Applications of Primeness in Near Rings to Malone Trivial Near Rings

Fatma Münevver Yiğiter
Department of Mathematics
Bozok University
66100, Yozgat, Turkey

Akin Osman Atagün
Department of Mathematics
Bozok University
66100, Yozgat, Turkey
aoatagun@erciyes.edu.tr

Funda Taşdemir
Department of Mathematics
Bozok University
66100, Yozgat, Turkey

Abstract
In this paper we study applications of completely prime, 3-prime near-rings to Malone trivial near-rings by using the properties right permutable, left permutable, medial, left self distributive (LSD) and right self distributive (RSD).

Mathematics Subject Classification: 16Y30

Keywords: completely prime near-ring, 3-prime near-ring, Malone trivial near-ring

1 Introduction
Some different generalizations of primeness for rings have been introduced for near-rings. In [9] Holcombe defined three different concepts of primeness, which he called 0-, 1- and 2- prime. In [8] Groenewald obtained further results
for these and introduced a further notion which he called 3-primeness. Booth, Groenewald and Veldsman [7] gave another generalization of prime rings which they called equiprimeness or e-primeness. In [11] Veldsman showed that if $N$ is a zero-symmetric, RSD and 3-prime near-ring, then $N$ is a Malone trivial near-ring and using this result he obtained that an equiprime RSD (or LSD) near-ring is the two element field. In this study, we investigate relations between prime near-rings and Malone trivial near-rings.

2 Preliminary Notes

Throughout this paper, $N$ will always denote a right near-ring. It is assumed that the reader is familiar with the basic definitions of right near-ring, zero-symmetric near-ring and ideal [10].

Definition 2.1. We recall from Holcombe [9], Groenewald [8] and Booth [7]; an ideal $P$ is $v$-prime, if for $A, B \subseteq N$ with $A, B$ ideals of $N$ if $v = 0$
$A, B$ left ideals of $N$ if $v = 1$
$A, B$ $N$-subgroups of $N$ if $v = 2$,
the inclusion $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal $P$ of a near-ring $N$
 is called a 3-prime ideal if for all $a, b \in N$, $aNb \subseteq P$ implies $a \in P$ or $b \in P$.

If for all $a, b \in N$, $ab \in P$ implies $a \in P$ or $b \in P$, then $P$ is called a completely prime ideal [3]. In [1] some relations between 3-prime and completely prime ideals of a near-ring are given.
$P$ is called an equiprime ideal, if $a \in N \setminus P$ and $x, y \in N$ such that $anx − any \in P$ for all $n \in N$, then $x − y \in P$ [7]. If the zero ideal of $N$ is $v$-prime $(v = 0, 1, 2, 3, e, f)$, then $N$ is called a $v$-prime near-ring.
It is already known that if $P$, then $P$ is completely prime $\implies$ $P$ is 3-prime $\implies$ $P$ is 2-prime, $P$ is 1-prime $\implies$ $P$ is 0-prime and $P$ is 2-prime $\implies$ $P$ is 1-prime when $N$ is zero-symmetric. Furthermore, any equiprime ideal is 3-prime [11]. Also playing a role in this paper are the identities:
If for all $a, b, c, d \in N$, $abc = acb$ (resp. $abc = bac$, $abcd = acbd$), then $N$ is called a right permutable (resp. left permutable, medial) near-ring [5]. If $abc = abac$ (resp. $abc = acbc$), then $N$ is called a left self distributive-LSD- (resp. right self distributive -RSD-) near-ring [4]. Birkenmeier and Heatherly [4] showed that 3-prime ideals in an LSD or RSD zero-symmetric near-ring are also completely prime.
3 Applications of Primeness to Malone Trivial Near-rings

Definition 3.1. [11] Let $G$ be any group with $|G| \geq 2$ and let $S \subseteq G \setminus \{0\}$. Let $M$ be the near ring on $G$ with multiplication given by: $ab = a$ if $b \in S$, $ab = 0$ if $b \notin S$. Then $M$ is called a Malone trivial near-ring.

Proposition 3.2. Let $N$ be a Malone trivial near ring. Then $N$ is a right self distributive near ring.

Proof. Let $N$ be a Malone trivial near-ring and $a, b, c \in N$.

$$abc = \begin{cases} ab, & \text{if } c \neq 0 \\ 0, & \text{if } c = 0 \end{cases} = \begin{cases} a, & \text{if } c \neq 0 \text{ and } b \neq 0 \\ 0, & \text{if } c = 0 \text{ or } b = 0 \end{cases} \quad (1)$$

$$acbc = \begin{cases} acb, & \text{if } c \neq 0 \\ 0, & \text{if } c = 0 \end{cases} = \begin{cases} a, & \text{if } c \neq 0 \text{ and } b \neq 0 \\ 0, & \text{if } c = 0 \text{ or } b = 0 \end{cases} \quad (2)$$

Then $abc = acbc$. By (1) and (2), $N$ is an RSD near-ring. □

Proposition 3.3. Let $N$ be a Malone trivial near-ring. Then $N$ is a 3-prime near-ring.

