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A Note on Prime Radicals in $\Gamma-$Near Rings

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Abstract

In 1969, Murata, Kurata and Marubayashi defined $f-$prime ideal in rings and discussed related results. Groenewald and Potgieter generalised these results to near rings in 1984. In the same year, Satyanarayana Bhavanari introduced the concept of Γ-near ring. Later, in 1999, he defined $f-s-$prime ideal, $f-s-$prime radical in Γ-near ring. In this paper, some results of near rings are extended for $f-s-$ prime radical in Γ-near ring and it has been shown that for any two ideals A, B of a Γ-near ring, $A \subseteq B \Rightarrow f-s-rad(A) \subseteq f-s-rad(B)$, but converse need not be true.

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1 Introduction

Throughout this article, M denotes a zero symmetric Γ-near ring, for a fixed non-empty set Γ. For any two non-empty subsets A and B of M, the set $\{a\alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}$ is denoted by AB, and the set $\{a \in A \mid a \notin B\}$ is denoted by $A \setminus B$. For any subset X of M, $< X >$ denotes the smallest ideal of M containing X. For preliminary definitions and results related to near rings, we refer Pilz[5]. Let M be a Γ- near ring. Then a normal subgroup I of (M,

$+$) is said to be (i) a left ideal if $a\alpha(b+i) - a\alpha b \in I$, for all a, $b \in M$, $\alpha \in \Gamma$ and $i \in I$; (ii) a right ideal if $i\alpha a \in I$, for all $a \in M$, $\alpha \in \Gamma$ and $i \in I$; (iii) an ideal if it is both left and right ideal of M. M is said to be a zero-symmetric Γ-near ring if $a\alpha 0{=}0$, for all $a \in M$ and $\alpha \in \Gamma$, where 0 is the additive identity in M. A subset H of M is said to be an $m-$system if, for every $h_1, h_2 \in H$, there exist $h_1^1 \in < h_1 >$, $h_2^1 \in < h_2 >$, and $\alpha \in \Gamma$ such that $h_1^1 \alpha h_2^1 \in H$. A subset H of M is said to be $f-$system if H contains an $m-$system H^*, called kernel of H, such that for every $h \in H$, $f\ (h) \cap H^* \neq \phi$. An ideal A of M is said to be $f-$prime if $M \setminus A$ is an $m-$system. A subset H of M is said to be a multiplicative set if $a, b \in H \Rightarrow a\alpha b \in H$, for some $\alpha \in \Gamma$. A subset H of M is said to be an $f-s-$system if H contains a multiplicative set H^* (kernel of H) such that $f\ (h) \cap H^* \neq \phi$, for all $h \in H$. An Ideal A of M is said to be an $f-s-$prime ideal if $M \setminus A$ is an $f-s-$system in M and A is said to be completely prime ideal of M if $M \setminus A$ is a multiplicative set.

2 Preliminary Notes

Remark 2.1 *In a Γ- near ring M, an m-system is a multiplicative set. But converse need not be true.*

Let M be the near ring of all $n \times n$ matrices over Z. Take $\Gamma = \{.\}$, where "." denotes matrix multiplication in M. Then M is a Γ-near ring. Now, let $H = \{a, a^2, a^3, ...a^n = 0\}$. Then H is a multiplicative set.

But H is not an $m-$system. Take $a = \begin{pmatrix} 0 & 1 & 1 & . & 1 & 1 \\ 0 & 0 & 1 & . & 1 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & 1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix}$ and

$y = a.b = \begin{pmatrix} 1 & 1 & 1 & . & 1 & 1 \\ 1 & 1 & 1 & . & 1 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix}$, where $b = \begin{pmatrix} 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & 0 \\ 1 & 1 & . & . & 1 & 1 \end{pmatrix} \in M$

Now take $b_1 = \begin{pmatrix} 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & 1 & 1 \\ 0 & 0 & 0 & . & 0 & 0 \end{pmatrix} \in M,$

then $y_1 = a.b_1 = \begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 1 & 1 & & . & 1 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix}$. Therefore, $y, y_1 \in < a >$.

