a $R$-vector space $V$, is called (linearly) independent over $R$ $\iff a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0$ and $a_1 a_2 a_3 \ldots a_n \neq 0$ imply $x_1 + x_2 + \ldots + x_n$ and a subset $S$ of non zero elements
of V is said to be independent subset of V ⇔ every finite subset of S of V is linearly independent over R

Definition 1.7. If S is a subset of V, we say S spans V ⇔ each x ∈ V can be written as \( x = \sum_{g \in S \cup \{0\}} a_g g \) where \( a_g a_h = 0 \) if \( g, h \in S \cup \{0\} \) and \( g \neq h \), \( a_g = 0 \) for almost all \( g \in S \cup \{0\} \) and \( \sum_{g \in S \cup \{0\}} |a_g| = 1 \)

Definition 1.8. An independent subset of V which spans V is called basis for V

Theorem 1.9. G is a subgroup of V where \( G = G^* \cup \{0\} \)

Lemma 1.10. If \( g \in G^* \) and \( r \in U \) then \( |rg| = 1 \)

Theorem 1.11. The representation of each \( x \in V \) in terms of a basis group is unique. If G is a basic group for V then \( a_{x,g} \) represents the uniquely determined coefficient of any \( g \in G \) in the representation of \( x \) in terms of \( G \) and \( x = \sum_{g \in G} a_{x,g} g \)

2 Linear Homomorphisms of R vector spaces

Raja gopala rao [2] introduced the concept of linear endomorphisms in R Vector Spaces. We consider the notion of linear Homomorphisms from one R-vector Space to another.

Now we begin with the following

Definition 2.1. Let \( V \) and \( W \) be R vector spaces over a regular ring R. A mapping \( T : V \rightarrow W \) is called linear homomorphism of \( V \) into \( W \) if for all \( x, y \in V \) and \( a, b \in R \)

1. \( T(ax + by) = aTx + bTy \) if \( ab = 0 \)

The set of linear homomorphisms will be denoted by Hom(\( V, W \)).

Definition 2.2. Let \( V, W \) be R vector spaces. A mapping \( T : V \rightarrow W \) is called strongly linear homomorphism of \( V \) into \( W \) provided

2. \( T(ax + by) = aTx + bTy \) for all \( x, y \in V \) and \( a, b \in R \)
The set of all strongly linear homomorphisms will be denoted by $\text{Hom}^\prime(V,W)$

**Definition 2.3.** Let $V, W$ be $\mathbb{R}$ vector spaces. A one to one mapping $T$ from $V$ onto $W$ is called an isomorphism provided

1. $T(x + y) = Tx + Ty$
2. $T(ax) = aTx$ for $a \in U$

**Theorem 2.4.** Let $V$ and $W$ be $\mathbb{R}$-Vector spaces. A mapping $T$ of $V$ into $W$ is an element of $\text{Hom}(V,W)$ $\iff T(ax) = aT(x)$ for all $x \in V$ and $a \in \mathbb{R}$

**Proof.** $\Rightarrow$: Let $T \in \text{Hom}(V,W)$. If $x \in V$ and $a \in \mathbb{R}$ then

$$T(ax) = T(ax + 0x) = aTx + 0 = aTx$$

$\Leftarrow$: Let $T(ax) = aTx$. Let $x, y \in V$ and $a, b \in \mathbb{R}$ such that $ab = 0$

Consider $T(ax + by) = T((ax + by) = (a + 1) T(ax + by) = aT(ax + by) + T((1 - a) (ax + by))$

$$= T(a \cdot T(ax + by)) + T((1 - a) (ax + by))$$

$$= T((a \cdot ax) + (a \cdot by)) + T((1 - a) (ax) + (1 - a) (by))$$

$$= T((ax + by) + T(0x + (b - a)y)$$

$$= T(ax) + T(by)$$

$$= aTx + bTy$$

Now we give a necessary and sufficient condition for linear transformation to be strongly linear in the following

**Theorem 2.5.** Let $V, W$ be $\mathbb{R}$-vector spaces. Let $T \in \text{Hom}(V,W)$. A necessary and sufficient condition for $T$ is strongly linear is that $T(x + y) = Tx + Ty$ for all $x, y \in V$

**Proof.** Necessary condition follows by putting $a = b = 1$ in definition 2.2
For sufficiency, let $T \in \text{Hom}(V, W)$ and $x, y \in V$.

