On Weak \((\alpha, \delta)\)-Compatible Rings

Ouyang Lunqun\(^1\) and Liu Jingwang

Department of Mathematics
Hunan University of Science and Technology
Xiangtan, Hunan 411201, P.R. China

Abstract

For a ring endomorphism \(\alpha\) and an \(\alpha\)-derivation \(\delta\), we introduce the notion of weak \((\alpha, \delta)\)-compatible rings, that is a generalization of \(\alpha\)-rigid rings and \((\alpha, \delta)\)-compatible rings. We first observe the basic properties of weak \((\alpha, \delta)\)-compatible rings, and extend the class of weak \((\alpha, \delta)\)-compatible rings through various ring extensions. We next study on the relationship between the ideal quotient property of the ring \(R\) and that of the Ore extension \(R[x; \alpha, \delta]\) in case \(R\) is weak \((\alpha, \delta)\)-compatible.

Mathematics Subject Classification: 16D25; 16D40

Keywords: weak \((\alpha, \delta)\)-compatible ring; ore extension ring; nilpotent element

1 Introduction

Throughout this paper, \(R\) denotes an associative ring with unity, \(\alpha : R \rightarrow R\) is an endomorphism, and \(\delta\) an \(\alpha\)-derivation of \(R\), that is, \(\delta\) is an additive map such that \(\delta(ab) = \delta(a)b + \alpha(a)\delta(b)\), for \(a, b \in R\). We denote \(S = R[x; \alpha, \delta]\) the Ore extension whose elements are the polynomials over \(R\), the addition is defined as usual and the multiplication subject to the relation \(xa = \alpha(a)x + \delta(a)\) for any \(a \in R\). Recall that a ring \(R\) is reduced if \(R\) has no nonzero nilpotent elements. Observe that reduced rings are abelian (i.e., all idempotents are central). Let \(R\) be a ring, the prime radical (i.e., the intersection of all prime ideals) of \(R\) and the set of all nilpotent elements in \(R\) are denoted by \(P(R)\) and \(\text{nil}(R)\), respectively. A ring \(R\) is called 2-primal if \(P(R) = \text{nil}(R)\), and a ring \(R\) is said to be an \(NI\) ring if \(\text{nil}(R)\) forms an ideal. Given a skew polynomial \(h(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x; \alpha, \delta]\), we denote by \(C_h\) the subset of \(R\) consisting of the coefficients of \(h(x)\), and for a subset \(V \subseteq R[x; \alpha, \delta]\), \(C_V = \bigcup_{h \in V} C_h\). Let \(I\) be a subset of \(R\), \(I[x; \alpha, \delta]\) means \(\{a_0 + a_1x + \cdots + a_lx^l \in \)

\(^1\text{ouyanglqtxy@163.com}\)
$R[x; \alpha, \delta] \mid a_i \in I} \subseteq R[x; \alpha, \delta]$, that is, for any skew polynomial $h(x) = h_0 + h_1 x + \cdots + h_t x^t \in R[x; \alpha, \delta]$, $h(x) \in I[x; \alpha, \delta]$ if and only if $h_i \in I$ for all $0 \leq i \leq t$. In particular, We say that $h(x) \in \text{nil}(R)[x; \alpha, \delta]$ if and only if $C_h \subseteq \text{nil}(R)$. If $h(x)$ is a nilpotent element of $R[x; \alpha, \delta]$, then we say that and $h(x) \in \text{nil}(R[x; \alpha, \delta])$.

According to Krempa [6], an endomorphism $\alpha$ of a ring $R$ is said to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring $R$ $\alpha$-rigid if there exists a rigid endomorphism $\alpha$ of $R$. Note that any rigid endomorphism of a ring $R$ is a monomorphism and $\alpha$-rigid rings are reduced rings by Hong et al.[3]. Properties of $\alpha$-rigid rings have been studied in Krempa [6], Hong [3], and Hirano [2].

Following E. Hashemi and A. Moussavi [1], a ring $R$ is $\alpha$-compatible if for each $a, b \in R, ab = 0 \iff a\alpha(b) = 0$. Moreover, $R$ is said to be $\delta$-compatible if for each $a, b \in R, ab = 0 \implies a\delta(b) = 0$. If $R$ is both $\alpha$-compatible and $\delta$-compatible, we say that $R$ is $(\alpha, \delta)$-compatible. In this case, clearly the endomorphism $\alpha$ is injective, and $R$ is $\alpha$-rigid if and only if $R$ is $\alpha$-compatible and reduced [1]. Thus the $\alpha$-compatible ring is a generalization of $\alpha$-rigid ring to the more general case where $R$ is not assumed to be reduced.

As a generalization of $(\alpha, \delta)$-compatible rings, in this paper, we introduce the notion of weak $(\alpha, \delta)$-compatible rings. We first extend the class of weak $(\alpha, \delta)$-compatible rings through various ring extensions. We next investigate the basic property of weak $(\alpha, \delta)$-compatible rings. As a consequence, some properties of $(\alpha, \delta)$-compatible rings are generalized to a more general setting.

2 Weak $(\alpha, \delta)$- compatible rings

Our focus in this section is to introduce the concept of an weak $(\alpha, \delta)$-compatible ring and study it properties.

**Definition 2.1** For an endomorphism $\alpha$ and an $\alpha$-derivation $\delta$, we say that $R$ is weak $\alpha$-compatible if for each $a, b \in R, ab = 0 \iff a\alpha(b) \in \text{nil}(R)$. Moreover, $R$ is said to be weak $\delta$-compatible if for each $a, b \in R, ab = 0 \implies a\delta(b) \in \text{nil}(R)$. If $R$ is both weak $\alpha$-compatible and weak $\delta$-compatible, we say that $R$ is weak $(\alpha, \delta)$-compatible.

Clearly every subring of a weak $(\alpha, \delta)$-compatible ring is also weak $(\alpha, \delta)$-compatible. If $R$ is reduced, then $R$ is a weak $(\alpha, \delta)$-compatible ring if and only if $R$ is an $(\alpha, \delta)$-compatible ring. The definition of weak $(\alpha, \delta)$-compatible rings is quite natural, in the light of its similarity with the notion of $(\alpha, \delta)$-compatible rings, where in Proposition 2.4 and Example 2.5, we will show that all $(\alpha, \delta)$-compatible rings are weak $(\alpha, \delta)$-compatible, but there exists
a weak \((\alpha, \delta)\)-compatible ring which is not \((\alpha, \delta)\)-compatible. Thus the weak \((\alpha, \delta)\)-compatible ring is a true generalization of \((\alpha, \delta)\)-compatible ring.