But $y\alpha y_1 = y.y_1 = \begin{pmatrix} n-1 & n-1 & . & . & n-1 & n-1 \\ n-1 & n-1 & n-1 & . & n-1 & n-1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ n-1 & n-1 & . & . & n-1 & n-1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix} \notin H$, for all $\alpha \in \Gamma$.

Proposition 2.2 *An ideal A of M is a completely prime ideal if and only if $a\alpha a_1 \in A \Rightarrow a \in A$ or $a_1 \in A$, for all a, $a_1 \in M$, and for all $\alpha \in \Gamma$.*

Proof. Let A be a completely prime ideal. Then $M \setminus A$ is a multiplicative set. Let $m_1\alpha m_2 \in A$. If $m_1 \notin A$ and $m_2 \notin A \Rightarrow m_1 \in M \setminus A$ and $m_2 \in M \setminus A$. So, by definition of multiplicative set, $m_1\alpha m_2 \in M \setminus A$, for some $\alpha \in \Gamma$, which is a contradiction. Hence $m_1 \in A$ or $m_2 \in A$.

Conversely, let $a\alpha a_1 \in A \Rightarrow a \in A$ or $a_1 \in A$, for all a, $a_1 \in M$ and for all $\alpha \in \Gamma$. In order to prove A is a completely prime ideal of M, we prove $M \setminus A$ is a multiplicative set. If possible, let $a\alpha a_1 \notin M \setminus A$, for all a, $a_1 \in M \setminus A$ and $\alpha \in \Gamma$. Then $a\alpha a_1 \in A$, for all $\alpha \in \Gamma$. Then $a \in A$ or $a_1 \in A$, which is a contradiction. Hence the result. ■

Remark 2.3 *It is known that every completely prime ideal of M is an $f-s-$prime ideal (Note (2.2), Bhavanari [1]). But converse need not be true.*

For this, we consider $\mathbb{Z}$, the ring of integers, and $M = \mathbb{Z}$. Take $\Gamma = \{.\}$. Then M is a Γ-near ring. Let $P =< p^2 >$ and $S^* = \{q, q^2 \dots\}$, where p and q are distinct prime numbers. Take $f(a) =< \{a, q\} >$, for each $a \in M$. Then $M \setminus P$ contains the multiplicative set S^* such that $f(a) \cap S^* \neq \phi$, for all $a \in M \setminus P$. So, P is an $f-s-$prime ideal. Here, $P =< p^2 >$ is not a completely prime ideal, because, for $a = 2p$ and $b = 2p$, we have $a.b \in P$ but, neither a nor b in P.

Proposition 2.4 *The intersection of a linearly ordered set of completely prime ideal of M is a completely prime ideal.*

Proof. Let $\mathcal{F}$ be a linearly ordered set of completely prime ideals of M. Let $P = \cap P_i$, for all $P_i \in \mathcal{F}$.

Suppose a, $b \in M$ such that $a\alpha b \in P$, for all $\alpha \in \Gamma$. If possible, let $a \notin P$ and $b \notin P$. Then $a \notin P_1$ and $b \notin P_2$, for some $P_1, P_2 \in \mathcal{F}$. But, $\mathcal{F}$ is a linearly ordered set, so, either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.

Let $P_1 \subseteq P_2$. Then $a \notin P_1$ and $b \notin P_1$, and this implies, $a\alpha b \notin P_1$, for some $\alpha \in \Gamma$, because, P_1 is a completely prime ideal, which is a contradiction. Hence the result.

■

Corollary 2.5 *Every completely prime ideal A of M contains a minimal completely prime ideal.*

Proposition 2.6 *Let I be a direct summand of M. Let P be a completely prime ideal of M. Then $P \cap I$ is a completely prime ideal of M.*

Proof. Let a, $b \in M$ such that $a\alpha b \in P \cap I$, for all $\alpha \in \Gamma$. Then $a\alpha b \in P$ and $a\alpha b \in I$, for all $\alpha \in \Gamma$. But, P is a completely prime ideal, so, $a \in P$ or $b \in P$. In order to prove $a \in P \cap I$ or $b \in P \cap I$, we claim that both a, $b \in I$. Since, I is a direct summand of M, so, there exists an ideal $K \subseteq M$ such that $I + K = M$, and $I \cap K = (0)$. Therefore, $a\alpha b \in I \Rightarrow \ a\alpha b \notin K$, for some $\alpha \in \Gamma$. Now, if $a \notin I$, then $a \in K$. But K is a right ideal. So, $a\alpha b \in K$, for all $a \in K$, $\alpha \in \Gamma$, $b \in M$, a contradiction. Hence, $a \in I$.

