On Non-degenerate Jordan Triple Systems

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Abstract

We demonstrate that all $JB^*$-triples (hence, all $JB^*$-algebras and all $C^*$-algebras) are non-degenerate Jordan triple systems but a Jordan triple system may not be non-degenerate; any two elements $x$ and $y$ in a non-degenerate Jordan triple system are von Neumann regular with generalized inverses of each other whenever the induced Bergmann operator $B(x, y)$ (or equivalently, $B(y, x)$) vanishes; however, the converse is not true even in case of $C^*$-algebras. L. G. Brown and G. K. Pedersen introduced a notion of quasi-invertible elements in a $C^*$-algebra, which plays a significant role in studying geometry of the unit ball. Recently, the present authors began a study of BP-quasi invertible elements in the general setting of $JB^*$-triples; here, it is deduced further that $B(x, y)$ on a $JB^*$-triple $J$ vanishes if and only if $x$ and $y$ are BP-quasi inverses of each other. Thus, von Neumann regularity for BP-quasi invertible elements in a $JB^*$-triple is a necessity but not a sufficiency.

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1 Introduction

L. G. Brown and G. K. Pedersen [1] jointly introduced a notion of quasi invertible elements in a $C^*$-algebra (henceforth, called BP-quasi invertible elements), which plays a significant role in studying geometry of the unit ball. A generalization of $C^*$-algebras, called $JB^*$-triples, was introduced by W. Kaup, who proved that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a $JB^*$-triple (cf. [4, 5, 11]). Inspired by the work of L. G. Brown and G. K. Pedersen [1, 2], the present authors [10] initiated a study of BP-quasi invertible elements in the general setting of $JB^*$-triples; it is proved that a non-zero element $u$ in a unital $C^*$-algebra is BP-quasi invertible if and only if it is von Neumann regular and admits a generalized inverse $v$ such that the Bergmann operator $B(u, v^*)$ vanishes [10, Theorem 3.1]; this led the authors to formulate an exact analogue of BP-quasi invertible elements in the general setting of $JB^*$-triples.

$JB^*$-triples is a special class of more general systems, called Jordan triple systems (cf. [8, 7, 4, 11]). An other special class of such systems is non-degenerate Jordan triple systems (cf. [8, 6]. In this article, we show that all $JB^*$-triples (hence, all $JB^*$-algebras and all $C^*$-algebras) are non-degenerate Jordan triple systems but a Jordan triple system may not be non-degenerate. For any two elements $x$ and $y$ in a non-degenerate Jordan triple system, we prove that the Bergmann operator $B(x, y)$ vanishes if and only if $B(y, x)$ vanishes, implying that the elements $x$ and $y$ are von Neumann regular with generalized inverses of each other; however, the converse is not true even in case of $C^*$-algebras. From these facts, we deduce further that $B(x, y)$ on a $JB^*$-triple $\mathcal{J}$ vanishes if and only if $x$ and $y$ are BP-quasi inverses of each other; consequently, von Neumann regularity for BP-quasi invertible elements in a $JB^*$-triple is a necessity but not a sufficiency.

2 Preliminaries

Our notation and basic terminology is standard as appeared in [10] (or [4, 11]). Recall that the underlying binary product $\circ$ of a Jordan algebra $\mathcal{J}$ induces a triple product $\{xyz\} := (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y$. A $JB^*$-algebra is a Banach space $\mathcal{J}$ which is a complex Jordan algebra with Jordan binary product $\circ$ equipped with an involution $*$ satisfying $\|x \circ y\| \leq \|x\|\|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$ for all $x, y \in \mathcal{J}$. If, in addition, $\mathcal{J}$ has a unit $e$ with $\|e\| = 1$ then it is called a unital $JB^*$-algebra. Thus, any $C^*$-algebra with associative product $ab$ is a $JB^*$-algebra under the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$.

A Jordan triple system is a vector space $\mathcal{J}$ over a field of characteristic not 2, endowed with a triple product $\{xyz\}$ which is linear and symmetric in the outer variables $x, z$, and linear or anti-linear in the inner variable $y$. 
satisfying the Jordan triple identity: \( \{ xu \{ yvz \} \} + \{ \{ xy \} uz \} - \{ yv \{ xuz \} \} = \{ x \{ yuv \} z \} \) (cf. [7, page 8] or [8, 11]). A \( JB^* \)-triple is a complex Banach space \( J \) together with a continuous, sesquilinear, operator-valued map \( (x, y) \in J \times J \mapsto L_{x,y} \in J \) that defines a triple product \( L_{x,y}z := \{ xy \} z \) in \( J \) making it a Jordan triple system such that each \( L_{x,x} \) is a positive hermitian operator on \( J \) and \( \| \{ xx^* x \} \| = \| x \|^3 \) for all \( x \in J \) (cf. [4, p. 504] or [11, page 336]). Thus, every \( JB^* \)-algebra is a Jordan triple system; more precisely, it is a \( JB^* \)-triple under the triple product \( \{ xy^* z \} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^* \). In particular, every \( C^* \)-algebra is a \( JB^* \)-triple under the triple product \( \{ xy^* z \} := \frac{1}{2}(xy^* z + zy^* x) \).