Lemma 2.2 Let \(R\) be a weak \((\alpha, \delta)\)-compatible ring. Then we have the following:

1. If \(ab \in \text{nil}(R)\), then \(a\alpha^n(b) \in \text{nil}(R), \alpha^m(a)b \in \text{nil}(R)\) for all positive integers \(m, n\).
2. If \(\alpha^k(a)b \in \text{nil}(R)\) for some positive integer \(k\), then \(ab \in \text{nil}(R)\).
3. If \(a\alpha^s(b) \in \text{nil}(R)\) for some positive integer \(s\), then \(ab \in \text{nil}(R)\).
4. If \(ab \in \text{nil}(R)\), then \(\alpha^n(a)\delta^m(b) \in \text{nil}(R)\), and \(\delta^s(a)\alpha^t(b) \in \text{nil}(R)\) for all positive integers \(m, n, s, t\).

Proof

(1) If \(ab \in \text{nil}(R)\), then \(a\alpha(b) \in \text{nil}(R)\) by definition. Since \(a\alpha(b) \in \text{nil}(R)\), we get \(a\alpha^2(b) \in \text{nil}(R)\). Continuing this procedure yields that \(a\alpha^n(b) \in \text{nil}(R)\) for any positive integer \(n\). If \(ab \in \text{nil}(R)\), then \(ba \in \text{nil}(R)\) and so \(ba^m(a) \in \text{nil}(R)\) for any positive integer \(m\).

(2) If \(\alpha^k(a)b \in \text{nil}(R)\), then \(ba^k(a) = b\alpha(\alpha^{k-1}(a)) \in \text{nil}(R)\). Thus \(ba^k(a) \in \text{nil}(R)\) since \(R\) is weak \((\alpha, \delta)\)-compatible. Continuing this procedure yields that \(ba \in \text{nil}(R)\) and so \(ab \in \text{nil}(R)\).

(3) The proof is straightforward.

(4) If \(ab \in \text{nil}(R)\), then \(a_\delta^m(b) \in \text{nil}(R)\) for any positive integer \(m\) since \(R\) is weak \(\delta\)-compatible. Then by (1), we obtain \(\alpha^n(a)\delta^m(b) \in \text{nil}(R)\) for any positive integers \(m, n\). If \(ab \in \text{nil}(R)\), then \(ba \in \text{nil}(R)\), then \(\alpha^t(b)\delta^s(a) \in \text{nil}(R)\) and so \(\delta^s(a)\alpha^t(b) \in \text{nil}(R)\) for all positive integers \(s, t\).

The next Lemma appears in [1].

Lemma 2.3 Let \(R\) be an \((\alpha, \delta)\)-compatible ring. Then we have the following:

1. If \(ab = 0\), then \(a\alpha^n(b) = \alpha^n(a)b = 0\) for any positive integer \(n\).
2. If \(\alpha^k(a)b = 0\) for some positive integer \(k\), then \(ab = 0\).
3. If \(ab = 0\), then \(\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)\) for any positive integers \(m, n\).

Theorem 2.4 All \((\alpha, \delta)\)-compatible rings are weak \((\alpha, \delta)\)-compatible.

Proof Suppose \(R\) is an \((\alpha, \delta)\)-compatible ring and \(ab \in \text{nil}(R)\) for \(a, b \in R\). Then there exists a positive integer \(n\) such that \((ab)^n = 0\). In the following computations, we use freely the condition that \(R\) is \((\alpha, \delta)\)-compatible: \(0 = (ab)^n = abab \cdots ab \Rightarrow a\alpha(bab \cdots ab) = 0 \Rightarrow a\alpha(b)\alpha(abab \cdots ab) = 0 \Rightarrow a\alpha(b)\alpha(abab \cdots ab) = 0 \Rightarrow \cdots \Rightarrow a\alpha(b) \in \text{nil}(R)\). If \(a\alpha(b) \in \text{nil}(R)\), then there exists some positive integer \(m\) such that \((a\alpha(b))^m = \)
0. Thus \(0 = (a\alpha(b))^m = \alpha(\alpha(b)a\alpha(b) \cdots a\alpha(b) \Rightarrow a\alpha(b)a\alpha(b) \cdots a\alpha(b)ab = 0 \Rightarrow a\alpha(b) \cdots a\alpha(bab) = 0 \Rightarrow a\alpha(b) \cdots \) 
\(a\alpha(b)ab = 0 \Rightarrow \cdots \Rightarrow ab \in \text{nil}(R)\). Hence \(R\) is weak \(\alpha - \text{compatible}\).

In order to show that \(R\) is weak \(\delta - \text{compatible}\), it is enough to show that \(abc = 0\) implies \(a\delta(b)c = 0\). We have \(abc = 0 \Rightarrow \alpha(abd(c) = 0 \Rightarrow \alpha(a)\alpha(b)\delta(c) = 0 \Rightarrow \alpha(a)(\alpha(b)\delta(c)) = 0 \Rightarrow a\alpha(b)\delta(c) = 0\). On the other hand, we have \(abc = 0 \Rightarrow 0 = a\delta(bc) = a(\alpha(b)\delta(c) + \delta(b)c) = a\alpha(b)\delta(c) + a\delta(b)c\). Thus we obtain \(a\delta(b)c = 0\). Hence \(R\) is weak \(\delta - \text{compatible}\). Therefore \(R\) is weak \((\alpha, \delta)\)-compatible.

We know that \((\alpha, \delta)\)-compatible rings are weak \((\alpha, \delta)\)-compatible, but the converse is not true in general. We have the following counterexample for this situation. Thus a weak \((\alpha, \delta)\)-compatible ring is a true generalization of an \((\alpha, \delta)\)-compatible ring.

**Example 2.5** Let \(R\) be a ring and let 
\[
R_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in R \right\}.
\]

Let \(\alpha : R_2 \rightarrow R_2\) be an endomorphism defined by 
\[
\alpha \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix},
\]
and let \(\delta : R_2 \rightarrow R_2\) be the zero mapping. Clearly \(R_2\) is a weak \((\alpha, \delta)\)-compatible ring. Since \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \alpha \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = 0\), but \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \neq 0\), \(R\) is not \((\alpha, \delta)\)-compatible.

**Theorem 2.6** Let \(\alpha\) be an endomorphism and \(\delta\) an \(\alpha\)-derivation of a ring \(R\). Then \(R\) is \(\alpha\)-rigid if and only if \(R\) is weak \((\alpha, \delta)\)-compatible and reduced.