If $b \notin I$, then $b \in K$, and K being left ideal, $a\alpha b = a\alpha(0 + b) - a\alpha 0 \in K$, for all $a \in M$, $\alpha \in \Gamma$, $b \in K$, which is a contradiction. Hence the claim. ■

3 $f - s-$PRIME RADICAL

Definition 3.1 *An $f - s-$ prime radical of an ideal A of M is defined to be the set of all those elements $x \in M$, for which every $f - s-$ system that contains x must contains an element of A and it is denoted by $f - s - rad(A)$.*

Lemma 3.2 *Let A be an ideal of M and $K(K^*)$ be an $f - s-$system such that $A \cap K = \phi$. Then A is contained in a maximal ideal P, which does not meet K. In this case, ideal P is an $f - s-$prime ideal.*

Proof. See Lemma (2.4) in Bhavanari [1].

■

Theorem 3.3 *Let A be an ideal of M. Then the $f - s$-prime radical of an ideal A is the intersection of all $f - s-$prime ideals containing A.*

Proof. Let A be an ideal of M. We claim that $f-s-rad(A) \subseteq \cap P$, where intersection is taken over all the $f-s-$prime ideals P containing A. If possible, let $f-s-rad(A) \subsetneq \cap P$. This implies, $f-s-rad(A) \subsetneq P$, for some $f-s-$prime ideals P which contains A. So, there exists some $x \in f-s-rad(A)$ such that $x \notin P$, i.e. $x \in M \setminus P$. But P, being $f-s-$prime ideal, $M \setminus P$ is an $f-s-$system and $x \in f-s-rad(A)$. So, by definition of $f-s-rad(A)$, we have $(M \setminus P) \cap A \neq \phi$. But, this contradicts the fact that $A \subseteq P$. Hence the claim.

Conversely, Let $a \notin f-s-rad(A)$. We claim that $a \notin P$, for some $f-s-$prime ideal P containing A. Since $a \notin f-s-rad(A)$, so, there exists an $f-s-$system $S(S^*)$ such that $a \in S$ and $S \cap A = \phi$. Then by using the Lemma (3.2), there exists an $f-s-$prime ideal P_1 containing A such that $P_1 \cap A = \phi$. So, $a \notin P_1$. ■

Corollary 3.4 *The $f-s-$prime radical of an ideal of M is an ideal of M.*

Proof. Let $S(S^*)$ be an $f-s-$system in M. Let A be an ideal of M such that $S \cap A = \phi$.

Consider $N = \{B \mid B$ is a multiplicative set such that $S^* \subseteq B$, and $B \cap A = \phi\}$. By using Zorn's Lemma, it is easy to prove that there exists a maximal multiplicative set S_1^* such that $S^* \subseteq S_1^*$ and $S_1^* \cap A = \phi$. Now, take $S_1 = \{x \in M \mid f(x) \cap S_1^* \neq \phi\} \cap (M \setminus A)$. Then S_1 is an $f-s-$system with kernel S_1^* and $S_1 \cap A = \phi$. So, by using the Lemma (3.2), there exists a maximal ideal P which contains A and $P \cap S_1 = \phi$, and this ideal P is an $f-s-$prime ideal. So, by the Lemma (2.4) of [1], $M \setminus P$ is an $f-s-$system with kernel $S_1^* + P$ such that $(M \setminus P) \cap A = \phi$. Also, S_1^* is a maximal multiplicative set in N such that $S_1^* \cap A = \phi$. Therefore, $S_1^* + P = S_1^*$ and $M \setminus P = S_1 \Rightarrow P = M \setminus S_1$, but P, being a maximal ideal, is an ideal and S_1 is an arbitrary $f-s-$system. So, $\cap P = \cap (M \setminus S_1)$, where intersection is taken over all $f-s-$system containing A.

