A Note on Prime Radicals
in $\Gamma$-Near Rings

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Abstract

In 1969, Murata, Kurata and Marubayashi defined $f$-prime ideal in rings and discussed related results. Groenewald and Potgieter generalised these results to near rings in 1984. In the same year, Satyanarayana Bhavanari introduced the concept of $\Gamma$-near ring. Later, in 1999, he defined $f-s$-prime ideal, $f-s$-prime radical in $\Gamma$-near ring. In this paper, some results of near rings are extended for $f-s$-prime radical in $\Gamma$-near ring and it has been shown that for any two ideals $A, B$ of a $\Gamma$-near ring, $A \subseteq B \Rightarrow f-s-rad(A) \subseteq f-s-rad(B)$, but converse need not be true.

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1 Introduction

Throughout this article, $M$ denotes a zero symmetric $\Gamma$-near ring, for a fixed non-empty set $\Gamma$. For any two non-empty subsets $A$ and $B$ of $M$, the set $\{a \alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}$ is denoted by $AB$, and the set $\{a \in A \mid a \notin B\}$ is denoted by $A \setminus B$. For any subset $X$ of $M$, $<X>$ denotes the smallest ideal of $M$ containing $X$. For preliminary definitions and results related to near rings, we refer Pilz[5]. Let $M$ be a $\Gamma$-near ring. Then a normal subgroup $I$ of $(M,$
+) is said to be (i) a left ideal if \(a\alpha(b + i) - aab \in I\), for all \(a, b \in M\), \(\alpha \in \Gamma\) and \(i \in I\); (ii) a right ideal if \(i\alpha a \in I\), for all \(a \in M\), \(\alpha \in \Gamma\) and \(i \in I\); (iii) an ideal if it is both left and right ideal of \(M\). \(M\) is said to be a zero-symmetric \(\Gamma\)-near ring if \(a\alpha 0 = 0\), for all \(a \in M\) and \(\alpha \in \Gamma\), where 0 is the additive identity in \(M\). A subset \(H\) of \(M\) is said to be \(f\)-system if \(H\) contains an \(m\)-system \(H^*\), called kernel of \(H\), such that for every \(h \in H\), \(f(h) \cap H^* \neq \phi\). An ideal \(A\) of \(M\) is said to be \(f\)-prime if \(M \setminus A\) is an \(m\)-system. A subset \(H\) of \(M\) is said to be a multiplicative set if \(a, b \in H \Rightarrow aab \in H\), for some \(\alpha \in \Gamma\). A subset \(H\) of \(M\) is said to be an \(f - s\)-system if \(H\) contains a multiplicative set \(H^*\) (kernel of \(H\)) such that \(f(h) \cap H^* \neq \phi\), for all \(h \in H\). An Ideal \(A\) of \(M\) is said to be an \(f - s\)-prime ideal if \(M \setminus A\) is an \(f - s\)-system in \(M\) and \(A\) is said to be completely prime ideal of \(M\) if \(M \setminus A\) is a multiplicative set.

## 2 Preliminary Notes

**Remark 2.1** In a \(\Gamma\)-near ring \(M\), an \(m\)-system is a multiplicative set. But converse need not be true.

Let \(M\) be the near ring of all \(n \times n\) matrices over \(Z\). Take \(\Gamma = \{.\}\), where "." denotes matrix multiplication in \(M\). Then \(M\) is a \(\Gamma\)-near ring. Now, let \(H = \{a, a^2, a^3, \ldots \ldots a^n = 0\}\). Then \(H\) is a multiplicative set.

\[
\begin{pmatrix}
0 & 1 & 1 & . & 1 & 1 \\
0 & 0 & 1 & . & 1 & 1 \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & . & . & 0 & 0 \\
\end{pmatrix}
\]

But \(H\) is not an \(m\)-system. Take \(a = \begin{pmatrix}
1 & 1 & 1 & . & 1 & 1 \\
1 & 1 & 1 & . & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & . & 1 & 1 \\
0 & 0 & . & 0 & 0 \\
\end{pmatrix}
\) and \(b = \begin{pmatrix}
0 & 0 & . & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & . & 0 & 0 \\
1 & 1 & . & 1 & 1 \\
\end{pmatrix}
\) \(\in M\).

