On the Cycle Basis Join of Two Graphs

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Abstract

The basis number, $b(G)$, of a graph $G$ is defined to be the least integer $d$ such that $G$ has a $d$-fold basis for its cycle space.

A minimum cycle basis (MCB) for graph $G$ is a cycle basis with minimum length.

In this paper, we give an MCB and an upper bound for the basis number of join of two graphs.

Mathematics Subject Classification: 05C38

Keywords: Cycle basis, Basis number, Join of graphs

1 Introduction

Let $G(V,E)$ be a graph. The set $P(E)$ of all subsets of $E$ forms an $|E|$-dimensional vector space over $\mathbb{GF}(2)$ with vector addition $X \oplus Y := (X \cup Y) - (X \cap Y)$ and scalar multiplication $1 \cdot X = X$, $0 \cdot X = \emptyset$, for all $X, Y \in P(E)$.

A (generalized) cycle is a subgraph such that every vertex has even degree. Such graphs are also known as Eulerian subgraphs. We represent a (generalized) cycle by its edge set $C$ and write $V_C$ for its vertex set. An elementary cycle is a connected minimal subgraph such that every vertex in $V_C$ has degree two. The set $C$ of all generalized cycles forms a subspace of $(P(E), \oplus, \cdot)$ which is called the cycle space $C(G)$ of $G$. A basis $B$ of the cycle space $C$ is called a cycle basis of $G(V,E)$. The dimension of the cycle space is the cyclomatic number or first Betti number $\dim(C) = \nu(G) = |E| - |V| + 1$.

The length $|C|$ of a generalized cycle $C$ is the number of its edges. The length $\ell(B)$ of a cycle basis $B$ is the sum of the lengths of its generalized cycles, $\ell(B) = \sum_{C \in B} |C|$.

A minimum cycle basis (MCB) $M$ for graph $G$ is a cycle basis with minimum length.

A basis for $C(G)$ is called a k-fold basis if each edge of $G$ occurs in at most $k$ of the cycles in the basis. The basis number of $G$ denoted by $b(G)$ is the smallest integer $k$ such that $C(G)$ has a $k$-fold basis. $f_B(e)$ is the number of the cycles in basis $B$ that edge $e$ of $G$ occurs in them.

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2 Preliminary Notes

Minimum length bases of the cycle space of a graph (MCBs) have a variety of applications in science and engineering, for example, in structural flexibility analysis [12], electrical networks [5], and in chemical structure storage and retrieval systems [6]. Brief surveys and extensive references can be found in [10, 9, 8]. In general, minimum cycle bases are not very well behaved under graph operations. Neither the total length $\ell(G)$ nor the length of the longest cycle $\lambda(G)$ in a MCB of $G$ are minor monotone. Hence, there does not seem to be a general way of extending MCBs of a certain collection of partial graphs of $G$ to an MCB of $G$. Consequently, not much is known about the length $\ell(G)$ of the MCB. The sharp upper bound $\ell(K_m) = 3(m - 1)(m - 2)/2$ for graphs with $m$ vertices is proved in [10, Thm.6]. For 2-connected outerplanar and planar graphs we have $\ell(G) \leq 3m - 6$ and $\ell(G) \leq 6m - 15$, respectively [13, Thm.11]. A global upper bound $\ell(G) \leq \nu(G) + \kappa(T(G))$, where $\nu(G)$ is the cyclomatic number of $G$ and $\kappa(T(G))$ the connectivity of the tree graph of $G$, is derived in [14].

The basis number was introduced by Schmeichel [16] in 1981. MacLane in [15] proved that graph $G$ is planar if and only if $b(G) \leq 2$. In 1981, Schmeichel [17] proved that $b(K_n) = 3$ whenever $n \geq 5$ and $b(K_{n,m}) \geq 4$ for each $n$ and $m$. In 1982, Bank and Schmeichel [4] proved that $b(Q_n) = 4$ whenever $n \geq 7$. Many papers appeared to investigate the basis number of certain graphs, especially the graph products, see [1], [2], [3] and [11].

Getting new graphs from known graphs through different kinds of operations on graphs originated as early as the beginning of graph theory as an independent subject. Join of graphs is natural way to enlarge the space of the graphs that we study it in this paper.

