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IP-Dimensions of Modules and Rings

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Abstract

In this paper, we introduce and investigate the IP dimensions of modules and rings. The relations between the IP dimension and other dimensions are discussed.

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1 Introduction

Throughout this paper, all rings are associative and all modules -if not specified otherwise- are left and unitary. Let R be a ring, and let M be an R-module. As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. We use also $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$ to denote, respectively, the classical global and weak dimension of R. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ and $E(M)$ denote the envelope injective of the R-module M.

In [3], the authors defined and studied a refinement of flat dimension, which they called the IF-*dimension* of modules. Recall that, for an R-module M, $\text{IF-d}_R(M) \leq n$ if $\mathrm{Tor}_R^i(I, M) = 0$ for all right injective R-module I. In this paper, in order to make a real analogy with the classical dimensions, we define and study a refinement of the injective dimension. Thus, for an R-module M, let $\text{IP-d}_R(M)$ denotes the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^i(I, M) = 0$ for all injective modules I and all $i > n$. If no such integer exists, set $\text{IP-d}_R(M) = \infty$. We call this dimension of IP-*dimension* of modules. If $\text{IP-d}_R(M) = 0$ then M will be called IP-*module*.

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Recall that a ring R is called *quasi-Frobenius* if it is right or left self-injective and right or left artinian.

The aim of section 2 is to study the IP-dimension of modules. In section 3, we define and investigate the IP dimension of rings.

2 IP-dimension of modules

Recall that a subclass $\mathfrak{X}$ of R-modules is called *injectively resolving* if $\mathfrak{X}$ contain all injective R-modules, and for every short exact sequence $0 \to X' \to X \to X" \to 0$ with $X' \in \mathfrak{X}$ the conditions $X" \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent. The first result of this paper studies the injectively resolving of the class of IP-modules.

Proposition 2.1 *The class of all* IP*-modules is injectively resolving, closed under arbitrary direct products and under direct summands.*

Proof. Clearly every injective module is IP-injective. Consider $0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$ an exact sequence of modules where M' is an IP-module. From the long exact sequence of homology induced from the functor $\mathrm{Hom}_R(I,-)$ where I is an injective module, it is easy to see that M is an IP-module if and only if $M"$ is an IP-module since $\mathrm{Ext}^i_R(I,M') = 0$ for all $i > 0$. Hence, The class of all IP-modules is injectively resolving. Moreover, since $\mathrm{Ext}^i_R(I,-)$ commutes with direct products, the other assertions holds. □

The next result gives some characterizations of the IP-dimension of modules.

Proposition 2.2 *For any R-module M and any positive integer n, the following assertions are equivalent.*

1. $\mathrm{IP\text{-}d}_R(M) \leq n$.

2. $\mathrm{Ext}^i_R(I,M) = 0$ *for all $i > n$ and all R-modules I with finite injective dimension.*

3. *If* $0 \to M \to E_0 \to ... \to E_{n-1} \to E_n \to 0$ *is exact with* $E_0, .., E_{n-1}$ *are* IP *modules, then* E_n *is an* IP *module.*

Proof. (1) ⇔ (2) The proof of this equivalence is standard homological algebra.

(2) ⇒ (1) Obvious.

(1) ⇒ (2) The proof will be by induction on $m := \mathrm{id}_R(I)$. The induction start is given by (3). If $m > 0$, pick the short exact sequence $0 \longrightarrow$

$I \longrightarrow E(I) \longrightarrow E(I)/I \longrightarrow 0$. Clearly, $\mathrm{id}_R(E(I)/I) = m-1$. Thus, $\mathrm{Ext}^i_R(E(K)/I, M) = 0$ for all $i > n$. From the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R((E(I), M) \longrightarrow \mathrm{Ext}^i_R(I, M) \longrightarrow \mathrm{Ext}^{i+1}_R(E(I)/I, M) \longrightarrow \dots$$

it is clear that $\mathrm{Ext}^i_R(I, M) = 0$ for all $i > n$. $\square$

As we said bellow, an injective module is trivially IP-injective, while the inverse implication is not true, as shown by the following example.

Example 2.3 *Consider the local quasi-Frobenius ring $R = k[X]/(X^2)$ where k is a field, and denote by $\overline{X}$ the the residue class in R of X. Then, $\overline{X}$ is an* IP *R-module which is not injective.*

Proof. Since R is quasi-Frobenius, every injective module I is projective. Then, $\mathrm{Ext}^i_R(I, \overline{X}) = 0$ for all $i > 0$. Thus, $\overline{X}$ is an IP R-module. Now, if we suppose that $\overline{X}$ is injective. Then, it is projective. But R is local. So, $\overline{X}$ must be a free which is absurd by the fact that $\overline{X}^2 = 0$. So, we conclude that $\overline{X}$ is not injective, as desired. $\square$

The next proposition shows that IP dimension is a refinement of the injective dimension.