**Proof** It follows from \([1, \text{Lemma 2.2}]\).

The following results will give more examples of weak \((\alpha, \delta)\)-compatible rings.

Let \(\delta\) be an \(\alpha\)-derivation of \(R\). We defined by \(T_n(R)\) the \(n\) by \(n\) upper triangular matrix ring over \(R\). The endomorphism \(\alpha\) of \(R\) is extended to the endomorphism \(\overline{\alpha} : T_n(R) \rightarrow T_n(R)\) defined by \(\overline{\alpha}(\alpha_{ij}) = (\alpha(a_{ij}))\), also the \(\alpha\)-derivation \(\delta\) is extended to the \(\overline{\delta}\)-derivation \(\overline{\delta} : T_n(R) \rightarrow T_n(R)\) defined by \(\overline{\delta}(\alpha_{ij}) = (\delta(a_{ij}))\), for each \((a_{ij}) \in T_n(R)\). Then we have the following results:

**Theorem 2.7** The following statements are equivalent:

1. \(R\) is weak \((\alpha, \delta)\)-compatible.
2. \(T_n(R)\) is weak \((\overline{\alpha}, \overline{\delta})\)-compatible.
proof (1)⇒(2) Suppose that
\[
\left( \begin{array}{cccc}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 0 & a_{22} & \cdots & a_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & a_{nn}
\end{array} \right)^{\alpha} \left( \begin{array}{cccc}
 b_{11} & b_{12} & \cdots & b_{1n} \\
 0 & b_{22} & \cdots & b_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & b_{nn}
\end{array} \right) = 0
\]
for some positive integer \( s \). Then \((a_{ii}a_{ii})^{s} = 0\) for all \( 0 \leq i \leq n \). Hence there is some positive integer \( t_i \) such that \((a_{ii}b_{ii})^{t_i} = 0\) since \( R \) is weak \( \alpha \)-compatible. Let \( t = \text{Max}\{t_i\}, 0 \leq i \leq n \). Then
\[
\left( \begin{array}{cccc}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 0 & a_{22} & \cdots & a_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & a_{nn}
\end{array} \right)^{t_n} \left( \begin{array}{cccc}
 b_{11} & b_{12} & \cdots & b_{1n} \\
 0 & b_{22} & \cdots & b_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & b_{nn}
\end{array} \right) = 0.
\]
Now assume that
\[
\left( \begin{array}{cccc}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 0 & a_{22} & \cdots & a_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & a_{nn}
\end{array} \right)^{s} \left( \begin{array}{cccc}
 b_{11} & b_{12} & \cdots & b_{1n} \\
 0 & b_{22} & \cdots & b_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & b_{nn}
\end{array} \right) = 0.
\]
Then \((a_{ii}b_{ii})^{s} = 0\) for all \( 0 \leq i \leq n \). Hence there exist some positive integers \( t_i \), and \( p_i \) such that \((a_{ii}a_{ii})^{t_i} = 0\) and \((a_{ii}b_{ii})^{p_i} = 0\) since \( R \) is weak \( (\alpha, \delta) \)-compatible. Let \( t = \text{Max}\{t_i\}, 0 \leq i \leq n \), and \( p = \text{Max}\{p_i\}, 0 \leq i \leq n \). Then
\[
\left( \begin{array}{cccc}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 0 & a_{22} & \cdots & a_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & a_{nn}
\end{array} \right)^{t_n} \left( \begin{array}{cccc}
 b_{11} & b_{12} & \cdots & b_{1n} \\
 0 & b_{22} & \cdots & b_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & b_{nn}
\end{array} \right) = 0.
\]
and
\[
\left( \begin{array}{cccc}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 0 & a_{22} & \cdots & a_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & a_{nn}
\end{array} \right)^{p_n} \left( \begin{array}{cccc}
 b_{11} & b_{12} & \cdots & b_{1n} \\
 0 & b_{22} & \cdots & b_{2n} \\
 \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & \cdots & b_{nn}
\end{array} \right) = 0.
\]
Therefore \( T_n(R) \) is weak \( (\alpha, \delta) \)-compatible.

(2)⇒(1). It is trivial.

Let \( R \) be a ring and let
\[
S_n(R) = \left\{ \left( \begin{array}{cccc}
 a & a_{12} & \cdots & a_{1n} \\
 0 & a & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & a
\end{array} \right) \mid a, a_{ij} \in R \right\}
\]
with $n \geq 2$; and let

$$T(R, n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \middle| a_i \in R \right\}$$

with $n \geq 2$, and let $T(R, R)$ be the trivial extension of $R$ by $R$. Any endomorphism $\alpha$ of $R$ can be extended to an endomorphism $\overline{\alpha}$ of $S_n(R)$ (or $T(R, n)$, or $T(R, R)$) defined by $\overline{\alpha}(\alpha_{ij}) = (\alpha(a_{ij}))$, and any $\alpha$-derivation $\delta$ can be extended to an $\overline{\delta}$-derivation of $S_n(R)$ (or $T(R, n)$, or $T(R, R)$) defined by $\overline{\delta}(\delta(a_{ij}))$.

Using the same method in the proof of Proposition 2.7, we obtain the following results.

**Theorem 2.8** Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Then the following conditions are equivalent:

1. $R$ is weak $(\alpha, \delta)$-compatible.
2. $S_n(R)$ weak $(\overline{\alpha}, \overline{\delta})$-compatible.
3. $T(R, n)$ is weak $(\overline{\alpha}, \overline{\delta})$-compatible.
4. $T(R, R)$ is weak $(\overline{\alpha}, \overline{\delta})$-compatible.

**Lemma 2.9** If $R$ is a 2-primal ring and $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then $f(x) \in \text{nil}(R[x])$ if and only if $a_i \in \text{nil}(R)$ for each $0 \leq i \leq n$, that is, we have $\text{nil}(R[x]) = \text{nil}(R)[x]$ when $R$ is a 2-primal ring.