Then $T(ax+by) = T(ax) + T(by)$ (by hypothesis)
$$= T\left[ a \left( a +1 - \left| a \right| \right)x + b \left( b +1 - \left| b \right| \right)y \right]$$
$$= \left| a \right| T\left[ (a +1 - \left| a \right|)x + b \left( b +1 - \left| b \right| \right)y \right]$$
$$= \left| a \right| T \left[ ((a +1 - \left| a \right|)Tx + \left| b \right| (b +1 - \left| b \right|)Ty \right]$$
$$= a Tx + bTy.$$ 

Thus $T$ is strongly linear homomorphism.

**Theorem 2.6.** Let $V, W$ be R vector spaces, $T$ is an isomorphism from $V$ into $W$. If $G^*$ is a Basis for $V$ then $T(G^*)$ is a basis for $T(V)$.

**Proof.** Let $G^*$ be a basis for $V$. Since $0 \not\in G^*$, it is clear that $0 \not\in T(G^*)$.

Let $x \in V$. Then $x = \sum_{g \in G^*} a_{x,g} g$ and $a_{x,g} a_{x,h} = 0$ for $g \neq h$ and $a_{x,g} = 0$ for almost all $g \in G$.

Hence $Tx = T\left( \sum_{g \in G} a_{x,g} g \right) = \sum_{g \in G} \left( T a_{x,g} g \right)$
$$= \sum_{g \in G} T \left[ a_{x,g} \left( a_{x,g} +1 - \left| a_{x,g} \right| \right)g \right]$$
(by 4 of lemma 2)
$$= \sum_{g \in G} a_{x,g} T \left[ (a_{x,g} +1 - \left| a_{x,g} \right|)g \right]$$
$$= \sum_{g \in G} a_{x,g} Tg \left( \text{by def 2.3} \right)$$
$$= \sum_{Tg \in T(G^*)} a_{x,g} Tg$$

Hence $T(G^*)$ spans $T(V)$.

2. Let $Tg_1, Tg_2, \ldots, Tg_n \in T(G^*)$ and $a_1 Tg_1 + a_2 Tg_2 + \ldots + a_n Tg_n = 0$ and $a_1, a_2, \ldots, a_n \neq 0$ for $a_1, a_2, \ldots, a_n \in R$. 
Then \(0 = a_1 Tg_1 + a_2 Tg_2 + \ldots + a_n Tg_n\)
\[
\begin{align*}
&= \left[ \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_n \\
\end{array} \right] \begin{array}{c}
(a_1 + 1 - a_1) g_1 + \ldots + (a_n + 1 - a_n) g_n \\
\end{array} \\
&= T \left[ \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_n \\
\end{array} \right] \begin{array}{c}
(a_1 + 1 - a_1) g_1 + \ldots + (a_n + 1 - a_n) g_n \\
\end{array} \\
&= T \left[ \begin{array}{c}
a_1 g_1 \\
a_2 g_2 \\
\vdots \\
a_n g_n \\
\end{array} \right] \\
&= T(a_1 g_1 + \ldots + a_n g_n) = 0 = T0
\end{align*}
\]
Now \(T(a_1 g_1 + \ldots + a_n g_n) = 0 = T0\)
\(\Rightarrow\) \(a_1 g_1 + \ldots + a_n g_n = 0\) (since \(T\) is one one ). Since \(G^*\) is linearly independent, we have that \(g_1 + g_2 + \ldots + g_n = 0\)
\(\Rightarrow\) \(T(g_1 + g_2 + \ldots + g_n) = T0\)
\(\Rightarrow\) \(T(g_1) + T(g_2) + \ldots + T(g_n) = 0\). This shows that \(T(G^*)\) is linearly independent.
Thus \(T(G^*)\) is a basis for \(T(V)\).
\[
\begin{align*}
&= \left[ \begin{array}{c}
a \\
b \\
\end{array} \right] \begin{array}{c}
(a + 1 - a) x + (b + 1 - b) y \\
\end{array} \\
&= \left[ \begin{array}{c}
a \\
b \\
\end{array} \right] \begin{array}{c}
[(a + 1 - a) T x + (b + 1 - b) T y \\
\end{array} = a T x + b T y.
\end{align*}
\]

References


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