Proposition 2.4 *For any R-module M,* $\mathrm{IP\text{-}d}_R(M) \leq \mathrm{id}_R(M)$ *with equality if* $\mathrm{id}_R(M)$ *is finite.*

Proof. The first inequality follows from the fact that every injective module is an IP module. Now, set $\mathrm{id}_R(M) = n < \infty$ and suppose that $\mathrm{IP\text{-}d}_R(M) = m < n$. Thus, there is an R-module N such that $\mathrm{Ext}^n_R(N, M) \neq 0$. Clearly, N can not be injective. Hence, we can consider the short exact sequence $0 \longrightarrow N \longrightarrow E(N) \longrightarrow E(N)/N \longrightarrow 0$. We have the exact sequence

$$0 = \mathrm{Ext}^n_R(E(N), M) \longrightarrow \mathrm{Ext}^n_R(N, M) \longrightarrow \mathrm{Ext}^{n+1}_R(E(N)/N, M) \rightarrow \mathrm{Ext}^{n+1}_R(E(N), M) = 0$$

Therefore, $\mathrm{Ext}^{n+1}_R(E(N)/N, M) \neq 0$, a contradiction since $\mathrm{id}_R(M) = n$. $\square$

The proof of the next proposition is standard homological algebra.

Proposition 2.5 *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules. If two of* $\mathrm{IP\text{-}d}_R(A)$, $\mathrm{IP\text{-}d}_R(B)$, *and* $\mathrm{IP\text{-}d}_R(C)$ *are finite, so is the third. Moreover,*

1. $\mathrm{IP\text{-}d}_R(B) \leq \sup\{\mathrm{IP\text{-}d}_R(A), \mathrm{IP\text{-}d}_R(C)\}$.
2. $\mathrm{IP\text{-}d}_R(A) \leq \sup\{\mathrm{IP\text{-}d}_R(B), \mathrm{IP\text{-}d}_R(C) + 1\}$.
3. $\mathrm{IP\text{-}d}_R(C) \leq \sup\{\mathrm{IP\text{-}d}_R(B), \mathrm{IP\text{-}d}_R(A) - 1\}$.

The next corollary is an immediate consequence of Proposition 2.5

Corollary 2.6 *Let R be a ring and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules. If B is* IP *module and* $\text{IP-d}\,(A) > 0$ *then,* $\text{IP-d}\,(A) = \text{IF-d}\,(C) + 1$.

Proposition 2.7 *Let R be ring.*

1. *If M is an* IF *R-module then M^+ is an* IP *right R-module*

2. *If R is right coherent and M is an* IP *R-module then M^+ is an* IF *right R-module.*

Proof. (1) Assume that M is an IF R-module. Let I be an injective right R-module. For all $i > 0$, $\text{Ext}^i_R(I, M^+) \cong (\text{Tor}^i_R(I, M))^+ = 0$. Thus, M^+ is an IP right R-module.

(2) Assume that R is right coherent and M is an IP R-module. For each injective left R-module I and all $i > 0$ we have $0 = (\text{Ext}^i_R(I, M))^+ = \text{Tor}^i_R(M^+, I)$. Thus, M^+ is an IF right R-module. □

3 IP dimension of rings

The IP dimension of modules gives a rise to a global dimension over rings, as follows.

Definition 3.1 *The left* IP *dimension of a ring R, $l.\text{IPD}\,(R)$, is defined by setting*

$$l.\text{IPD}\,(R) = \sup\{\text{IP-d}\,(M) \mid M \text{ is a (left) } R\text{-module}\}.$$

Similarly for right module we define the right IP *dimension, $r.\text{IPD}\,(R)$ (when R is commutative we drop the unneeded letters r and l).*

The next result gives a characterization of $l.\text{IPD}\,(R)$.

Theorem 3.2 *Let R be a ring and n be a positive integer. The following are equivalent.*

1. $l.\text{IPD}\,(R) \leq n$.

2. $\text{Ext}^{n+1}_R(I, M) = 0$ *for all R-modules M and modules I with injective dimension.*

3. $\text{Ext}^{n+1}_R(I, M) = 0$ *for all R-modules M and all injective modules I.*

4. $\text{pd}_R(I) \leq n$ *for all modules I with finite injective dimension.*

5. $\mathrm{pd}_R(I) \leq n$ *for all injective modules* I.

consequently, $l.\mathrm{IPD}\,(R) = \sup\{\, \mathrm{pd}_R(I) \mid I$ is an injective R-module$\}$.