**Proof** Suppose $f(x) = \sum_{i=0}^{n} a_i x^i \in \text{nil}(R[x])$. Then by [5, Proposition 1.3], we obtain $a_i \in \text{nil}(R)$ for each $0 \leq i \leq n$, and so $\text{nil}(R[x]) \subseteq \text{nil}(R)[x]$. Now assume that $f(x) = \sum_{i=0}^{n} a_i x^i \in \text{nil}(R)[x]$. Consider the finite subset $\{a_0, a_1, \cdots, a_n\} \subseteq \text{nil}(R)$. Since $R$ is a 2-primal ring, there exists a positive integer $k$ such that any product of $k$ elements $a_1 a_2 \cdots a_k$ from $\{a_0, a_1, \cdots, a_n\}$ is zero. Hence we obtain $(f(x))^{k+1} = 0$, and so $f(x) \in \text{nil}(R[x])$. Hence $\text{nil}(R)[x] \subseteq \text{nil}(R[x])$. Therefore we obtain $\text{nil}(R)[x] = \text{nil}(R[x])$.

Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Then the map $\overline{\alpha} : R[x] \rightarrow R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \rightarrow \sum_{i=0}^{m} \alpha(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$, and clearly this map extends $\alpha$, and the $\alpha$-derivation $\delta$ of $R$ is also extended to $\overline{\delta} : R[x] \rightarrow R[x]$ defined by $\overline{\delta}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \delta(a_i) x^i$. We can easily see that $\overline{\delta}$ is an $\overline{\alpha}$-derivation of $R[x]$.

**Theorem 2.10** Let $R$ be a 2-primal and weak $(\alpha, \delta)$-compatible ring. Then $R[x]$ is a weak $(\overline{\alpha}, \overline{\delta})$-compatible ring.
Proof Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with

$$f(x)g(x) = \sum_{k=0}^{m+n} (\sum_{i+j=k} a_i b_j) x^k \in \text{nil}(R[x]).$$

Then we have the following by Lemma 2.9:

$$\sum_{i+j=k} a_i b_j \in \text{nil}(R), \quad k = 0, 1, \cdots, m + n.$$

By induction on $i + j$, we can show that $a_i b_j \in \text{nil}(R)$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Since $R$ is weak $(\alpha, \delta)$-compatible, we obtain $a_i \alpha(b_j) \in \text{nil}(R), a_i \delta(b_j) \in \text{nil}(R)$. Thus

$$f(x)\overline{\alpha}(g(x)) = (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} \alpha(b_j) x^j) = \sum_{k=0}^{m+n} (\sum_{i+j=k} a_i \alpha(b_j)) x^k \in \text{nil}(R[x]),$$

and

$$f(x)\overline{\delta}(g(x)) = (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} \delta(b_j) x^j) = \sum_{k=0}^{m+n} (\sum_{i+j=k} a_i \delta(b_j)) x^k \in \text{nil}(R[x])$$

by Lemma 2.9. Now assume that

$$f(x)\overline{\alpha}(g(x)) = (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} \alpha(b_j) x^j) = \sum_{k=0}^{m+n} (\sum_{i+j=k} a_i \alpha(b_j)) x^k \in \text{nil}(R[x]).$$

By a similar proof as above, we can see that $f(x)g(x) \in \text{nil}(R[x])$. Hence $R[x]$ is a weak $(\overline{\alpha}, \overline{\delta})$-compatible ring.

Let $\delta$ be an $\alpha$-derivation of $R$. For integers $i, j$ with $0 \leq i \leq j$, $f_i^j \in \text{End}(R, +)$ will denote the map which is the sum of all possible words in $\alpha, \delta$ built with $i$ letters $\alpha$ and $j - i$ letters $\delta$. For instance, $f_0^0 = 1, f_1^1 = \alpha, f_0^1 = \delta^1$ and $f_{j-1}^j = \alpha^{j-1} \delta + \alpha^{j-2} \delta \alpha + \cdots + \delta \alpha^{j-1}$. The next Lemma appears in [7, Lemma 4.1].

**Lemma 2.11** For any integer $n$ and $r \in R$, we have $x^n r = \sum_{i=0}^{n} f_i^n(r) x^i$ in the ring $R[x; \alpha, \delta]$.

**Lemma 2.12** Let $\delta$ be an $\alpha$-derivation of $R$. If $R$ is a weak $(\alpha, \delta)$-compatible NI ring, then $ab \in \text{nil}(R)$ implies $a f_i^j(b) \in \text{nil}(R)$ for all $j \geq i \geq 0$ and $a, b \in R$.

**Proof** If $ab \in \text{nil}(R)$, then $a \alpha^i(b) \in \text{nil}(R)$ and $a \delta^j(b) \in \text{nil}(R)$ for all $i \geq 0$ and $j \geq 0$ because $R$ is weak $(\alpha, \delta)$-compatible. Then $a f_i^j(b) \in \text{nil}(R)$ for all $j \geq i \geq 0$. 

Lemma 2.13 Let \( R \) be a weak \((\alpha, \delta)\)-compatible NI ring, and \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \text{nil}(R[x; \alpha, \delta]) \). Then \( a_i \in \text{nil}(R) \) for all \( 0 \leq i \leq n \).

Proof Suppose \( f(x) \in \text{nil}(R[x; \alpha, \delta]) \). There exists some positive integer \( k \) such that \( f(x)^k = (a_0 + a_1 x + \cdots + a_n x^n)^k = 0 \). Then

\[
0 = f(x)^k = \text{"lower terms"} + a_n \alpha^n(a_n)\alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n)x^{nk}.
\]

Hence \( a_n \alpha^n(a_n)\alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0 \in \text{nil}(R) \). Note that

\[
\begin{align*}
\alpha^n(a_n)\alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n) &= 0 \\
\Rightarrow & a_n \alpha^n(a_n)\alpha^{2n}(a_n) \cdots \alpha^{(k-2)n}(a_n) a_n \in \text{nil}(R) \\
\Rightarrow & a_n \alpha^n(a_n)\alpha^{2n}(a_n) \cdots \alpha^{(k-3)n}(a_n) a_n a_n \in \text{nil}(R) \\
\Rightarrow & \cdots \Rightarrow a_n \in \text{nil}(R).
\end{align*}
\]