Since, $\cap (M \setminus S_1) = f-s-rad(A)$, hence, $f-s-rad(A)$ is an ideal.

■

Definition 3.5 *An $f-s-$prime ideal P is said to be minimal $f-s-$prime ideal belonging to an ideal A if $A \subseteq P$ and there exists a kernel S^* for the $f-s-$system $M \setminus P$ such that S^* is a maximal multiplicative set which does not meet A.*

Remark 3.6 *Every $f-s-$prime ideal in M, which contains the ideal A of M, contains a minimal $f-s-$prime ideal belonging to an ideal A.*

Proof. Let $S(S^*)$ be an $f-s-$system such that $S \cap A = \phi$. Then, by Corollary (3.4), there is an $f-s-$system $S_1(S_1^*)$ such that S_1^* is the maximal multiplicative set and $S_1 \cap A = \phi$. But, $S^* \subseteq S_1^* \Rightarrow S \subseteq S_1$ and, so, $M \setminus S_1 \subseteq M \setminus S$. Since, $A \subseteq M \setminus S_1$, therefore, $M \setminus S$ is an $f-s-$prime ideal and $M \setminus S_1$, is a minimal $f-s-$prime ideal belonging to an ideal A.

■

Proposition 3.7 *The $f-s-$radical of an ideal A of M is equal to the intersection of all minimal $f-s-$prime ideals belonging to A.*

Proof. Let P be an $f-s-$prime ideal in M. Then, from Corollary (3.4), we have $S_1(S_1^*)$ is an $f-s-$system in M and $P = M \setminus S_1 \Rightarrow \cap P = \cap (M \setminus S_1)$, where $M \setminus S_1$ is a minimal $f-s-$prime ideal. So, this implies that intersection of $f-s-$prime ideal P containing A is equal to the intersection of all minimal $f-s-$prime ideal P belonging to A.

Therefore, $f-s-rad(A) =$ Intersection of all minimal $f-s-$prime ideal P belonging to A.

■

Definition 3.8 *Let $S(S^*)$ be an $f-s-$system in M. Then the kernel S^* of S is said to be densed in S if $S \cap A \neq \phi \Rightarrow S^* \cap A \neq \phi$, for any ideal A in M.*

Proposition 3.9 *Let A and B be two ideals in M. Then*

(a) $A \subseteq B \Rightarrow f-s-rad(A) \subseteq f-s-rad(B)$.

(b) $f-s-rad(f-s-rad(A)) = f-s-rad(A)$.

(c) $f-s-rad(A \cap B) = f-s-rad(A) \cap f-s-rad(B)$, if every $f-s-$system in M has a densed kernel in S.

Proof. The proofs of (a) and (b) follow quite easily.

(c) Let $x \in f-s-rad(A \cap B)$. Then, every $f-s-$system containing x, must contains an element of $A \cap B$. This implies $x \in f-s-rad(A)$ and $x \in f-s-rad(B)$. So, $x \in f-s-rad(A) \cap f-s-rad(B)$.

Hence, $f-s-rad(A \cap B) \subseteq f-s-rad(A) \cap f-s-rad(B)$.

Conversely, let $x \in f-s-rad(A) \cap f-s-rad(B)$, this implies that $x \in f-s-rad(A)$ and $x \in f-s-rad(B)$.

So, every $f-s-$system which contains x, must contains an element of A and also an element of B.

Let $S(S^*)$ be an $f-s-$system which contains x and containing an element of A as well as an element of B. Then, $S \cap A \neq \phi$ and $S \cap B \neq \phi$. Also, as every $f-s-$system S in M has a densed kernel S^* in S, we have $S^* \cap A \neq \phi$ and $S^* \cap B \neq \phi$.