Proof. The implication (2) ⇒ (3), (3) ⇒ (4) and (4) ⇒ (5) are obvious.

(1) ⇒ (2) Since $\mathrm{IPD}\,(R) \leq n$, for each R-module M we have $\text{IP-d}_R(M) \leq n$ and then, by Proposition 2.2, $\mathrm{Ext}^i_R(I, M) = 0$ for all $i > n$ and all R-module I with finite injective dimension. Thus, (2) holds clearly.

(5) ⇒ (1) Let M be an R-module. For each injective R-module I and all $i > n$, $\mathrm{Ext}^i_R(I, M) = 0$ since $\mathrm{pd}_R(I) \leq n$. Thus, $\text{IP-d}\,(M) \leq n$. So, $\mathrm{IPD}\,(R) \leq n$. □

Corollary 3.3 *Let R be a ring. The following are equivalent.*

1. *R is quasi-Frobenius.*

2. $l.\mathrm{IPD}\,(R) = 0$

3. $r.\mathrm{IPD}\,(R) = 0$.

Proof. Follows from Theorem 3.2 and [6, Theorem7.56]. □

In [5], the authors introduced the left IF dimension of a ring R, $l.\mathrm{IFD}\,(R)$, as

$$l.\mathrm{IFD}\,(R) = \sup\{\mathrm{fd}_R(I) \mid I \text{ is an injective (left) } R\text{-module}\}.$$

For such dimension, Ding and Chen [5] gave a various characterizations. We have the following corollary.

Corollary 3.4 *For any ring R, $l.\mathrm{IFD}\,(R) \leq l.\mathrm{IPD}\,(R) \leq l.\mathrm{gldim}(R)$. Moreover, if $l.\mathrm{gldim}(R) < \infty$ then $l.\mathrm{IPD}\,(R) = l.\mathrm{gldim}(R)$.*

Proof. Follows from Proposition 2.4 and Theorem 3.2. □

Corollary 3.5 *Let R be a commutative Noetherian ring such that* $\mathrm{id}_R(R) < \infty$. *Then,* $\mathrm{IFD}\,(R) = \mathrm{IPD}\,(R)$.

Proof. Follows from Theorem 3.2 and [5, Theorem 3.19] □

If R is any ring such $l.\mathrm{wdim}(R) < l.\mathrm{gldim}(R) < \infty$ then $l.\mathrm{IPD}\,(R) = l.\mathrm{gldim}(R) \neq l.\mathrm{wdim}(R) = l.\mathrm{IFD}\,(R)$ (by Corollary 3.4 and [5, Corollary 3.3]. Even if $l.\mathrm{wdim}(R) = \infty$ the dimensions $l.\mathrm{IPD}\,(-)$ and $l.\mathrm{IFD}\,(-)$ may be different, as shown by the following example.

Example 3.6 *Consider the quasi-Frobenius ring $R = k[X]/(X^2)$ where k is a field, and let S be a commutative non-Noetherian Von Neumann regular and hereditary ring (see [2]). Then,* $\mathrm{wdim}(R \times S) = \infty$ *and* $\mathrm{IPD}\,(R \times S) = 1$ *and* $\mathrm{IFD}\,(R \times S) = 0$.

Proof. Clearly, $\mathrm{wdim}(R \times S) = \infty$ since $\mathrm{wdim}(R) = \infty$. Let I be an arbitrary injective $R \times S$-module. We can see that $I \cong \mathrm{Hom}_{R\times S}(R \times S, I) \cong \mathrm{Hom}_{R\times S}(R, I) \times \mathrm{Hom}_{R\times S}(S, I)$ and that $I_1 = \mathrm{Hom}_{R\times S}(R, I)$ is an injective R-module and that $I_2 = \mathrm{Hom}_{R\times S}(S, I)$ is an injective S-module. Clearly, I_1 is projective since R is quasi-Frobenius. Also, since S is a Von Neumann regular hereditary ring, I_2 is flat and $\mathrm{pd}_R(I_2) \leq 1$. Using [1, Lemma 3.7] and [7, Lemma 2.5 (2)], we have $\mathrm{fd}_{R\times S}(I_1 \times I_2) = \sup\{\mathrm{fd}_R(I_1), \mathrm{fd}_S(I_2)\} = 0$ and $\mathrm{pd}_{R\times S}(I_1 \times I_2) = \sup\{\mathrm{pd}_R(I_1), \mathrm{pd}_S(I_2)\} \leq 1$. Consequently, $\mathrm{IFD}\,(R \times S) = 0$ and $\mathrm{IPD}\,(R \times S) \leq 1$. If $\mathrm{IPD}\,(R \times S) = 0$ then, by Corollary 3.3, S is a quasi-Frobenius ring and so S is Noetherian, a contradiction. □

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