So by Lemma 2.12, \( a_n = 1 \cdot a_n \in \text{nil}(R) \) implies \( 1 \cdot f_i^j(a_n) = f_i^j(a_n) \in \text{nil}(R) \) for all \( 0 \leq i \leq j \). Let \( Q = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \). Then we have

\[
0 = (Q + a_n x^n)^k \\
= (Q + a_n x^n)(Q + a_n x^n) \cdots (Q + a_n x^n) \\
= (Q^2 + Q \cdot a_n x^n + a_n x^n \cdot Q + a_n x^n \cdot a_n x^n)(Q + a_n x^n) \cdots (Q + a_n x^n) \\
= \cdots = Q^k + \Delta,
\]

where \( \Delta \in R[x; \alpha, \delta] \). Note that the coefficients of \( \Delta \) can be written as sums of monomials in \( a_i \) and \( f_i^j(a_j) \), where \( a_i, a_j \in \{a_0, a_1, \cdots, a_n\} \) and \( v \geq u \geq 0 \) are nonnegative integers, and each monomial has \( a_n \) or \( f_i^j(a_n) \). Since \( \text{nil}(R) \) is an ideal, we obtain that \( \Delta \in \text{nil}(R)[x; \alpha, \delta] \). Thus we obtain

\[
(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1})^k = \text{"lower terms"} + a_{n-1} \alpha^{n-1}(a_{n-1}) \cdots \alpha^{(n-1)(k-1)}(a_{n-1}) x^{(n-1)k} \in \text{nil}(R)[x; \alpha, \delta].
\]

Hence \( a_{n-1} \alpha^{n-1}(a_{n-1}) \cdots \alpha^{(k-1)(n-1)}(a_{n-1}) \in \text{nil}(R) \) and so \( a_{n-1} \in \text{nil}(R) \). Using induction on \( n \) we obtain \( a_i \in \text{nil}(R) \) for all \( 0 \leq i \leq n \).

McCoy [8, Theorem 2] proved that if \( R \) is a commutative ring, then whenever \( g(x) = b_0 + b_1 x + \cdots + b_n x^n \) is a zero divisor in \( R[x] \) there exists a nonzero \( c \in R \) such that \( b_i c = 0 \) for all \( 0 \leq i \leq n \). We shall extend this result as follows:

Theorem 2.14 Let \( R \) be a weak \((\alpha, \delta)\)-compatible NI ring. If \( f(x) = \sum_{i=0}^m a_i x^i \), and \( g(x) = \sum_{j=0}^n b_j x^j \in R[x; \alpha, \delta] \setminus \{0\} \) be such that \( f(x)g(x) \in \text{nil}(R[x; \alpha, \delta]) \), then there exists \( r \in R \setminus \{0\} \) such that \( a_i r \in \text{nil}(R) \) for all \( 0 \leq i \leq m \).
On weak \((\alpha, \delta)\)-compatible rings

Proof Let \(f(x) = \sum_{i=0}^{m} a_i x^i\), \(g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]\) be such that \(f(x)g(x) \in \text{nil}(R[x; \alpha, \delta])\). Then

\[
f(x)g(x) = \sum_{k=0}^{m+n} \left( \sum_{s+t=k}^{m} (\sum_{i=s}^{m} a_i f_s^i(b_t)) \right) x^k = \sum_{k=0}^{m+n} \Delta_k x^k \in \text{nil}(R[x; \alpha, \delta]).
\]

Then we have the following equations:

\[
\begin{align*}
\Delta_{m+n} &= a_m \alpha^m(b_n), \\
\Delta_{m+n-1} &= a_m \alpha^m(b_{n-1}) + a_{m-1} \alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n), \\
\Delta_{m+n-2} &= a_m \alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} a_i f_{i}^m(b_{n-1}) + \sum_{i=m-2}^{m} a_i f_{i}^{m-2}(b_n), \\
&\vdots \\
\Delta_k &= \sum_{s+t=k}^{m} (\sum_{i=s}^{m} a_i f_s^i(b_t)),
\end{align*}
\]

where \(\Delta_{m+n} \in \text{nil}(R)\), \(\Delta_{m+n-1} \in \text{nil}(R)\), \(\Delta_{m+n-2} \in \text{nil}(R)\), \ldots, \(\Delta_k \in \text{nil}(R)\) by Lemma 2.13. Then by analogy with the proof of [9, Proposition 3.8], we obtain \(a_i b_j \in \text{nil}(R)\) for each \(0 \leq i \leq n\) and \(0 \leq j \leq n\). Since \(g(x) \in R[x; \alpha, \delta] \setminus \{0\}\), without loss of generality, we can assume that \(b_n \neq 0\). Let \(r = b_n \in R \setminus \{0\}\). Then \(a_i r \in \text{nil}(R)\) for all \(0 \leq i \leq m\).

Applying the same method in the proof of Proposition 2.14, we obtain the following result.

**Corollary 2.15** Let \(R\) be a weak \((\alpha, \delta)\)-compatible \(NI\) ring. Then we have the following:

1. If \(f(x)g(x) \in \text{nil}(R[x; \alpha, \delta])\) where \(f(x) = \sum_{i=0}^{m} a_i x^i\), \(g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]\), then \(a_i b_j \in \text{nil}(R)\) for each \(i, j\).

2. If \(f(x)g(x) \in \text{nil}(R[x; \alpha, \delta])\) where \(f(x) = \sum_{i=0}^{m} a_i x^i\), \(g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]\), then \(a_i b_j \in \text{nil}(R)\) for each \(i, j\).

**Theorem 2.16** Let \(R\) be a weak \((\alpha, \delta)\)-compatible \(NI\) ring. Then for each idempotent element \(e \in R\), we have \(\delta(e) \in \text{nil}(R)\) and \(\alpha(e) = e + u\) where \(u \in \text{nil}(R)\).

Proof Since \(e^2 = e\), we have \(\delta(e) = \alpha(e) \delta(e) + \delta(e)e\). Let \(f(x) = \delta(e) + \alpha(e)x\) and \(g(x) = (e-1)+(e-1)x \in R[x; \alpha, \delta]\). Then \(f(x)g(x) = \delta(e)e - \delta(e) + \alpha(e) \delta(e) + (\delta(e)e - \delta(e) + \alpha(e) \delta(e))x = 0\). Thus we have \(\delta(e)(e-1) = \delta(e)e - \delta(e) \in \text{nil}(R)\) by Corollary 2.15. Now suppose that \(h(x) = \delta(e) - (1 - \alpha(e))x\) and \(k(x) = e + ex \in R[x; \alpha, \delta]\). Then \(h(x)k(x) = 0\). Hence \(\delta(e)e \in \text{nil}(R)\) by Corollary 2.15, and so \(\delta(e) \in \text{nil}(R)\).
Now take \( p(x) = (1 - e) + (1 - e)\alpha(e)x \) and \( q(x) = e + (e - 1)\alpha(e)x \in R[x; \alpha, \delta] \). Then
\[
\begin{align*}
p(x)q(x) & = [(1 - e) + (1 - e)\alpha(e)x][e + (e - 1)\alpha(e)x] \\
& = (1 - e)\alpha(e)\delta(e) + (1 - e)\alpha(e)x \cdot (e - 1)\alpha(e)x \\
& = (1 - e)\alpha(e)\delta(e) + (1 - e)\alpha(e)x\alpha(e)x - (1 - e)\alpha(e)x\alpha(e)x \\
& = (1 - e)\alpha(e)\delta(e) + (1 - e)\alpha(e)x\alpha(e)x \in \text{nil}(R)[x; \alpha, \delta]
\end{align*}
\]