\(y = a.b = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 1 & . & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\) \(\in M\),

Now take \(b_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\) \(\in M\),
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then $y_1 = a.b_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$. Therefore, $y, y_1 \in \langle a \rangle$.

But $y_\alpha y_1 = y.y_1 = \begin{pmatrix} n-1 & n-1 & \cdots & n-1 & n-1 \\ n-1 & n-1 & n-1 & n-1 & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & \cdots & n-1 & n-1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \notin H$, for all $\alpha \in \Gamma$.

**Proposition 2.2** An ideal $A$ of $M$ is a completely prime ideal if and only if $a_\alpha a_1 \in A \Rightarrow a \in A$ or $a_1 \in A$, for all $a, a_1 \in M$, and for all $\alpha \in \Gamma$.

**Proof.** Let $A$ be a completely prime ideal. Then $M \setminus A$ is a multiplicative set. Let $m_1 \alpha m_2 \in A$. If $m_1 \notin A$ and $m_2 \notin A \Rightarrow m_1 \in M \setminus A$ and $m_2 \in M \setminus A$. So, by definition of multiplicative set, $m_1 \alpha m_2 \in M \setminus A$, for some $\alpha \in \Gamma$, which is a contradiction. Hence $m_1 \in A$ or $m_2 \in A$.

Conversely, let $a_\alpha a_1 \in A \Rightarrow a \in A$ or $a_1 \in A$, for all $a, a_1 \in M$ and for all $\alpha \in \Gamma$. In order to prove $A$ is a completely prime ideal of $M$, we prove $M \setminus A$ is a multiplicative set. If possible, let $a_\alpha a_1 \notin M \setminus A$, for all $a, a_1 \in M \setminus A$ and $\alpha \in \Gamma$. Then $a_\alpha a_1 \in A$, for all $\alpha \in \Gamma$. Then $a \in A$ or $a_1 \in A$, which is a contradiction. Hence the result. 

**Remark 2.3** It is known that every completely prime ideal of $M$ is an $f$–$s$–prime ideal (Note (2.2), Bhavanari [1]). But converse need not be true.

For this, we consider $Z$, the ring of integers, and $M = Z$. Take $\Gamma = \{\cdot\}$. Then $M$ is a $\Gamma$-near ring. Let $P = \langle p^2 \rangle$ and $S^* = \{q, q^2, \ldots\}$, where $p$ and $q$ are distinct prime numbers. Take $f(a) = \langle \{a, q\} \rangle$, for each $a \in M$. Then $M \setminus P$ contains the multiplicative set $S^*$ such that $f(a) \cap S^* \neq \phi$, for all $a \in M \setminus P$. So, $P$ is an $f$–$s$–prime ideal. Here, $P = \langle p^2 \rangle$ is not a completely prime ideal, because, for $a = 2p$ and $b = 2p$, we have $a.b \in P$ but, neither $a$ nor $b$ in $P$.

**Proposition 2.4** The intersection of a linearly ordered set of completely prime ideal of $M$ is a completely prime ideal.
Proof. Let $F$ be a linearly ordered set of completely prime ideals of $M$. Let $P = \cap P_i$, for all $P_i \in F$.
Suppose $a, b \in M$ such that $aab \in P$, for all $\alpha \in \Gamma$. If possible, let $a \notin P$ and $b \notin P$. Then $a \notin P_1$ and $b \notin P_2$, for some $P_1, P_2 \in F$. But, $F$ is a linearly ordered set, so, either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.
Let $P_1 \subseteq P_2$. Then $a \notin P_1$ and $b \notin P_2$, and this implies, $aab \notin P_1$, for some $\alpha \in \Gamma$, because, $P_1$ is a completely prime ideal, which is a contradiction. Hence the result.