Suppose, $a \in S^* \cap A$ and $b \in S^* \cap B$. Then a, $b \in S^*$ and S^* is a multiplicative set, so, $a\alpha b \in S^*$, for some $\alpha \in \Gamma$. Also, A is a right ideal of M. So, $a\alpha b \in A$, where $a \in A$ and $b \in M$. Again, B is a left ideal of M, so, $a\alpha(0+b) - a\alpha 0 \in B$ (since $a\alpha 0 = 0$, for all $a \in M$ and $\alpha \in \Gamma$ in a zero symmetric Γ near ring). So, $a\alpha b \in B$.

Therefore, $a\alpha b \in A \cap B \Rightarrow a\alpha b \in S^* \cap (A \cap B)$. So, $S^* \cap (A \cap B) \neq \phi$. Also, we have $S \subseteq S^*$, so, $S \cap (A \cap B) \neq \phi$. But S is an arbitrary $f-s-$system containing x must contains an element of $A \cap B$. So, $x \in f-s-rad(A \cap B)$

$\Rightarrow f-s-rad(A) \cap f-s-rad(B) \subseteq f-s-rad(A \cap B)$.

Hence, $f-s-rad(A) \cap f-s-rad(B) = f-s-rad(A \cap B)$.

The converse of Proposition [3.9 (a)] is not true. Towards this, we have the following Example

■

Example 3.10 *Let $X = \{a, b\}$ and $G = \{\overline{0}, \overline{1}, ...\overline{5}\}$ mod 6. We define $M = \{f | f : X \longrightarrow G$ such that $f(a) = \overline{0}\} = \{f_0, f_1.....f_5\}$ where $f_i(b) = \overline{i}$. Consider the mapping g_0 from G to X defined by $g_0(i) = a$, for all $i \in G$. Take $\Gamma = \{g_0\}$. Then M is a Γ-near ring.*

Now, we calculate $f-s-$system in Γ- near ring M. For this, we consider all the subsets of M, which are as follows:
$\phi, \{f_0\}, \{f_1\}, \{f_2\}, \{f_3\}, \{f_4\}, \{f_5\}, \{f_0, f_1\}, \{f_0, f_2\}, \{f_0, f_3\}, \{f_0, f_4\}, \{f_0, f_5\},$
$\{f_1, f_2\}, \{f_1, f_3\}, \{f_1, f_4\}, \{f_1, f_5\}, \{f_2, f_3\}, \{f_2, f_4\}, \{f_2, f_5\}, \{f_3, f_4\}, \{f_3, f_5\},$
$\{f_4, f_5\}\{f_0, f_1, f_2\}, \{f_0, f_1, f_3\}, \{f_0, f_1, f_4\}, \{f_0, f_1, f_5\}, \{f_0, f_2, f_3\}, \{f_0, f_2, f_4\},$
$\{f_0, f_2, f_5\}, \{f_0, f_3, f_4\}, \{f_0, f_3, f_5\}, \{f_0, f_4, f_5\}, \{f_1, f_2, f_3\}, \{f_1, f_2, f_4\},$
$\{f_1, f_2, f_5\}, \{f_2, f_3, f_4\}, \{f_2, f_3, f_5\}, \{f_3, f_4, f_5\}, \{f_1, f_3, f_4\}, \{f_1, f_3, f_5\},$
$\{f_1, f_4, f_5\}, \{f_2, f_4, f_5\}, \{f_0, f_1, f_2, f_3\}, \{f_0, f_1, f_2, f_4\}, \{f_0, f_1, f_2, f_5\},$
$\{f_0, f_1, f_3, f_4\}, \{f_0, f_1, f_3, f_5\}, \{f_1, f_2, f_3, f_4\}, \{f_1, f_2, f_3, f_5\}, \{f_2, f_3, f_4, f_5\},$
$\{f_0, f_2, f_4, f_5\}, \{f_0, f_2, f_3, f_4\}, \{f_0, f_2, f_3, f_5\}, \{f_0, f_1, f_4, f_5\}, \{f_0, f_3, f_4, f_5\},$
$\{f_1, f_2, f_4, f_5\}, \{f_1, f_3, f_4, f_5\}, \{f_0, f_1, f_2, f_3, f_4\}, \{f_0, f_1, f_2, f_4, f_5\},$
$\{f_0, f_1, f_3, f_4, f_5\}, \{f_0, f_2, f_3, f_4, f_5\}, \{f_1, f_2, f_3, f_4, f_5\}, \{f_0, f_1, f_2, f_3, f_5\},$
$\{f_0, f_1, f_2, f_3, f_4, f_5\}$.