Proof. Assume that $N$ is a Malone trivial near-ring and $a, b \in N$ such that $aNb = 0$. Let $a \neq 0$. Since $aNb = 0$, then for all $n \in N$ $anb = 0$. In particular for $n = a$, $aab = 0$. Since $N$ is Malone and $a \neq 0$, this implies that $b = 0$. Thus $N$ is a 3-prime near-ring. □

Proposition 3.4. Let $N$ be a Malone trivial near-ring. Then $N$ is a left self distributive near-ring.

Proof. Let $N$ be a Malone trivial near-ring. For all $a, b, c \in N$

$$abc = \begin{cases} ab, & \text{if } c \neq 0 \\ 0, & \text{if } c = 0 \end{cases} = \begin{cases} a, & \text{if } c \neq 0 \text{ and } b \neq 0 \\ 0, & \text{if } c = 0 \text{ or } b = 0 \end{cases} \quad (3)$$

$$abac = \begin{cases} aba, & \text{if } c \neq 0 \\ 0, & \text{if } c = 0 \end{cases} = \begin{cases} a, & \text{if } b \neq 0 \text{ and } c \neq 0 \\ 0, & \text{if } c = 0 \text{ or } b = 0 \end{cases} \quad (4)$$

Then $abc = abac$. By (3) and (4), $N$ is an LSD near-ring. □

Proposition 3.5. [11] Let $N$ be a right self distributive 3-prime near-ring. Then $N$ is a Malone trivial near-ring.
Corollary 3.6. A near-ring $N$ is right self distributive and 3-prime if and only if $N$ is a Malone trivial near-ring.

Proof. Let $N$ be an RSD 3-prime near-ring. Then $N$ is a Malone trivial near-ring by Proposition 3.5. Conversely, let $N$ be a Malone trivial near-ring. Then by Proposition 3.2 and Proposition 3.3, $N$ is 3-prime and RSD.

Corollary 3.7. Let $N$ be an RSD equiprime near-ring. Then $N$ is a Malone trivial near-ring.

Proof. Since $N$ is an equiprime near-ring, $N$ is 3-prime near-ring [11]. Also $N$ is a Malone trivial near-ring by Proposition 3.5.

Proposition 3.8. Let $N$ be a Malone trivial near-ring. Then $N$ is an LSD 3-prime near-ring.

Proof. Let $N$ be a Malone trivial near-ring. $N$ is an LSD 3-prime near-ring by Proposition 3.3 and Proposition 3.4.

Remark 3.9. If $N$ is an LSD near-ring, it does not need to be a Malone trivial near-ring.

The following example illustrates Proposition 3.4 and Remark 3.9.

Example 3.10. Let $N = \{0, 1, 2, 3\}$ be the near-ring which has addition and multiplication tables as follows:

Table 1: Addition and Multiplication Tables of $N$

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

| .  | 0  | 1  | 2  | 3  |
|----|----|----|----|
| 0  | 0  | 0  | 0  | 0  |
| 1  | 0  | 0  | 0  | 0  |
| 2  | 0  | 0  | 2  | 2  |
| 3  | 0  | 0  | 2  | 2  |

It is easily seen that, this near-ring is LSD and left permutable, but not 3-prime and Malone.

Proposition 3.11. Let $N$ be an LSD right permutable near-ring. If $N$ is c-prime, then $N$ is a Malone trivial near-ring.

Proof. Let $N$ be an LSD and right permutable near-ring. Also let $b \neq 0$ and $c \neq 0$. Since $N$ is LSD and right permutable, then $abc = acb = acab = acacb = acabc = acbc$ for $a, b, c \in N$. So, $abc = acbc$. Then $(ab - acb)c = 0$. Since $N$ is c-prime this implies that either $ab - acb = 0$ or $c = 0$. Since $c \neq 0$, $ab - acb = 0$. It follows that $(a - ac)b = 0$. Using c-primeness of $N$ and $b \neq 0$, $a - ac = 0$ i.e $a = ac$. Thus, $N$ is a Malone trivial near-ring.

Proof. Since $x0 = 0$, for $\forall x \in N$, $N$ is a zero-symmetric near-ring. If $x, y \neq 0$, then $xNy = \{0, x\}$. When $xNy = 0$, either $x = 0$ or $y = 0$. So $N$ is a 3-prime near-ring. Now let $0 \neq x, y \in N$ and $x \neq y$. If $n \neq 0$, then $xn = xny = x$. If $n = 0$, then $xn = xny = 0$. So $xn = xny$ for $\forall n \in N$. But $x \neq y$, so $N$ is not an equiprime near-ring.

Example 3.13. Let $(N, +)$ be the Klein’s four group with multiplication defined as per Scheme 1, p.408, [10];

Table 2: Multiplication Table of $N$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
</tbody>
</table>

It can be easily seen that $N$ is a near-ring which is Malone trivial but not equiprime.