Corollary 2.5 Every completely prime ideal $A$ of $M$ contains a minimal completely prime ideal.

Proposition 2.6 Let $I$ be a direct summand of $M$. Let $P$ be a completely prime ideal of $M$. Then $P \cap I$ is a completely prime ideal of $M$.

Proof. Let $a, b \in M$ such that $aab \in P \cap I$, for all $\alpha \in \Gamma$. Then $aab \in P$ and $aab \in I$, for all $\alpha \in \Gamma$. But, $P$ is a completely prime ideal, so, $a \in P$ or $b \in P$. In order to prove $a \in P \cap I$ or $b \in P \cap I$, we claim that both $a, b \in I$.
Since, $I$ is a direct summand of $M$, so, there exists an ideal $K \subseteq M$ such that $I + K = M$, and $I \cap K = (0)$. Therefore, $aab \in I \Rightarrow aab \notin K$, for some $\alpha \in \Gamma$. Now, if $a \notin I$, then $a \in K$. But $K$ is a right ideal. So, $aab \in K$, for all $a \in K, \alpha \in \Gamma, b \in M$, a contradiction. Hence, $a \in I$.
If $b \notin I$, then $b \in K$, and $K$ being left ideal, $aab = aa(0 + b) - aa0 \in K$, for all $a \in M, \alpha \in \Gamma, b \in K$, which is a contradiction. Hence the claim.

3 $f - s$-PRIME RADICAL

Definition 3.1 An $f - s$- prime radical of an ideal $A$ of $M$ is defined to be the set of all those elements $x \in M$, for which every $f - s$- system that contains $x$ must contains an element of $A$ and it is denoted by $f - s - rad(A)$.

Lemma 3.2 Let $A$ be an ideal of $M$ and $K(K^*)$ be an $f - s$-system such that $A \cap K = \phi$. Then $A$ is contained in a maximal ideal $P$, which does not meet $K$. In this case, ideal $P$ is an $f - s$-prime ideal.

Proof. See Lemma (2.4) in Bhavanari [1].

Theorem 3.3 Let $A$ be an ideal of $M$. Then the $f - s$-prime radical of an ideal $A$ is the intersection of all $f - s$-prime ideals containing $A$.  

Proof. Let $A$ be an ideal of $M$. We claim that $f - s - \text{rad}(A) \subseteq \cap P$, where intersection is taken over all the $f - s$-prime ideals $P$ containing $A$. If possible, let $f - s - \text{rad}(A) \nsubseteq \cap P$. This implies, $f - s - \text{rad}(A) \nsubseteq P$, for some $f - s$-prime ideals $P$ which contains $A$. So, there exists some $x \in f - s - \text{rad}(A)$ such that $x \notin P$, i.e. $x \in M \setminus P$. But $P$, being $f - s$-prime ideal, $M \setminus P$ is an $f - s$-system and $x \in f - s - \text{rad}(A)$. So, by definition of $f - s - \text{rad}(A)$, we have $(M \setminus P) \cap A \neq \emptyset$. But, this contradicts the fact that $A \subseteq P$. Hence the claim.

Conversely, Let $a \notin f - s - \text{rad}(A)$. We claim that $a \notin P$, for some $f - s$-prime ideal $P$ containing $A$. Since $a \notin f - s - \text{rad}(A)$, so, there exists an $f - s$-system $S(S^*)$ such that $a \in S$ and $S \cap A = \emptyset$. Then by using the Lemma (3.2), there exists an $f - s$-prime ideal $P_1$ containing $A$ such that $P_1 \cap A = \emptyset$. So, $a \notin P_1$. 

Corollary 3.4 The $f - s$-prime radical of an ideal of $M$ is an ideal of $M$.