Clearly, ϕ is an $f-s-$system. We note that $< f_i >= \{f_0\}$, since $f_i^k = f_i \Gamma f_i \Gamma .....\Gamma f_i$. Now, $\{f_0\}$ is an $f-s-$system with a multiplicative set $\{f_0\}$ itself and satisfying $f(x) \cap \{f_0\} \neq \phi$, for all $x \in \{f_0\}$.
But $\{f_1\}$ is not an $f-s-$system, since there does not exist any $g_0 \in \Gamma$ such that $f_1 g_0 f_1 \in \{f_1\}$. Which shows that there does not exist any multiplicative set contained or equal to $\{f_1\}$.
Similarly, we can show that $\{f_2\}, \{f_3\}, \{f_4\}, \{f_5\}$ are not $f-s-$systems. Now, $\{f_0, f_1\}$ is an $f-s-$system, as $\{f_0\}$ is a multiplicative set satisfying $f(x) \cap \{f_0\} \neq \phi$, for all $x \in \{f_0, f_1\}$. Similarly, $\{f_0, f_2\}, \{f_0, f_3\}, \{f_0, f_4\}, \{f_0, f_5\}$ are $f-s-$systems. But $\{f_1, f_2\}$ is not an $f-s-$system as there does not exist

any multiplicative set $S \subseteq \{f_1, f_2\}$ *such that* $f(x) \cap S \neq \phi$, *for all* $x \in \{f_1, f_2\}$. *Now, clearly only those subsets of* M *which contains* f_0 *as an element, are* $f-s-$*systems. Hence, the following are the* $f-s-$*systems in* M*-:*

$\phi, \{f_0\}, \{f_1\}, \{f_0, f_1\}, \{f_0, f_2, \{f_0, f_3\}, \{f_0, f_4\}, \{f_0, f_5\}, \{f_0, f_1, f_2\}, \{f_0, f_1, f_3\}, \{f_0, f_1, f_4\}, \{f_0, f_1, f_5\}, \{f_0, f_2, f_3\}, \{f_0, f_2, f_4\}, \{f_0, f_2, f_5\}, \{f_0, f_3, f_4\}, \{f_0, f_3, f_5\}, \{f_0, f_4, f_5\}, \{f_0, f_1, f_2, f_3\}, \{f_0, f_1, f_2, f_4\}, \{f_0, f_1, f_2, f_5\}, \{f_0, f_1, f_3, f_4\}, \{f_0, f_1, f_3, f_5\}, \{f_0, f_2, f_4, f_5\}, \{f_0, f_2, f_3, f_4\}, \{f_0, f_2, f_3, f_5\}, \{f_0, f_1, f_4, f_5\}, \{f_0, f_3, f_4, f_5\}, \{f_0, f_1, f_2, f_3, f_4\}, \{f_0, f_1, f_2, f_4, f_5\}, \{f_0, f_1, f_3, f_4, f_5\}, \{f_0, f_2, f_3, f_4, f_5\}, \{f_0, f_1, f_2, f_3, f_5\}, \{f_0, f_1, f_2, f_3, f_4, f_5\}$.

Consider $A = \{f_0, f_2, f_4\}$, *which is an ideal of* Γ*-near ring* M. *Then, by using the Definition of* $f-s-rad$ *of an ideal, we can calculate that* $f-s-rad(A) = \{f_0, f_1, f_2, f_3, f_4, f_5\}$
And if we consider $B = \{f_0, f_1, f_3\}$, *then clearly* B *is an ideal of the* Γ*-near ring* M. *Also,* $f-s-rad(B) = \{f_0, f_1, f_2, f_3, f_4, f_5\}$.
So, $f-s-rad(A) = f-s-rad(B)$, *but neither* $A \subseteq B$ *nor* $B \subseteq A$.