since \( \delta(e) \in \text{nil}(R) \) and \( \text{nil}(R) \) is an ideal of \( R \). Hence \((1 - e) \cdot (e - 1)\alpha(e) = e\alpha(e) - \alpha(e) \in \text{nil}(R) \) by Corollary 2.15. Now suppose \( t(x) = e + e(1 - \alpha(e))x \) and \( s(x) = (1 - e) - e(1 - \alpha(e))x \). Then
\[
\begin{align*}
t(x)s(x) & = [e + e(1 - \alpha(e))x][(1 - e) - e(1 - \alpha(e))x] \\
& = -e(1 - \alpha(e))\delta(e) - e(1 - \alpha(e))x \cdot e(1 - \alpha(e))x \\
& = -e(1 - \alpha(e))\delta(e) - e(1 - \alpha(e))xex + e(1 - \alpha(e))x\alpha(e)x \\
& = -e(1 - \alpha(e))\delta(e) - e(1 - \alpha(e))\delta(e)x + e(1 - \alpha(e))\delta(e)e\alpha(e)x.
\end{align*}
\]

Since \( \delta(e) \in \text{nil}(R) \), we have \( t(x)s(x) \in \text{nil}(R)[x; \alpha, \delta] \). Hence \( e \cdot e(1 - \alpha(e)) = e - e\alpha(e) \in \text{nil}(R) \). Thus \( e - \alpha(e) \in \text{nil}(R) \). Hence \( \alpha(e) = e + u \) where \( u \in \text{nil}(R) \).

**Theorem 2.17** Let \( R \) be a weak \((\alpha, \delta)-\)compatible ring and \( \text{nil}(R) \) an ideal of \( R \). Then for each \( e^2 = e \in R \) and \( a \in R \), we have \( ea = ae + u \) where \( u \in \text{nil}(R) \).

**Proof** By Proposition 2.16, we have \( \alpha(e) = e + u, \ u \in \text{nil}(R) \) and \( \delta(e) \in \text{nil}(R) \). Consider the polynomials \( f(x) = e - e\alpha(1 - e)x \) and \( g(x) = 1 - e + e\alpha(1 - e)x \in R[x; \alpha, \delta] \). Then we have
\[
\begin{align*}
f(x)g(x) & = [e - e\alpha(1 - e)x][1 - e + e\alpha(1 - e)x] \\
& = ea(1 - e)xe - ea(1 - e)x\alpha(1 - e)x.
\end{align*}
\]

Since \( \text{nil}(R) \) is an ideal and \( u \in \text{nil}(R) \) and \( \delta(e) \in \text{nil}(R) \), we have
\[
ea(1 - e)xe \ = ea(1 - e)\alpha(e)x + ea(1 - e)\delta(e) \\
= ea(1 - e)ux + ea(1 - e)\delta(e) \in \text{nil}(R)[x; \alpha, \delta].
\]

Likewise we have \( ea(1 - e)x\alpha(1 - e)x \in \text{nil}(R)[x; \alpha, \delta] \). So by Corollary 2.15, we have \( e \cdot ea(1 - e) \in \text{nil}(R) \). Hence \( ea - eae \in \text{nil}(R) \). Next let \( h(x) = 1 - e - (1 - e)aex \) and \( k(x) = e + (1 - e)aex \). Then we have \( h(x)k(x) \in \text{nil}(R)[x; \alpha, \delta] \). Thus by Corollary 2.15, we obtain \((1 - e) \cdot (1 - e)ae = ae - eae \in \text{nil}(R) \). Thus \( ea - ae \in \text{nil}(R) \) and so \( ea = ae + u \) where \( u \in \text{nil}(R) \).

Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). An ideal \( I \) of \( R \) is said to be weak \((\alpha, \delta)\)-compatible provided that \( ab \in \text{nil}(R) \iff a\delta(b) \in \text{nil}(R) \), and \( ab \in \text{nil}(R) \rightarrow a\delta(b) \in \text{nil}(R) \) for any \( a, b \in I \).
Theorem 2.18  Let $R$ be an abelian NI ring, $\alpha$ an endomorphism, and $\delta$ an $\alpha-$derivation of $R$. Then the following statements are equivalent:

1) $R$ is a weak $(\alpha, \delta)$-compatible ring.

2) For each idempotent $e \in R$ such that $\alpha(e) = e + u$ with $u \in \text{nil}(R)$ and $\delta(e) \in \text{nil}(R)$, $eR$ and $(1 - e)R$ are weak $(\alpha, \delta)$-compatible ideals.

Proof  It suffices to show that (2) $\implies$ (1). Let $a, b \in R$ and $ab \in \text{nil}(R)$. Then we have $eab \in \text{nil}(R)$ because $R$ is an abelian ring. Since $\alpha(e) = e + u$ with $u \in \text{nil}(R)$ and $\delta(e) \in \text{nil}(R)$, $eR$ and $(1 - e)R$ are weak $(\alpha, \delta)$-compatible, we have

$eaa(e)b = ea\alpha(\alpha(e)b) = ea(e + u)\alpha(b) = eaa(b) + eau\alpha(b) \in \text{nil}(R)$,

and $eaa\delta(e)b = ea\alpha(e)\delta(b) + \delta(e)b = ea\alpha(e)b + ea\delta(e)b = ea(e + u)\delta(b) + ea\delta(e)b = ea\delta(b) + eau\delta(b) + eaa\delta(e)b \in \text{nil}(R)$. Thus $eaa(b) \in \text{nil}(R)$ and $eaa\delta(b) \in \text{nil}(R)$. Likewise we obtain $(1 - e)a\alpha(b) \in \text{nil}(R)$, and $(1 - e)a\delta(b) \in \text{nil}(R)$. Hence we have $ab \in \text{nil}(R)$. Therefore $R$ is a weak $(\alpha, \delta)$-compatible ring.