Proof. Let $S(S^*)$ be an $f - s$-system in $M$. Let $A$ be an ideal of $M$ such that $S \cap A = \emptyset$.

Consider $N = \{B \mid B$ is a multiplicative set such that $S^* \subseteq B$, and $B \cap A = \emptyset\}$. By using Zorn’s Lemma, it is easy to prove that there exists a maximal multiplicative set $S_1^*$ such that $S^* \subseteq S_1^*$ and $S_1^* \cap A = \emptyset$. Now, take $S_1 = \{x \in M \mid f(x) \cap S_1^* \neq \emptyset\} \cap (M \setminus A)$. Then $S_1$ is an $f - s$-system with kernel $S_1^*$ and $S_1 \cap A = \emptyset$. So, by using the Lemma (3.2), there exists a maximal ideal $P$ which contains $A$ and $P \cap S_1 = \emptyset$, and this ideal $P$ is an $f - s$-prime ideal. So, by the Lemma (2.4) of [1], $M \setminus P$ is an $f - s$-system with kernel $S_1^* + P$ such that $(M \setminus P) \cap A = \emptyset$. Also, $S_1^*$ is a maximal multiplicative set in $N$ such that $S_1^* \cap A = \emptyset$. Therefore, $S_1^* + P = S_1^*$ and $M \setminus P = S_1 \Rightarrow P = M \setminus S_1$, but $P$, being a maximal ideal, is an ideal and $S_1$ is an arbitrary $f - s$-system. So, $\cap P = \cap (M \setminus S_1)$, where intersection is taken over all $f - s$-system containing $A$.

Since, $\cap (M \setminus S_1) = f - s - \text{rad}(A)$, hence, $f - s - \text{rad}(A)$ is an ideal.

Definition 3.5 An $f - s$-prime ideal $P$ is said to be minimal $f - s$-prime ideal belonging to an ideal $A$ if $A \subseteq P$ and there exists a kernel $S^*$ for the $f - s$-system $M \setminus P$ such that $S^*$ is a maximal multiplicative set which does not meet $A$.

Remark 3.6 Every $f - s$-prime ideal in $M$, which contains the ideal $A$ of $M$, contains a minimal $f - s$-prime ideal belonging to an ideal $A$. 
**Proof.** Let $S(S^*)$ be an $f - s$–system such that $S \cap A = \phi$. Then, by Corollary (3.4), there is an $f - s$–system $S_1(S^*)$ such that $S_1^*$ is the maximal multiplicative set and $S_1 \cap A = \phi$. But, $S^* \subseteq S_1^* \Rightarrow S \subseteq S_1$ and, so, $M \setminus S_1 \subseteq M \setminus S$. Since, $A \subseteq M \setminus S_1$, therefore, $M \setminus S$ is an $f - s$–prime ideal and $M \setminus S_1$, is a minimal $f - s$–prime ideal belonging to an ideal $A$.

**Proposition 3.7** The $f - s$–radical of an ideal $A$ of $M$ is equal to the intersection of all minimal $f - s$–prime ideals belonging to $A$.

**Proof.** Let $P$ be an $f - s$–prime ideal in $M$. Then, from Corollary (3.4), we have $S_1(S^*)$ is an $f - s$–system in $M$ and $P = M \setminus S_1 \Rightarrow \cap P = \cap (M \setminus S_1)$, where $M \setminus S_1$ is a minimal $f - s$–prime ideal. So, this implies that intersection of $f - s$–prime ideal $P$ containing $A$ is equal to the intersection of all minimal $f - s$–prime ideal $P$ belonging to $A$.

Therefore, $f - s - \text{rad}(A) = \text{Intersection of all minimal } f - s - \text{prime ideal } P$ belonging to $A$.

**Definition 3.8** Let $S(S^*)$ be an $f - s$–system in $M$. Then the kernel $S^*$ of $S$ is said to be densed in $S$ if $S \cap A \neq \phi \Rightarrow S^* \cap A \neq \phi$, for any ideal $A$ in $M$.