### 3 Main Results

For any elements \( x, y, z \) in a Jordan triple system with triple product \( \{ \ldots \} \), note that the operators \( P_z := \{ xxz \} \) and \( L_{x,y}z := \{ xyz \} \) are the Jordan triple system analogues of the Jordan algebra operators \( U_{xz} := \{ xzx \} \) and \( V_{xy}z := \{ yzx \} \). Hence, in case of a \( JB^* \)-algebra, \( L_{x,y} = V_{xy}^* \), \( P_z = \{ xx^* x \} = U_{xz}^* \) for all \( z \) (see above).

An element \( x \) in a Jordan triple system \( J \) is said to be trivial if \( P_x J = \{ 0 \} \). A Jordan triple system is called non-degenerate if it has no non-zero trivial element, that is, \( x = 0 \) whenever \( P_x = 0 \) (cf. [8]). Jordan algebras can be considered as Jordan triple systems under the induced triple product with the identity map as involution and so \( P_x = U_x \) in such a case (cf. [11, page 318]). Thus, a Jordan algebra is non-degenerate if and only if \( U_x = 0 \) implies \( x = 0 \) (cf. [7, 4.8]).

**Lemma 1** Any \( JB^* \)-triple is non-degenerate. Thus, all \( JB^* \)-algebras and all \( C^* \)-algebras are non-degenerate Jordan triple systems.

**Proof.** We already know that all \( JB^* \)-algebras including all \( C^* \)-algebras are \( JB^* \)-triples and that \( JB^* \)-triples are Jordan triple systems. Let \( J \) be any fixed \( JB^* \)-triple and let \( x \in J \) such that \( P_x = 0 \). Then \( \{ x, y^*, x \} = 0 \) for \( y \in J \). In particular, \( \{ x, x^*, x \} = 0 \). Hence \( \| x \|^3 = \| \{ x, x^*, x \} \| = 0 \) and so \( x = 0 \). Thus, \( J \) is non-degenerate.

For any fixed elements \( x, y \) in a Jordan triple system \( J \), the well known Bergmann operator \( B(x, y) \) on \( J \) is defined by \( B(x, y) := I - 2L_{x,y} + P_x P_y \) where \( I \) denotes the identity operator on \( J \) (cf. [6, 2.11]). Thus, \( B(x, y) = I - 2V_{xy} + U_x U_y \) for any elements \( x, y \) in a Jordan algebra (with identity map as involution). In case of a \( JB^* \)-algebra \( J \), it takes the form \( B(x, y) = I - 2V_{xy}^* + U_x U_y^* \), which translates to \( B(x, y) = (1 - xy^*) z (1 - y^* x) \) for all \( z \in J \) if \( J \) is a \( C^* \)-algebra.

**Theorem 2** For any elements \( x \) and \( y \) in a non-degenerate Jordan triple system \( J \), \( B(x, y) = 0 \) if and only if \( B(y, x) = 0 \).
Proof. If the Bergmann operator $B(x, y) = 0$, then the identity $P_{B(a,b)c} = B(a,b)P_{B(b,a)}$ (cf. [9] or [6, JP26]) gives $P_{B(y,x)z} = B(y,x)P_z B(x,y) = 0$ for all $z \in \mathcal{J}$. Hence, $P_{B(y,x)z} = 0$ for all $z \in \mathcal{J}$. This together with the non-degeneracy of $\mathcal{J}$ gives $B(y,x) = 0$.

An element $x$ in a Jordan algebra $\mathcal{J}$ is called von Neumann regular if $U_x y = x$ for some $y \in \mathcal{J}$; this condition is equivalent to $yx = x$ if $\mathcal{J}$ is a $C^*$-algebra. In such a case, $y$ is called a generalized inverse of $x$ (cf. [3, 7]).