Example 3.11 *Let* $M = \{0, a, b, c\}$ *be a Klein's four group with addition and multiplication defined as follows:*

+	0	a	b	c
0	0	a	b	c
a	a	0	b	c
b	b	c	0	a
c	c	b	a	0

and

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

Then we try to find $f-s-$*radical for the ideals* $A_1 = \{o, a\}$ *and* $A_2 = \{o, b\}$. *Take* $\Gamma = \{.\}$ *and* $f\{x\} =< x >$, *for all* $x \in M$. *Then clearly,* M *is a* Γ*-near ring. Now, we find all the* $f-s-$*system in* M .
The following are all the subsets of M.
$\phi, M, \{0\}, \{a\}, \{b\}, \{c\}, \{0, a\}, \{0, b\}, \{0, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{0, a, b\}, \{0, b, c\}, \{0, a, c\}, \{a, b, c\}$

Clearly, ϕ *is an* $f-s-$*system. Also,* $\{0\}$ *is a multiplicative set as well as an* $f-s-$*system with kernel itself. Similarly,* $\{a\}$ *is also an* $f-s-$*system with kernel itself as* $a.a = a \in \{a\}$ *and* $f(x) \cap \{a\} \neq \phi$, *for all* $x \in \{a\}$.
But $\{b\}$ *is not an* $f-s-$*system since for* $b \in \{b\}, b.b = 0 \notin \{b\}$, *which shows that* $\{b\}$ *is not a multiplicative set. Similarly, we can check that* $\{c\}$ *is not an* $f-s-$ *system.*
Now, $\{0, a\}$ *is an* $f-s-$*system with kernel* $\{0\}$, *because* $f\{a\} =< a >= \{o, a, o, a\}$. *So,* $f(x) \cap \{0\} \neq \phi$, *for all* $x \in \{o, a\}$. *Similarly* $\{0, a\}, \{0, b\}, \{a, c\}$,

$\{0,c\},\{0,a,b\},\{o,a,c\},M$ are $f-s-$systems with kernel $\{0\}$.
But $\{a,b\},\{b,c\},\{0,b,c\},\{a,b,c\}$, are not $f-s-$systems as there does not exist any Kernel for these sets. So, following are the $f-s-$systems in M:
$\phi, M, \{0\}, \{a\}, \{0,a\}, \{0,b\}, \{a,c\}, \{0,a,b\}, \{0,a,c\}$
Now, if $A_1=\{o,a\}$ is an ideal of M, then $f-s-rad(A_1)=\{0,a,b,c\}$ and for ideal $A_2=\{o,b\}$, the $f-s-rad(A_2)=\{0,a,b,c\}$.
So, $f-s-rad(A_1)=f-s-rad(A_2)$. But neither $A_1\subseteq A_2$ nor $A_2\subseteq A_1$.

Remark 3.12 *In the above Example [3.11], we see that if $a\in f-s-rad(A)$, then a need not be an element of an ideal A. e.g. $c\in f-s-rad(A_1)$, but $c\notin A_1$, even though $c^2=c.c=a\in A_1$. Now, we generalise this observation as follows:*

Proposition 3.13 *If $a\in f-s-rad(A)\Rightarrow \exists\ k\in N: a^k\in A$, where A is an ideal of M.*

Proof. Consider $K=\{a,a^2,a^3,...\}$ and let $K^*=\{a^2,a^4.....\}$. Then K^* is a multiplicative set such that $K^*\subseteq K$.
Let $f(a)=<a^2>$. Then $f(a)\cap K^*\neq\phi$, for all $a\in K$, this shows that K is an $f-s-$system in M.
Let $a\in f-s-rad(A)$ and if $a^k\notin A$, for any $k\in N$, then $A\cap K=\phi$ and by using the Lemma (3.2), we have $P\cap K=\phi$. So, $a\notin P$, where P is an $f-s-$prime ideal containing A. This implies that $a\notin f-s-rad(A)$, a contradiction. Hence the result.
■

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