**Proposition 3.9** Let $A$ and $B$ be two ideals in $M$. Then

(a) $A \subseteq B \Rightarrow f - s - \text{rad}(A) \subseteq f - s - \text{rad}(B)$.

(b) $f - s - \text{rad}(f - s - \text{rad}(A)) = f - s - \text{rad}(A)$.

(c) $f - s - \text{rad}(A \cap B) = f - s - \text{rad}(A) \cap f - s - \text{rad}(B)$, if every $f - s$–system in $M$ has a densed kernel in $S$.

**Proof.** The proofs of (a) and (b) follow quite easily.

(c) Let $x \in f - s - \text{rad}(A \cap B)$. Then, every $f - s$–system containing $x$, must contains an element of $A \cap B$. This implies $x \in f - s - \text{rad}(A)$ and $x \in f - s - \text{rad}(B)$. So, $x \in f - s - \text{rad}(A \cap f - s - \text{rad}(B))$.

Hence, $f - s - \text{rad}(A \cap B) \subseteq f - s - \text{rad}(A) \cap f - s - \text{rad}(B)$.

Conversely, let $x \in f - s - \text{rad}(A) \cap f - s - \text{rad}(B)$, this implies that $x \in f - s - \text{rad}(A)$ and $x \in f - s - \text{rad}(B)$.

So, every $f - s$–system which contains $x$, must contains an element of $A$ and also an element of $B$.

Let $S(S^*)$ be an $f - s$–system which contains $x$ and containing an element of $A$ as well as an element of $B$. Then, $S \cap A \neq \phi$ and $S \cap B \neq \phi$. Also, as every $f - s$–system $S$ in $M$ has a densed kernel $S^*$ in $S$, we have $S^* \cap A \neq \phi$ and $S^* \cap B \neq \phi$. 


Suppose, \( a \in S^* \cap A \) and \( b \in S^* \cap B \). Then \( a, b \in S^* \) and \( S^* \) is a multiplicative set, so, \( aab \in S^* \), for some \( \alpha \in \Gamma \). Also, \( A \) is a right ideal of \( M \). So, \( aab \in A \), where \( a \in A \) and \( b \in M \). Again, \( B \) is a left ideal of \( M \), so, \( a\alpha(0 + b) − αa0eB \) (since \( a\alpha0 = 0 \), for all \( a \in M \) and \( \alpha \in \Gamma \) in a zero symmetric \( \Gamma \)-near ring). So, \( a\alpha b \in B \).

Therefore, \( aab \in A \cap B \Rightarrow aab \in S^* \cap (A \cap B) \). So, \( S^* \cap (A \cap B) \neq \emptyset \). Also, we have \( S \subseteq S^* \), so, \( S \cap (A \cap B) \neq \emptyset \). But \( S \) is an arbitrary \( f - s \)-system containing \( x \) must contains an element of \( A \cap B \). So, \( x \in f - s - rad(A \cap B) \)

\[ f - s - rad(A) \cap f - s - rad(B) \subseteq f - s - rad(A \cap B) \]

Therefore, \( f - s - rad(A) \cap f - s - rad(B) = f - s - rad(A \cap B) \).

The converse of Proposition [3.9 (a)] is not true. Towards this, we have the following Example

**Example 3.10** Let \( X = \{a, b\} \) and \( G = \{\overline{0}, \overline{1}, \ldots, \overline{5}\} \mod 6 \). We define \( M = \{f|f : X \rightarrow G \text{ such that } f(a) = \overline{0}\} = \{f_0, f_1, \ldots, f_3\} \) where \( f_i(b) = \overline{i} \).

Consider the mapping \( g_0 \) from \( G \) to \( X \) defined by \( g_0(i) = a \), for all \( i \in G \).