A study of von Neumann regularity for Jordan triple systems was initiated by K. Meyberg: an element $x$ in a Jordan triple system $\mathcal{J}$ is called von Neumann regular if there exists $y \in \mathcal{J}$, called generalized inverse of $x$, such that $xyx = x$ if $\mathcal{J}$ is a $C^*$-algebra. In such a case, $y$ is called a generalized inverse of $x$ (cf. [9, 3]). If $\mathcal{J}$ is a $JB^*$-algebra, then $x = P_{xy} = \{yx\}^{x}$ and so $y$ is a generalized inverse of $x$ in $\mathcal{J}$ considered as a $JB^*$-algebra if and only if $y^*$ is a generalized inverse of $x$ in the $JB^*$-algebra $\mathcal{J}$. In particular, $x$ has generalized inverse $y^*$ in a $C^*$-algebra $\mathcal{A}$ if and only if $x$ is generalized inverse of $y$ in $\mathcal{A}$ considered as $JB^*$-triple (see [3, page 198]).

Next result shows that von Neumann regularity of $x$ and $y$ is a necessity for the vanishing Bergmann operator $B(x,y)$ on a non-degenerate Jordan triple system.

Theorem 3 Let $x$ and $y$ be elements in a non-degenerate Jordan triple system $\mathcal{J}$. If $B(x, y) = 0$, then $x$ and $y$ are von Neumann regular with generalized inverses of each other.

Proof. We recall the identity $B(x, y)P_x = P_x B(y, x) = P_{x-p_{xy}}$ that holds in any Jordan triple system (cf. [8, 9] or [6, JP23]). This identity together with our hypothesis $B(x, y) = 0$ gives $P_{x-p_{xy}} = 0$. Hence, $P_{x} y = x$ by the non-degeneracy of $\mathcal{J}$. Further, since the system $\mathcal{J}$ is non-degenerate and $B(x, y) = 0$, Theorem 2 implies $B(y, x) = 0$. Hence by interchanging $x$ and $y$ in the above argument, we obtain $P_y x = y$. Thus, $x$ and $y$ are von Neumann regular elements of $\mathcal{J}$ with generalized inverses of each other.

Corollary 4 For any elements $x, y$ in a $JB^*$-triple, $B(x, y) = 0$ implies that $x$ and $y$ are generalized inverses of each other.

Proof. Follows from Lemma 1 and Theorem 3.

Corollary 5 For any elements $x, y$ in a $JB^*$-algebra, $B(x, y) = 0$ implies that $y^*$ is a generalized inverse of $x$ and $x^*$ is a generalized inverse $y$. In particular, the same result holds true for any $C^*$-algebra.

Proof. Immediate from Corollary 4 and the discussion before Theorem 3.

As noticed above that any Jordan algebra can be considered as a Jordan triple system under the induced triple product and identity map as involution, if $x$ and $y$ are any two elements in a non-degenerate Jordan algebra $\mathcal{J}$ with
Let $B(x, y) = 0$ the $x$ and $y$ are generalized inverses of each other by Theorem 3. There do exist degenerate Jordan triple systems. Recall that a non-zero element $x$ in a Jordan algebra $J$ with unit 1 is trivial if $U_x z = 0$ for all $z \in J$, and so $U_x 1 = x^2$ must also vanish. Further, if there is a scalar $\alpha$ in a field $F$ such that $\alpha^2 = 0$ then for any fixed $y \in J$, the element $x := \alpha y$ is a trivial element in $J$ because $U_x z = \{xzx\} = \{\alpha y\} = \alpha^2 \{zy\} = 0 \{zy\} = 0$. Thus any Jordan algebra over such a field $F$ is a degenerate Jordan triple system.

In [10], the present authors introduced an exact analogue of Brown-Pedersen’ quasi invertibility (cf. [1]) for general $JB^*$-triples: an element $x$ in a $JB^*$-triple $J$ is called BP-quasi invertible if $x$ is von Neumann regular with generalized inverse $y$ and $B(x, y)$ vanishes; such a generalized inverse of $x$ is called a BP-quasi inverse of $x$. Following result shows that the von Neumann regularity condition in this is redundant.

**Theorem 6** An element $x$ in a $JB^*$-triple $J$ is BP-quasi-invertible with BP-quasi inverse $y$ if and only if the Bergmann operator $B(x, y)$ vanishes. Or equivalently, $B(y, x)$ vanishes.

**Proof.** One way is clear from the definition of BP-quasi invertible elements. On the other way, if $B(x, y)$ vanishes for some elements $x, y \in J$, then $x$ is a von Neumann regular element with generalized inverse $y$ by Corollary 4. Hence, $x$ is a BP-quasi invertible with BP-quasi inverse $y$. The other equivalence is clear from Theorem 2.