The join of simple connected $(p_1, q_1)$ graph $G_1$ and $(p_2, q_2)$ graph $G_2$, written $G_1 \lor G_2$ is the graph obtained from the disjoint union $G_1 + G_2$ by adding the edges $\{(u, v) | u \in V(G_1), v \in V(G_2)\}$. It follows that $G_1 \lor G_2$ has $p_1 + p_2$ vertices and $q_1 + q_2 + p_1p_2$ edges [8]. Thus $\nu(G_1 \lor G_2) = q_1 + q_2 + p_1p_2 - p_1 - p_2 + 1$.

3 main results

In this section, we give an upper bound of the basis number of the join of two graphs in terms of their basis numbers.

**Lemma 1** Let $T$ be a tree with $p$ vertices ($p \geq 3$) if $v$ is any point which is not a vertex of $T$, and if $G = v \lor T$, then $b(G) = 2$.

**Proof.** Assume that $G$ is not planar. Then, by Kuratowski’s theorem, $G$ contains a subdivision of $K_5$ or $K_{3,3}$. Then $G - x$ cannot be acyclic graph for
any \( x \in V(G) \), while \( G - v \) is a tree. This is a contradiction, and hence \( G \) is planar. Therefore, \( b(G) \leq 2 \). If \( b(G) = 1 \), then \( G \) has a 1-fold basis, which implies that \( \dim C(G) \leq |E(G)|/3 \) since each cycle contains at least three edges. Since \( |E(G)| = 2p - 1 \) and \( \dim C(G) = p - 1 \), we have \( p - 1 \leq \frac{(2p-1)}{3} \), which implies that \( p \leq 2 \). This is a contradiction. Therefore, \( b(G) = 2 \).

**Lemma 2** Let \( H \) be any connected \((p,q)\) graph and let \( v \) be any vertex which is not a vertex of \( H \). If \( G = v \lor H \), then \( b(G) \leq b(H) + 2 \).

**Proof.** Let \( u_1, u_2, ..., u_p \) be the vertices of \( H \), \( B_1 \) be a \( b(H) \)-fold basis for \( C(H) \), and let \( B_2 \) be a 2-fold basis for \( v \lor T \) that \( T \) is a spanning tree of \( H \). Each cycle in \( v \lor T \) must contain an edge of the form \( vu_i \) for some \( i \in \{1, 2, ..., p\} \). Thus, these cycle are independent and are independent from the cycles in \( B_1 \). Then clearly \( B = B_1 \cup B_2 \) is an independent set of cycles with \( |B| = |B_1| \cup |B_2| = q - p + 1 + p - 1 = q = \dim C(G) \), hence \( B \) is a basis for \( C(G) \). If \( e = vu_i \), then \( f_B(e) \leq f_{B_2}(e) \leq 2 \), and if \( e \) is an edge of \( H \), then clearly \( f_B(e) \leq b(H) + 2 \). Thus, \( b(G) \leq b(H) + 2 \).

But for \( G_1 \lor G_2 \) we have following theorem.

**Theorem 3** Let \( G_1 \) and \( G_2 \) be two connected graphs. If \( G = G_1 \lor G_2 \), then \( B = B_F \lor B(G_2) \lor (\bigcup_{k=1}^{p_1} H_k) \) is a cycle basis for \( G \), that \( B_F \) is basis according to above lemma for \( F = u_k \lor G_1 \) that \( u_k \in V(G_2) \), \( k \in \{1, ..., p_1\} \), \( B(G_2) \) is basis of \( G_2 \) and \( H_k = v_k \lor T_{G_2} \) that \( T_{G_2} \) is spanning tree of \( G_2 \).

**Proof.** We have \( \dim C(G) = q_1 + q_2 - p_1 - p_2 + p_1 p_2 + 1 \). Let \( v_1, ..., v_{p_2} \) are vertices of \( G_1 \) and \( G_2 \), corresponding. Since each cycle in \( B_F \) has edge of the form \( v_i v_j \) for some \( i, j \), thus \( B_F \) is independent from \( H_k \) and \( B_{G_2} \). Each \( H_k = v_k \lor T_{G_2} \) must contain an edge of the form \( v_k u_i \) for \( k = 1, ..., p_1 \), \( i = 1, ..., p_2 \), thus \( \bigcup_{k=1}^{p_1} H_k \) is independent cycle set and is independent from \( B(G_2) \). Thus \( B \) is independent set and disjoint union of \( B_F \), \( B(G_2) \) and \( \bigcup_{k=1}^{p_1} H_k \), therefore \( |B| = |B_F| + |B(G_2)| + \bigcup_{k=1}^{p_1} H_k | = q_1 + q_2 - p_1 - p_2 + 1 + p_1 (p_2 - 1) = q_1 + q_2 - p_1 - p_2 + p_1 p_2 + 1 = \dim C(G) \). Thus \( B = B_F \lor B(G_2) \lor (\bigcup_{k=1}^{p_1} H_k) \) is a cycle basis for \( G \).