Take \( \Gamma = \{g_0\} \). Then \( M \) is a \( \Gamma \)-near ring.

Now, we calculate \( f - s \)-system in \( \Gamma \)-near ring \( M \). For this, we consider all the subsets of \( M \), which are as follows:

\[ \phi, \{f_0\}, \{f_1\}, \{f_2\}, \{f_3\}, \{f_4\}, \{f_5\}, \{f_0, f_1\}, \{f_0, f_2\}, \{f_0, f_3\}, \{f_0, f_4\}, \}

\[ \{f_0, f_5\}, \{f_1, f_2\}, \{f_1, f_3\}, \{f_1, f_4\}, \{f_1, f_5\}, \{f_2, f_3\}, \{f_2, f_4\}, \{f_2, f_5\}, \{f_3, f_4\}, \{f_3, f_5\}, \{f_4, f_5\}, \{f_0, f_1, f_3\}, \{f_0, f_1, f_4\}, \{f_0, f_2, f_3\}, \{f_0, f_2, f_4\}, \{f_0, f_2, f_5\}, \{f_0, f_3, f_4\}, \{f_0, f_3, f_5\}, \{f_0, f_4, f_5\}, \{f_1, f_2, f_3\}, \{f_1, f_2, f_4\}, \{f_1, f_2, f_5\}, \{f_1, f_3, f_4\}, \{f_1, f_3, f_5\}, \{f_1, f_4, f_5\}, \{f_2, f_3, f_4\}, \{f_2, f_3, f_5\}, \{f_2, f_4, f_5\}, \{f_3, f_4, f_5\}, \{f_1, f_5\}, \{f_2, f_4, f_5\}, \{f_0, f_1, f_2, f_3\}, \{f_0, f_1, f_2, f_4\}, \{f_0, f_1, f_2, f_5\}, \{f_0, f_1, f_3, f_4\}, \{f_0, f_1, f_3, f_5\}, \{f_0, f_1, f_4, f_5\}, \{f_0, f_2, f_3, f_4\}, \{f_0, f_2, f_3, f_5\}, \{f_0, f_2, f_4, f_5\}, \{f_0, f_3, f_4, f_5\}, \{f_0, f_4, f_5\}, \{f_0, f_5\}, \{f_0, f_1, f_2, f_3, f_4\}, \{f_0, f_1, f_2, f_3, f_5\}, \{f_0, f_1, f_2, f_4, f_5\}, \{f_0, f_1, f_3, f_4, f_5\}, \{f_0, f_2, f_3, f_4, f_5\}, \}

Clearly, \( \phi \) is an \( f - s \)-system. We note that \( \langle f_i \rangle = \{f_0\} \), since \( f_i^x = f_i \Gamma f_i \Gamma \ldots \Gamma f_i \). Now, \( \{f_0\} \) is an \( f - s \)-system with a multiplicative set \( \{f_0\} \) itself and satisfying \( f(x) \cap \{f_0\} \neq \emptyset \), for all \( x \in \{f_0\} \).

But \( \{f_1\} \) is not an \( f - s \)-system, since there does not exist any \( g_0 \in \Gamma \) such that \( f_1 g_0 f_1 \in \{f_1\} \). Which shows that there does not exist any multiplicative set contained or equal to \( \{f_1\} \).

Similarly, \( \{f_2\}, \{f_3\}, \{f_4\}, \{f_5\} \) are not \( f - s \)-systems. Now, \( \{f_0, f_1\} \) is an \( f - s \)-system, as \( \{f_0\} \) is a multiplicative set satisfying \( f(x) \cap \{f_0\} \neq \emptyset \), for all \( x \in \{f_0, f_1\} \). Similarly, \( \{f_0, f_2\}, \{f_0, f_3\}, \{f_0, f_4\}, \{f_0, f_5\} \) are \( f - s \)-systems. But \( \{f_1, f_2\} \) is not an \( f - s \)-system as there does not exist
any multiplicative set $S \subseteq \{f_1, f_2\}$ such that $f(x) \cap S \neq \phi$, for all $x \in \{f_1, f_2\}$. Now, clearly only those subsets of $M$ which contains $f_0$ as an element, are $f - s -$ systems. Hence, the following are the $f - s -$ systems in $M$:

\[
\begin{align*}
\phi, \{f_0\}, \{f_0, f_1\}, \{f_0, f_2\}, \{f_0, f_3\}, \{f_0, f_1, f_2\}, \{f_0, f_1, f_3\}, \\
\{f_0, f_1, f_4\}, \{f_0, f_1, f_5\}, \{f_0, f_2, f_3\}, \{f_0, f_2, f_4\}, \{f_0, f_2, f_5\}, \{f_0, f_3, f_4\}, \\
\{f_0, f_3, f_5\}, \{f_0, f_4, f_5\}, \{f_0, f_1, f_2, f_3\}, \{f_0, f_1, f_2, f_4\}, \{f_0, f_1, f_2, f_5\}, \{f_0, f_1, f_3, f_4\}, \\
\{f_0, f_1, f_3, f_5\}, \{f_0, f_2, f_3, f_4\}, \{f_0, f_2, f_3, f_5\}, \{f_0, f_2, f_4, f_5\}, \{f_0, f_3, f_4, f_5\}, \\
\{f_0, f_3, f_5, f_6\}, \{f_0, f_4, f_5, f_6\}, \{f_0, f_1, f_2, f_3, f_4\}, \{f_0, f_1, f_2, f_3, f_5\}, \{f_0, f_1, f_2, f_4, f_5\}, \{f_0, f_1, f_3, f_4, f_5\}, \\
\{f_0, f_1, f_3, f_5, f_6\}, \{f_0, f_2, f_3, f_4, f_5\}, \{f_0, f_2, f_3, f_5, f_6\}, \{f_0, f_2, f_4, f_5, f_6\}, \{f_0, f_3, f_4, f_5, f_6\}, \\
\{f_0, f_3, f_5, f_6, f_7\}, \{f_0, f_4, f_5, f_6, f_7\}, \{f_0, f_1, f_2, f_3, f_4, f_5\}, \{f_0, f_1, f_2, f_3, f_5, f_6\}, \{f_0, f_1, f_2, f_4, f_5, f_6\}, \{f_0, f_1, f_3, f_4, f_5, f_6\}.
\end{align*}
\]

Consider $A = \{f_0, f_2, f_4\}$, which is an ideal of the $\Gamma$-near ring $M$. Then, by using the Definition of $f - s - \text{rad}$ of an ideal, we can calculate that $f - s - \text{rad}(A) = \{f_0, f_1, f_2, f_3, f_4, f_5\}$

And if we consider $B = \{f_0, f_1, f_3\}$, then clearly $B$ is an ideal of the $\Gamma$-near ring $M$. Also, $f - s - \text{rad}(B) = \{f_0, f_1, f_2, f_3, f_4, f_5\}$.

So, $f - s - \text{rad}(A) = f - s - \text{rad}(B)$, but neither $A \subseteq B$ nor $B \subseteq A$.

**Example 3.11** Let $M = \{0, a, b, c\}$ be a Klein’s four group with addition and multiplication defined as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
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</tbody>
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<table>
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<th>.</th>
<th>0</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>a</td>
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<td>0</td>
<td>a</td>
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<td>b</td>
<td>b</td>
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<td>0</td>
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</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
</tbody>
</table>

Then we try to find $f - s -$ radical for the ideals $A_1 = \{a\}$ and $A_2 = \{a, b\}$. Take $\Gamma = \{\}\,$ and $f(x) = < x >$, for all $x \in M$. Then clearly, $M$ is a $\Gamma$-near ring. Now, we find all the $f - s -$ system in $M$.

The following are all the subsets of $M$.