Proof. Assume that $N$ is an RSD c-prime near-ring and $a, b \in N$ such that $0 \neq b \in N$. We need to show that $ab = a$. Since $N$ satisfies the RSD condition, then $abb = abbb$ and we have $(ab - abb)b = 0$. Because $N$ is c-prime, it implies that $ab - abb = 0$ or $b = 0$. By assumption we have $b \neq 0$, then $ab - abb = 0$. So $(a - ab)b = 0$. Again using c-primeness of $N$, $a - ab = 0$ or $b = 0$. Hence $a = ab$. Therefore, $N$ is a Malone trivial near-ring.

Corollary 3.15. If $N$ is a Malone trivial near-ring, then $N$ is a c-prime near-ring.

Proof. Since $N$ is a Malone trivial near-ring, then for every $x, y \in N$ such that $xy = 0$, we have $x = 0$ or $y = 0$. So $N$ is a c-prime near-ring.

Corollary 3.16. Let $N$ be an LSD c-prime near-ring. Then $N$ is not a Malone trivial near-ring, in general.

Example 3.17. Let the ring $N = (Z_3, +, \cdot)$. Then $N$ is LSD and c-prime but not a Malone trivial near-ring.
It is clear that if \( N \) is a commutative near-ring such that \( |N| \geq 3 \), then \( N \) is not a Malone trivial near-ring.

**Proposition 3.18.** Let \( N \geq 3 \) be a distributive, c-prime and medial near-ring. Then \( N \) is not a Malone trivial near-ring, in general.

*Proof.* Let \( a, b \in N \) such that \( a \neq 0, b \neq 0 \) and \( a \neq b \). Since \( N \) is medial for all \( a, b \in N \) \( abab = aabb \). Then \((aba - aab)b = 0\). Using c-primeness of \( N \), \( aba - aab = 0 \) or \( b = 0 \). By assumption \( b \neq 0 \), \( a(\bar{a} - ab) = 0 \). It follows that \( ab = ba \). Hence \( N \) is not a Malone trivial near-ring. \( \square \)

Following proposition shows that if \( N \) is a c-prime left permutabe near-ring such that \( |N| \geq 3 \), then \( N \) is not a Malone trivial near-ring.

**Proposition 3.19.** Let \( N \) be a left permutabe c-prime near-ring such that \( |N| \geq 3 \). Then \( N \) is a commutative near-ring and then \( N \) is not a Malone trivial near-ring.

*Proof.* Let \( a, b, c \in N \) such that \( c \neq 0 \) and \( a \neq b \). Since \( N \) is left permutabe, we have \( abc = bac \). Then \((ab - ba)c = 0\). Using c-primeness of \( N \), \( ab - ba = 0 \) or \( c = 0 \). Whence \( ab = ba \). Since \( a \neq b \), \( N \) is not a Malone trivial near-ring. \( \square \)

**Remark 3.20.** If \( N = N_a \) and \( N \) is right permutabe, \( N \) is not a Malone trivial near-ring. In fact, since every commutative ring is also a commutative near-ring, such near-ring \( N \) satisfies \( N = N_a \) and right permutability. Hence \( N \) needs not to be a Malone trivial near-ring.

Morever 3-primeness and right permutability don’t imply to be Malone. We have the following example:

**Example 3.21.** [2] Let the additive group \((\mathbb{Z}_6, +)\) under a multiplication given in the following table, \((\mathbb{Z}_6, +, \cdot)\) is a right permutabe near-ring. We say \( N = (\mathbb{Z}_6, +, \cdot) \).

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>
Let $P = \{0, 3\}$. Since $P$ is a 3-prime ideal of $N$, then $N/P$ is a 3-prime near-ring. It is easily seen that $N/P$ is also right permutable but not a Malone trivial near-ring.

**Proposition 3.22.** Let $N$ be a zero-symmetric, LSD and c-prime near-ring. Then $N$ is a Malone trivial near-ring.

**Proof.** Let $0 \neq a, b \in N$. To complete the proof, it is enough to show that $ba = b$. Since $N$ is LSD, $abb = abab$. So $(ab - aba)b = 0$. Using c-primeness of $N$, $ab - aba = 0$ or $b = 0$. By assumption $b \neq 0$, $ab - aba = 0$, i.e. $ab = aba$. Then $baa = b(ab)a = b(aba)a = baaa$, since $N$ is LSD. It follows $baa = baaa$. We have $(b - ba)aa = 0$. The fact that $N$ is c-prime, if $a \neq 0$ then $aa \neq 0$. Hence $b - ba = 0$, i.e $b = ba$. \qed

**Proposition 3.23.** If $N$ is a Malone trivial near-ring, then $N$ is a Boolean near-ring.

**Proof.** It is obvious. Hence omitted. \qed

**References**


Received: May, 2011