Let $U \subseteq R$, and $V \subseteq R$, we use $[U : V]$ to represent the set $\{x \in R \mid Vx \subseteq U\}$. Then for any $U \subseteq R$, we have

$[\text{nil}(R) : U] = \{x \in R \mid Ux \subseteq \text{nil}(R)\} = \{x \in R \mid xU \subseteq \text{nil}(R)\}$.

Given a ring $R$, we define

$N_R\text{Ann}_R(2^R) = \{[\text{nil}(R) : U] \mid U \subseteq R\},$

and

$N_{R[x, \alpha, \delta]}\text{Ann}_{R[x, \alpha, \delta]}(2^{R[x, \alpha, \delta]}) = \{[\text{nil}(R)[x; \alpha, \delta] : V] \mid V \subseteq R[x; \alpha, \delta]\}$.

Theorem 2.19  Let $R$ be a weak $(\alpha, \delta)$-compatible NI ring. Then

$\phi : N_R\text{Ann}_R(2^R) \longrightarrow N_{R[x, \alpha, \delta]}\text{Ann}_{R[x, \alpha, \delta]}(2^{R[x, \alpha, \delta]})$

defined by $\phi(I) = I[x; \alpha, \delta]$ for every $I \in N_R\text{Ann}_R(2^R)$ is bijective.

Proof  We first prove that $[\text{nil}(R)[x; \alpha, \delta] : U] = [\text{nil}(R) : U][x; \alpha, \delta]$ for any subset $U \subseteq R$. Since $[\text{nil}(R) : U][x; \alpha, \delta] \subseteq [\text{nil}(R)[x; \alpha, \delta] : U]$ is clear; now we show that $[\text{nil}(R)[x; \alpha, \delta] : U] \subseteq [\text{nil}(R) : U][x; \alpha, \delta]$. For any skew polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \text{nil}(R)[x; \alpha, \delta]$, we have $rf(x) = ra_0 + ra_1x + \cdots + ra_nx^n \in \text{nil}(R)[x; \alpha, \delta]$ for any $r \in U$. So $ra_i \in \text{nil}(R)$
for all $0 \leq i \leq n$ and all $r \in U$. Thus $a_i \in (\text{nil}(R) : U)$ for all $0 \leq i \leq n$ and so $f(x) \in (\text{nil}(R) : U)[x; \alpha, \delta]$. Then $(\text{nil}(R)[x; \alpha, \delta] : U) \subseteq (\text{nil}(R) : U)[x; \alpha, \delta]$. Hence $[\text{nil}(R)[x; \alpha, \delta] : U] = [\text{nil}(R) : U][x; \alpha, \delta]$. Thus $\phi$ is well defined. Obviously, $\phi$ is injective. In order to finish our proof, we must see that $\phi$ is surjective. To this end, let

$$[\text{nil}(R)[x; \alpha, \delta] : V] \in N_{R[x; \alpha, \delta]} \text{Ann}_{R[x; \alpha, \delta]}(2^{R[x; \alpha, \delta]}),$$

and

$$g(x) = b_0 + b_1 x + \cdots + b_n x^n \in [\text{nil}(R)[x; \alpha, \delta] : V]$$

where $V \subseteq R[x; \alpha, \delta]$. We then have $f(x)g(x) \in [\text{nil}(R)[x; \alpha, \delta] : U]$ for any $f(x) = a_0 + a_1 x + \cdots + a_m x^m \in U$. Thus $a_i b_j \in \text{nil}(R)$ for each $i, j$ by Corollary 2.15. So $b_j \in [\text{nil}(R) : C_V]$ for all $0 \leq j \leq n$, and hence $g(x) \in [\text{nil}(R) : C_V][x; \alpha, \delta]$, and so $[\text{nil}(R)[x; \alpha, \delta] : V] \subseteq [\text{nil}(R) : C_V][x; \alpha, \delta]$. On the other hand, it is easy to see that $[\text{nil}(R) : C_V][x; \alpha, \delta] \subseteq [\text{nil}(R)[x; \alpha, \delta] : V]$. Thus $[\text{nil}(R)[x; \alpha, \delta] : V] = [\text{nil}(R) : C_V][x; \alpha, \delta] = \phi([\text{nil}(R) : C_V])$. Hence $\phi$ is surjective. Therefore $\phi$ is bijective.

**Theorem 2.20** Let $R$ be a weak $(\alpha, \delta)$-compatible NI ring. If for each nonempty subset $X \not\subseteq \text{nil}(R)$, $[\text{nil}(R) : X]$ is generated as a right ideal by a nilpotent element, then for each nonempty subset $U \not\subseteq \text{nil}(R)[x; \alpha, \delta]$, $[\text{nil}(R)[x; \alpha, \delta] : U]$ is generated as a right ideal by a nilpotent element.

**Proof** Let $U$ be a nonempty subset of $R[x; \alpha, \delta]$ with $U \not\subseteq \text{nil}(R)[x; \alpha, \delta]$. Suppose $g(x) = b_0 + b_1 x + \cdots + b_n x^n \in [\text{nil}(R)[x; \alpha, \delta] : U]$. Then $f(x)g(x) \in [\text{nil}(R)[x; \alpha, \delta] : U]$ for each $f(x) = a_0 + a_1 x + \cdots + a_m x^m \in U$. Thus $a_i b_j \in \text{nil}(R)$ for each $i, j$ by Corollary 2.15, and so $b_j \in [\text{nil}(R) : C_U]$ for each $0 \leq j \leq n$. If $C_U \subseteq \text{nil}(R)$, then $U \not\subseteq [\text{nil}(R)[x; \alpha, \delta]$, a contradiction. Thus there exists a nilpotent element $w \in \text{nil}(R)$ such that $[\text{nil}(R) : C_U] = w R$. Hence $b_j = w r_j$ where $r_j \in R$. Thus $g(x) = b_0 + b_1 x + \cdots + b_n x^n = w(r_0 + r_1 x + \cdots + r_m x^m) \subseteq w R[x; \alpha, \delta]$, and hence $[\text{nil}(R)[x; \alpha, \delta] : U] \subseteq w R[x; \alpha, \delta]$. On the other hand, for each $f(x) = a_0 + a_1 x + \cdots + a_m x^m \in U$, and $h(x) = h_0 + h_1 x + \cdots + h_n x^n \in R[x; \alpha, \delta]$, we have

$$f(x) w h(x) = (\sum_{i=0}^{m} a_i x^i) (\sum_{j=0}^{n} w h_j x^j) = (\sum_{l=0}^{m+n} (\sum_{s+t=l}^{m+n} (\sum_{i=s}^{m} a_i f_s^i(w h_t))) x^l).$$

Since $w \in \text{nil}(R)$, $a_i w h_i \in \text{nil}(R)$ for all $0 \leq i \leq m$ and $0 \leq t \leq n$. Thus $a_i f_s^i(w h_t) \in \text{nil}(R)$, and so $f(x) w h(x) \in \text{nil}(R)[x; \alpha, \delta]$. Thus $w R[x; \alpha, \delta] \subseteq [\text{nil}(R)[x; \alpha, \delta] : U]$. Hence $[\text{nil}(R)[x; \alpha, \delta] : U] = w R[x; \alpha, \delta]$. Therefore $[\text{nil}(R)[x; \alpha, \delta] : U]$ is generated as a right ideal by a nilpotent element.
Thus and so statements are equivalent:

(1) For each nonempty subset \( X \not\subseteq \text{nil}(R) \), \([\text{nil}(R) : X]\) is generated as a right ideal by a nilpotent element.