$\phi, M, \{0\}, \{a\}, \{b\}, \{c\}, \{0, a\}, \{0, b\}, \{0, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{0, a, b\}, \{0, b, c\}, \{0, a, c\}, \{a, b, c\}$

Clearly, $\phi$ is an $f - s -$ system. Also, $\{0\}$ is a multiplicative set as well as an $f - s -$ system with kernel itself. Similarly, $\{a\}$ is also an $f - s -$ system with kernel itself as $a.a = a \in \{a\}$ and $f(x) \cap \{a\} \neq \phi$, for all $x \in \{a\}$.

But $\{b\}$ is not an $f - s -$ system since for $b \in \{b\}, b.b = 0 \notin \{b\},$ which shows that $\{b\}$ is not a multiplicative set. Similarly, we can check that $\{c\}$ is not an $f - s -$ system.

Now, $\{0, a\}$ is an $f - s -$ system with kernel $\{0\}$, because $f\{a\} =< a > = \{0, a, o, a\}$. So, $f(x) \cap \{0\} \neq \phi$, for all $x \in \{0, a\}$. Similarly $\{0, a\}, \{0, b\}, \{a, c\}$,
\{0,c\}, \{0,a,b\}, \{o,a,c\}, M are \(f-s\)-systems with kernel \(\{0\}\).

But \(\{a,b\}, \{b,c\}, \{0,b,c\}\), \(\{a,b,c\}\), are not \(f-s\)-systems as there does not exist any Kernel for these sets. So, following are the \(f-s\)-systems in \(M\):

\(\phi, M, \{0\}, \{a\}, \{0,a\}, \{a,c\}, \{0,a,b\}\), \(\{0,a,c\}\), \(\{0,a,b\}\), \(\{a,c\}\), \(\{0,a,b,c\}\), \(\{0\}\), \(\{a\}\), \(\{0,a\}\), \(\{a,c\}\), \(\{0,a,b\}\), \(\{0,a,c\}\), \(\{0,a,b,c\}\).

Now, if \(A_1 = \{o,a\}\) is an ideal of \(M\), then \(f-s-\text{rad}(A_1) = \{0,a,b,c\}\) and for ideal \(A_2 = \{o,b\}\), the \(f-s-\text{rad}(A_2) = \{0,a,b,c\}\).

So, \(f-s-\text{rad}(A_1) = f-s-\text{rad}(A_2)\). But neither \(A_1 \subseteq A_2\) nor \(A_2 \subseteq A_1\).

**Remark 3.12** In the above Example [3.11], we see that if \(a \in f-s-\text{rad}(A)\), then \(a\) need not be an element of an ideal \(A\). e.g. \(c \in f-s-\text{rad}(A_1)\), but \(c \notin A_1\), even though \(c^2 = c.c = a \in A_1\). Now, we generalise this observation as follows:

**Proposition 3.13** If \(a \in f-s-\text{rad}(A)\) \(\Rightarrow \exists k \in N : a^k \in A\), where \(A\) is an ideal of \(M\).

**Proof.** Consider \(K = \{a, a^2, a^3, \ldots\}\) and let \(K^* = \{a^2, a^4, \ldots\}\). Then \(K^*\) is a multiplicative set such that \(K^* \subseteq K\).

Let \(f(a) = \langle a^2 \rangle\). Then \(f(a) \cap K^* \neq \phi\), for all \(a \in K\), this shows that \(K\) is an \(f-s\)-system in \(M\).

Let \(a \in f-s-\text{rad}(A)\) and if \(a^k \notin A\), for any \(k \in N\), then \(A \cap K = \phi\) and by using the Lemma (3.2), we have \(P \cap K = \phi\). So, \(a \notin P\), where \(P\) is an \(f-s\)-prime ideal containing \(A\). This implies that \(a \notin f-s-\text{rad}(A)\), a contradiction. Hence the result.

\[
\blacksquare
\]

**References**


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