The relation of being BP-quasi inverse of some element in a $JB^*$-triple is symmetric:

**Corollary 7** In a $JB^*$-triple $J$, if $x$ is a BP-quasi invertible element with BP-quasi inverse $y$ then $y$ is BP-quasi invertible with BP-quasi inverse $x$. This remains no longer true when the BP-quasi invertibility is translated in terms of $JB^*$-algebras unless the elements $x$ and $y$ are self-adjoint.

**Proof.** If $x$ is a BP-quasi invertible element with BP-quasi inverse $y$ in $J$, then $B(x, y)$ vanishes. Since the $JB^*$-triple $J$ is non-degenerate by Lemma 1, $B(y, x)$ vanishes by Theorem 2. Hence by Theorem 6, $y$ is a BP-quasi invertible element with BP-quasi inverse $x$. Thus the result follows by the symmetry of the argument in $x$ and $y$. The other part is clear from the definition of a BP-quasi inverse and Theorem 6.

Next result improves [10, Theorem 3.1]:

**Theorem 8** For any element $u$ in a $C^*$-algebra $A$ with unit 1, the following statements are equivalent:

(i). $u$ is BP-quasi invertible in $A$;

(ii). There exist two elements $a, b \in A$ such that $(1 - ua)a(1 - bu) = \{0\}$;
(iii). \( u \) is von Neumann regular with generalized inverse \( v^* \) such that the Bergmann operator \( B(u, v) \) vanishes;
(iv). There exists \( v \in \mathcal{A} \) such that each of the Bergmann operators \( B(u, v), B(u, v^*), B(v, u^*), B(u^*, v), B(v^*, u), B(v^*, u^*) \) vanishes.

**Proof.** For the equivalence of statements \((i) - (iii)\), see [10, Proof of Theorem 3.1]. \((iv) \Rightarrow (iii)\) follows from Theorem 3 since every \( C^* \)-algebra is a non-degenerate Jordan triple system by Lemma 1. Conversely, if \((iii)\) holds true then \( v^* \) is a BP-quasi inverse of \( u \). So, there exists \( v = v^* \in \mathcal{A} \) as a BP-quasi inverse and hence generalized inverse of \( u \) such that the Bergmann operator \( B(u^*, v) \) vanishes by [10, Theorem 3.2]. Thus, by Lemma 1 and Theorem 2, the operators \( B(v, u) \) and \( B(v, u^*) \) both vanishes, too. Moreover, since \( \ast \) is an involution, we easily get \( B(x, y) = 0 \) if and only if \( B(x^*, y^*) = 0 \) for any \( x, y \in \mathcal{A} \). This complete the proof.

By Corollary 4, any BP-quasi invertible element in a \( JB^* \)-triple is necessarily von Neumann regular. The following example shows that even in case of \( C^* \)-algebras, the von Neumann regularity of an element \( x \) is not enough to guarantee BP-quasi invertibility of \( x \). Thus, in general, the set of all von Neumann regular elements properly contains the set of all BP-quasi invertible elements in a \( JB^* \)-triple.

**Example 9** In the \( C^* \)-algebra \( M_2(\mathcal{C}) \) of \( 2 \times 2 \) matrices with entries from the field \( \mathcal{C} \) of complex numbers, note that any generalized inverse of \[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]
is of the form \[
\begin{bmatrix}
x & y \\
1 & z
\end{bmatrix}
\] where \( x, y \) and \( z \) are any complex numbers. However, the Bergmann operator induced by such a pair gives
\[
\left( \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} - \begin{bmatrix}
1 & z \\
0 & 0
\end{bmatrix} \right) \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \left( \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} - \begin{bmatrix}
0 & x \\
0 & 1
\end{bmatrix} \right) = \begin{bmatrix}
-cz & czx \\
c & -cx
\end{bmatrix}
\]
for all \( a, b, c, d \in \mathcal{C} \), which is a non-zero matrix no matter what values are assigned to \( x, y \) and \( z \). Hence, the induced Bergmann operator does not vanish.

We conclude this article with the remark that the notion of BP-quasi invertibility extends naturally to any non-degenerate Jordan triple system as follows: an element \( x \) in a non-degenerate Jordan triple system \( \mathcal{J} \) is called BP-quasi invertible with BP-quasi inverse \( y \in \mathcal{J} \) if the Bergmann operator \( B(x, y) \) vanishes. Hence by Theorem 3, such elements in any non-degenerate Jordan triple system are von Neumann regular, too. However, this remains no longer true for degenerate Jordan triple systems (see the above example). We intend to present in our subsequent papers further results on BP-quasi invertibility in the general setting of non-degenerate Jordan triple systems.
References


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