(2) For each nonempty subset \( U \not\subseteq \text{nil}(R) [x; \alpha] \), \([\text{nil}(R) [x; \alpha] : U]\) is generated as a right ideal by a nilpotent element.

**Proof**  \( (1) \implies (2) \). It follows from Proposition 2.20.

\( (2) \implies (1) \). Let \( X \not\subseteq \text{nil}(R) \). There exists \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \text{nil}(R[x; \alpha]) \) such that \([\text{nil}(R)[x; \alpha] : X] = f(x) \cdot R[x; \alpha] \). We will show that \([\text{nil}(R) : X] = a_0 R \). Since \( f(x) \in \text{nil}(R[x; \alpha]) \), \( a_0 \in \text{nil}(R) \) by Corollary 2.15. Hence \( u_0 R \subseteq \text{nil}(R) \) for any \( u \in X \), and so \( a_0 R \subseteq [\text{nil}(R) : X] \). Assume that \( r \in [\text{nil}(R) : X] \), then \( r \in [\text{nil}(R)[x; \alpha] : X] = f(x) \cdot R[x; \alpha] \). There exists \( h(x) = h_0 + h_1 x + \cdots + h_l x^l \) such that \( r = f(x) h(x) \). Thus \( r = a_0 h_0 \in a_0 R \) and so \([\text{nil}(R) : X] \subseteq a_0 R \). Hence \([\text{nil}(R) : X] = a_0 R \) where \( a_0 \in \text{nil}(R) \).

**Theorem 2.22** Let \( R \) be a weak \((\alpha, \delta)\)-compatible NI ring. If for each \( p \not\subseteq \text{nil}(R) \), \([\text{nil}(R) : p]\) is generated as a right ideal by a nilpotent element, then for any \( f(x) \not\subseteq \text{nil}(R)[x; \alpha, \delta] \), \([\text{nil}(R)[x; \alpha, \delta] : f(x)]\) is generated as a right ideal by a nilpotent.

**Proof** Let \( f(x) = a_0 + a_1 x + \cdots + a_m x^m \not\subseteq \text{nil}(R)[x; \alpha, \delta] \). Suppose \( g(x) = b_0 + b_1 x + \cdots + b_n x^n \in [\text{nil}(R)[x; \alpha, \delta] : f(x)] \). Then \( f(x) g(x) \in \text{nil}(R)[x; \alpha, \delta] \).

Since \( R \) is a weak \((\alpha, \delta)\)-compatible NI ring, we obtain \( a_i b_j \in \text{nil}(R) \) for each \( i, j \) by Corollary 2.15. Thus \( b_j \in [\text{nil}(R) : a_i] \) for all \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). Since \( f(x) \not\subseteq \text{nil}(R)[x; \alpha, \delta] \), there exists some \( 0 \leq i \leq m \) such that \( a_i \not\in \text{nil}(R) \), and so \([\text{nil}(R) : a_i] = u R \) with \( u \in \text{nil}(R) \). Hence \( b_j = u r_j \), where \( r_j \in R \).

Thus \( g(x) = b_0 + b_1 x + \cdots + b_n x^n = u (r_0 + r_1 x + \cdots + r_n x^n) \in u R[x; \alpha, \delta] \), and hence \([\text{nil}(R)[x; \alpha, \delta] : f(x)] \subseteq u R[x; \alpha, \delta] \). On the other hand, for any \( h(x) = h_0 + h_1 x + \cdots + h_n x^n \in R[x; \alpha, \delta] \), we have

\[
f(x) u h(x) = \left( \sum_{i=0}^{m} a_i x^i \right) \left( \sum_{j=0}^{n} u h_j x^j \right) = \left( \sum_{l=0}^{m+n} \left( \sum_{s+t=l}^{m} (\sum_{i=s}^{m} a_i f_s^i (u h_t))) x^l \right).\]

Since \( u \in \text{nil}(R) \), \( a_i u h_t \in \text{nil}(R) \) for all \( 0 \leq i \leq m \) and \( 0 \leq t \leq n \). Thus \( a_i f^i_s (u h_t) \in \text{nil}(R) \), and so \( f(x) u h(x) \in \text{nil}(R)[x; \alpha, \delta] \). Hence \([\text{nil}(R)[x; \alpha, \delta] : f(x)] = u R[x; \alpha, \delta] \). Therefore \([\text{nil}(R)[x; \alpha, \delta] : f(x)]\) is generated as a right ideal by a nilpotent element.

By a similar proof as Proposition 2.21, we obtain the following:

**Theorem 2.23** Let \( R \) be a weak \((\alpha, \delta)\)-compatible NI ring. Then the following statements are equivalent:

(1) For each nonempty subset \( X \not\subseteq \text{nil}(R) \), \([\text{nil}(R) : X]\) is generated as a right ideal by a nilpotent element.

(2) For each nonempty subset \( U \not\subseteq \text{nil}(R) [x; \alpha] \), \([\text{nil}(R) [x; \alpha] : U]\) is generated as a right ideal by a nilpotent element.
(1) For each \( p \not\in \text{nil}(R) \), \([\text{nil}(R) : p]\) is generated as a right ideal by a nilpotent element.

(2) For each \( f(x) \not\in \text{nil}(R)[x; \alpha] \), \([\text{nil}(R)[x; \alpha] : f(x)]\) is generated as a right ideal by a nilpotent element.

Acknowledgements: This research is supported by the National Natural Science Foundation of China (11071062), Natural Science Foundation of Hunan Province (10jj3065) and Scientific Research Fundation of Hunan Provincial Education Department (10A033).

References


Received: May, 2011