**Theorem 4** Let \( G_1 \) and \( G_2 \) be two connected graphs. If \( G = G_1 \lor G_2 \), then

\[
\max\{b(G_1), b(G_2)\} \leq b(G) \leq \max\{b(G_1) + 2, (b(G_2) + 2)p_1\}.
\]

**Proof.** The left inequality is clear. Let \( B_1 \) be a \( b(G_1) \)-fold basis and \( B_2 \) be a \( b(G_2) \)-fold basis and \( B \) be a basis of \( G = G_1 \lor G_2 \) by theorem 3. Let \( e \in G = G_1 \lor G_2 \) then

- If \( e \in E(G_1) \) by lemma 2, \( f_B(e) \leq f_{B_1}(e) + 2 \)
- If \( e \in E(G_2) \) by lemma 2 and to consider construction of \( B \)
\[ f_B(e) \leq (f_{B_2}(e) + 2)p_1 \]

If \( e \in \{ u,v \mid u,v \in V(G_1) \} \) by construction of \( B \) and lemma 1, \( f_B(e) \leq 4 \)

If \( e \in \{ u,v \mid u,v \in V(G_1), u,v \in V(G_2) \text{ and } j \neq k \} \) by lemma 1, \( f_B(e) = 2 \)

Thus \( b(G) \leq \max \{ b(G_1) + 2, (b(G_2) + 2)p_1 \} \).

In this part we construct a minimal basis for join of two graphs, at the first for \( v \lor G \), we have

**Lemma 5** Let \( G \) is any connected \((p,q)\) graph and let \( v \) any vertex which is not a vertex of \( G \). If \( H = v \lor G \) then \( D = \cup \{ v \lor e \mid e \in G \} \) is a MCB of \( H \).

**Proof.** We have \( E(H) = E(v \lor G) = q + p \) and then \( \dim C(H) = q + p - (1 + p) + 1 = q \). on the other hand \( |D| = E(G) = q \). Thus it’s suffice to prove set \( D \) is independent. But each \( v \lor e \) is different at least on \( e \). Thus \( D = \{ v \lor e \mid e \in G \} \) is a cycle basis of \( v \lor G \). since each \( v \lor G \) is a 3-cycle then \( D \) is a MCB of \( H \).

for \( G = K_2 \lor C_4 \) by theorem 3, \( \ell(B) = 2(3(3)) + 4 = 22 \). Actually we have \( \ell(G) = \ell(B) = 21 \) with use a composed way for construct \( B \), that is, we consider four triangle for first vertex by lemma 5 and tree triangle for second vertex by lemma 2.

We do some editions on introduced cycle base in theorem 3 and construct a MCB. According to above discussion for \( V(G_1) \lor G_2 \), \( \{ v_1 \lor E(G_2) \} \cup \bigcup_{k=1}^{p_1} H_k \) that \( H_k = v_k \lor E(T_{G_2}) \) and \( T_{G_2} \) is spanning tree of \( G_2 \) is a minimum cycle basis and put \( u_k \lor E(G_1) \) instead \( B_F \) to earn a MCB for \( G = G_1 \lor G_2 \)

**Theorem 6** Let \( G_1 \) and \( G_2 \) be two connected graphs. If \( G = G_1 \lor G_2 \), then \( B = \{ u_k \lor G_1 \} \cup \{ v_l \lor G_2 \} \cup \bigcup_{k=1}^{p_1} H_k \) is a minimum cycle basis for \( G \), that \( k \in \{ 1, ..., p_2 \}, l \in \{ 1, ..., p_1 \} \), \( H_k = v_k \lor T_{G_2} \) and \( T_{G_2} \) is spanning tree of \( G_2 \).

**References**


Received: August, 2010